Fixed Point Theorem for Mappings Satisfying Implicit Relation on b-generalized Metric Spaces

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Abstract **- A fixed point theorem in b-generalized metric spaces is proved. The obtained result can be considered as a generalization of some well-known fixed point theorems in generalized metric spaces.**

Keywords: Fixed point, generalized metric space: b-metric space, implicit relation, b-generalized metric space.

1. INTRODUCTION

The concept of b-metric space was introduced by Bakhtin [1] in1989, who used it to prove a generalization of the Banach principle in spaces endowed with such kind of metric. Since then, this notion has been used by many authors to obtain various fixed point theorems. Aydi *et al*, in [2] proved common fixed point results for single-valued and multi-valued mappings satisfying $\varphi-contractions$ in b-metric spaces. Roshan *et al,* in [3] used the notation of almost generalized contractive mappings in ordered complete b-metric space and established some fixed and common fixed point results. In [4] Pacurar proved the existence and uniqueness of fixed point of $\varphi-contractions$ on b-metric spaces. Hussain and shah in [5] introduced the notation of cone b-metric space, generalizing both notations of b-metric spaces and cone metric spaces. Fixed point theorems of contractive mappings in cone b-metric spaces without assumption of the normality of a corresponding cone are proved by Huang and Xu in [6]. The setting of partially ordered b-metric spaces was used by Husain et al, in [7]to study tripled coincidence points of mappings which satisfy nonlinear contractive conditions, extending those results of Berinde and Borcut [8] for metric spaces to b-metric spaces. Using the concept of a g-monotone mapping, Shah and Hussain in [9] proved common fixed point theorems involving g-non-decreasing mappings in b-metric spaces.

In recent years Popa [10] have used implicit function rather than contraction conditions to prove fixed point theorems in metric spaces. Implicit function can cover several contraction conditions. Implicit relation on metric spaces have been used in many articles, (see e.g. [11], [12], [13], [14], [15]).

In 2000 Branciari [16] introduced the concept of generalized metric space (gms). Every metric space is a generalized metric space, but the converse need not be true [17]. Starting with the paper of Branciari [16], some classical metric fixed point theorems have been transferred to gms (see [18], [19], [20], [21], [22], [23]).

In [17] Gjino et al obtained result as an extension and generalization of some well-known fixed point theorems from metric spaces to generalized metric spaces, in this paper we generalized the main result in [17] from generalized metric spaces to b- generalized metric spaces.

2. PRELIMINARIES

Definition 2.1. Let X be nonempty set and $d: X \times X \to [0,\infty)$.A function d is called b-metric with constant (base) $s \geq 1$ if:

(1) $\mathrm{d(x, y)} = 0 \Leftrightarrow \mathrm{x} = \mathrm{y}$

(2) $\mathrm{d(x, y)} = d(y,x)$ *for* all $\mathrm{x, y} \in \mathrm{X}$

(3) $\mathrm{d(x, y)} \leq s[d(x,z) + d(z,y)]$ *for* all $\mathrm{x, y, z} \in \mathrm{X}$

The pair (X, d) is called b-metric space.

It is obvious that a b-metric space with base $s = 1$ is a metric space. (see, e.g., Singh and Prasad [24]).

Definition 2.2. Let X be nonempty set and $d : X^2 \to R^+$ a mapping such that for all $x, y \in X$ and for all distinct points $z, w \in X$ each of them different from x and y, then d is called generalized metric if: $(1)\ d(x, y) = 0 \Leftrightarrow x = y$

$(2)\ d(x, y) = d(y, x)$ *for* all $x, y \in X$

$(3)\ d(x, y) \le d(x, z) + d(z, w) + d(w, y)$ (tetrahedral inequality)

If d is a generalized metric, the pair (X, d) is called generalized metric space.

Definition 2.3. Let X be nonempty set and $d : X^2 \to R^+$ a mapping such that for all $x, y \in X$ and for all distinct points $z, w \in X$ each of them different from x and y, then d is called b-generalized metric with constant (base) $s \ge 1$ if: $(1)\ d(x, y) = 0 \Leftrightarrow x = y$

$(2)\ d(x, y) = d(y, x)$ *for* all $x, y \in X$

$(3)\ d(x, y) \le s[d(x, z) + d(z, w) + d(w, y)]$ (tetrahedral inequality)

If d is b-generalized metric, the pair (X, d) is called b-generalized metric space

It is obvious that a b-generalized metric with base $s = 1$ is a b-generalized metric space.

Definition 2.4. [25]. Let T be a self mapping of a metric space (X, d). If for all $x \in X$ every Cauchy sequence of the orbit $O_X(T) = \{x, Tx, Tx^2, ...\}$ is convergent in X, then the metric space (X, d) is said T-orbitally complete.

Every complete metric space is T-orbitally complete for any $T : X \to X$. An orbitally complete space may not be complete metric space [26].

We introduced a class of implicit relations which will give a general character to the main result theorem 3.1.

Definition 2.5. The set of all upper semi-continuous functions with 5 variables $f : R_+^5 \to R$ satisfying the properties:

(a) f is non decreasing in respect with each variable.

(b) $f(t, t, t, t, t) \le t$, $t \in R_+$

will be noted F_5 and every such function will be called F_5 function.

Some examples of F_5 function as follows:

(1) $f(t_1, t_2, t_3, t_4, t_5) = \max\{t_1, t_2, t_3, t_4, t_5\}$

(2) $f(t_1, t_2, t_3, t_4, t_5) = \max\{t_1 t_2, t_2 t_3, t_3 t_4, t_4 t_5, t_5 t_1\}$

(3) $f(t_1, t_2, t_3, t_4, t_5) = [\max\{t_1^p, t_2^p, t_3^p, t_4^p, t_5^p\}]^{1/p}$, $p > 0$

(4) $f(t_1, t_2, t_3, t_4, t_5) = [a_1 t_1^p + a_2 t_2^p + a_3 t_3^p + a_4 t_4^p + a_5 t_5^p]^{1/p}$, where $p > 0$ *and* $0 \le \sum_{i=1}^{5} a_i \le 1$

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(5) $f(t_1,t_2,t_3,t_4,t_5)=\frac{t+t+t}{3}$ or $f(t_1,t_2,t_3,t_4,t_5)=\frac{t_1+t_2}{2}$, ect.

.The notions of a convergent sequence and a Cauchy sequence are defined by Boriceanu [27].

Definition 2.6. Let $\{x_n\}$ be a sequence in a b-generalized metric space (X,d), it is called convergent if and only if there is $x\in X$ such that $d(x_n,x)\to 0$

when $n\to\infty$. $\{x_n\}$ is called a Cauchy sequence if and only if $d(x_m,x_n)\to 0$ when $m,n\to\infty$. A b-generalized metric space is said to be complete if and only if every Cauchy sequence in this space is convergent.

Definition 2.7. Let (X,d) and (X',d') be two b-generalized metric spaces with constant (base) s and s' respectively. A mapping $T:X\to X'$ is called continuous if for each sequence $\{x_n\}$ in X which converges to $x\in X$ with respect to d, then Tx_n converges to Tx with respect to d'

3.MAIN RESULTS

Theorem3.1. Let (X,d) be b-generalized metric space with constant (base) $s\geq 1$ and T a self of mapping of X satisfying the condition

$$d(x_n,x_m)\leq scf[d(x,y),d(x,Tx),d(y,Ty),d(y,Tx),d(y,T^2x) \tag{1}$$

for all $x,y\in X$, where $0<c<1$, and $0<sc<1$. If (X,d) is $T-orbitall$ complete, then T has a unique fixed point in X .

Proof: Choose any $x_0\in X$. Define the sequence (x_n) inductively as follows:

$x_n=Tx_{n-1}, n\in N$. By condition (1) ,

$$\begin{aligned}d(x_n,x_{n+1})=d(Tx_{n-1},Tx_n)&\leq scf[d(x_{n-1},x_n),d(x_{n-1},Tx_{n-1}),d(x_n,Tx_n),d(x_n,Tx_{n-1}),d(x_n,T^2x_{n-1})]\\&=scf[d(x_{n-1},x_n),d(x_{n-1},x_n),d(x_n,x_{n+1}),0,d(x_n,x_{n+1})]\leq scd(x_{n-1},x_n)\end{aligned}$$

and so

$$d(x_n,x_{n+1})\leq scd(x_{n-1},x_n),\ \mathrm{n}\in\mathrm{N} \tag{2}$$

$$d(x_n,x_{n+1})\leq sc^nd(x_0,x_1)\leq(sc)^nd(x_0,x_1),\qquad \mathrm{n}\in\mathrm{N} \tag{3}$$

And so

$$\lim_{n\to\infty}d(x_n,x_{n+1})=0 \tag{4}$$

By condition (1) and (3) we have

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$$\begin{aligned} d(x_n,x_{n+2}) = d(Tx_{n-1},Tx_{n+1}) &\le scf[d(x_{n-1},x_{n+1}),d(x_{n-1},Tx_{n-1}),d(x_{n+1},Tx_{n+1}),d(x_{n+1},Tx_{n-1}),d(x_{n+1},T^2x_{n-1})] \\ &= scf[d(x_{n-1},x_{n+1}),d(x_{n-1},x_n),d(x_{n+1},x_{n+2}),d(x_{n+1},x_n),0] \\ &\le sc\max[d(x_{n-1},x_{n+1}),(sc)^{n-1}d(x_0,x_1),(sc)^{n+1}d(x_0,x_1),(sc)^n d(x_0,x_1)] \\ &= \max[scd(x_{n-1},x_{n+1}),(sc)^n d(x_0,x_1)] \end{aligned} \tag{5}$$

Again by condition (1) and (3) we have

$$d(x_{n-1},x_{n+1}) = d(Tx_{n-2},Tx_n) \le \max[scd(x_{n-2},x_n),(sc)^{n-1}d(x_0,x_1)] \tag{6}$$

Using (6) in (5) we have

$$\begin{aligned} d(x_n,x_{n+2}) = d(Tx_{n-1},Tx_{n+1}) &\le \max[scd(x_{n-1},x_{n+1}),(sc)^n d(x_0,x_1)] \\ &\le \max\{s^2c^2d(x_{n-2},x_n),(sc)^n d(x_0,x_1),(sc)^n d(x_0,x_1)\} \\ &= \max\{(sc)^2 d(x_{n-2},x_n),(sc)^n d(x_0,x_1)\} \end{aligned} \tag{7}$$

Again by condition (1) and (3) we have

$$d(x_{n-2},x_n) = d(Tx_{n-2},Tx_n) \le \max[scd(x_{n-3},x_{n-1}),(sc)^{n-2}d(x_0,x_1)] \tag{8}$$

Using (8) in (7) we have

$$d(x_n,x_{n+2}) \le \max[(sc)^3 d(x_{n-3},x_{n-1}),(sc)^n d(x_0,x_1)] \tag{9}$$

Continue in this process we can write

$$d(x_n,x_{n+2}) \le \max[(sc)^n d(x_0,x_2),(sc)^n d(x_0,x_1)] \tag{10}$$

And so

$$d(x_n,x_{n+2}) \le (sc)^n l,\ n \in N$$

where $l = \max[d(x_0,x_2),d(x_0,x_1)]$

we divide the proof into two cases:

Case I: Suppose $x_p = x_q$ for some n $p,q \in N, p \ne q$. Let $p > q$. Then $T^p x_0 = T^{p-q}T^q x_0 = T^q x_0$. i.e. $T^n\alpha = \alpha$ where $n = p - q$ and $T^q x_0 = \alpha$. Now if $n > 1$ by (3) we have

$$d(\alpha,T\alpha) = sc^n d(T^n\alpha,T^{n+1}\alpha) \le (sc)^n d(\alpha,T\alpha)$$

Since $0 < sc < 1, d(\alpha,T\alpha) = 0$. So $T\alpha = \alpha$ and hence α is a fixed point of T .

Case II: Assume that $x_n \ne x_m$ for all $n \ne m$. Then $(x_n) = (T^n x_0)$ is a sequence of distinct point and for that $m > n+1$, we have:

(*) If $m>2$ and odd we can write $m=2k+1$, $k\geq 1$ (by rectangular property) we can show that

$$d(x_n,x_{n+m})\leq (sc)^n[d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})+d(x_{n+2},x_{n+3})+...+d(x_{n+2k},x_{n+2k+1})$$
$$\leq (sc)^n l+(sc)^{n+1}l+(sc)^{n+2}l+...+(sc)^{n+2k}l=(sc)^n l\frac{1-(sc)^{2k+1}}{1-sc}<(sc)^n\frac{l}{1-sc}$$

(**) If $m>2$ and even we can write $m=2k$, $k\geq 2$ by using the same arguments as before we can get

$$d(x_n,x_{n+m})\leq (sc)^n[d(x_n,x_{n+2})+d(x_{n+2},x_{n+3})+d(x_{n+3},x_{n+4})+...+d(x_{n+2k-1},x_{n+2k})$$
$$\leq (sc)^n l+(sc)^{n+1}l+(sc)^{n+2}l+...+(sc)^{n+2k}l=(sc)^n l\frac{1-(sc)^{2k+1}}{1-sc}<(sc)^n\frac{l}{1-sc}$$

Thus combining all the cases we have $d(x_n,x_{n+m})<(sc)^n\frac{l}{1-sc}$ for all $n,m\in N$.

Therefore, $\lim_{n\to\infty} d(x_n,x_{n+m})=0$ .It implies that (x_n) is a Cauchy sequence in X .

Since (X,d) is T - orbitaly complete, there exists a $\alpha\in X$ such that

$$\lim_{n\to\infty} x_n=\alpha \qquad (11)$$

To show the limit is unique, assume that $\alpha\neq\alpha'$,and $\lim_{n\to\infty} x_n=\alpha'$.

Since at $x_n\neq x_m$,for all at $n\neq m$, exist a subsequence (x_{n_k}) ,of (x_n), such that $x_{n_k}\neq\alpha$ and $x_{n_k}\neq\alpha'$ for all $k\in N$. Without lost of generality, assume that (x_n) is this subsequence. Then by Tetrahedral property of Definition 1.1 we obtain

$$d(\alpha,\alpha')\leq (sc)[d(\alpha,x_n)+d(x_n,x_{n+1})+d(x_{n+1},\alpha')]$$

Letting $n\to\infty$ we get $d(\alpha,\alpha')=0$, and so $\alpha=\alpha'$.

To prove α is a fixed point of all T, suppose $\alpha\neq\alpha'$, then there exist a subsequence (x_{n_k}) ,of (x_n), such that $x_{n_k}\neq T\alpha$ and $x_{n_k}\neq\alpha$ for all $k\in N$. Then by Tetrahedral property of Definition 1.1 we obtain

$$d(\alpha,T\alpha)\leq (sc)[d(\alpha,x_{n_k-1})+d(x_{n_k-1},x_{n_k+1})+d(x_{n_k+1},T\alpha)]$$

Then if $k\to\infty$ we get

$$d(\alpha,T\alpha)\leq \lim_{k\to\infty} d(x_{n_k},T\alpha) \qquad (12)$$

From (1) we have

$$d(x_n,T\alpha)=d(Tx_{n-1},T\alpha)\leq scf[d(x_{n-1},\alpha),d(x_{n-1},Tx_{n-1}),d(\alpha,T\alpha),d(\alpha,Tx_{n-1}),d(\alpha,T^2x_{n-1})]$$
$$=scf[d(x_{n-1},\alpha),d(x_{n-1},x_n),d(\alpha,T\alpha),d(\alpha,x_n),d(\alpha,x_{n+1})])$$

Letting $n \to \infty$ we get

$$d(x_n, T\alpha) = d(Tx_{n-1}, T\alpha) \le scf[0,0,d(\alpha,T\alpha),0,0]) \le sc(\alpha,T\alpha) \qquad (13)$$

From (12) and (13) we have

$$d(\alpha,T\alpha) \le \lim_{k\to\infty} d(x_{n_k},T\alpha) \le \lim_{n\to\infty} d(x_n,T\alpha) \le scd(\alpha,T\alpha)$$

Since $0 < sc < 1$, we have $d(\alpha,T\alpha) = 0$. So α is a fixed point of T .

To prove uniqueness of α (for case I and II in the same time). Assume that $\alpha \ne \alpha^{/}$ is also is a fixed point of a T . From and so (1)

$$d(\alpha,\alpha^{/}) \le d(T\alpha,T\alpha^{/}) \le sc[d(\alpha,T\alpha),0,0,d(\alpha^{/},\alpha),d(\alpha^{/}\alpha) \le scd(\alpha,\alpha^{/})$$

Since $0 < sc < 1$, we have $\alpha = \alpha^{///} = 0$. This complete the proof of the theorem.

4. Corollaries

For different f in Theorem 3.1 we get different theorems, same as for Theorem 2.1 in [17].

Corollary 4.1. Let (X,d) be b-generalized metric space with constant (base) $s \ge 1$ and T a self of mapping of X satisfying the condition

$$d(x_n,x_m) \le sc\max[d(x,y),d(x,Tx),d(y,Ty),d(y,Tx),d(y,T^2x)$$

for all $x,y \in X$, where $0 < c < 1$, and $0 < sc < 1$. If (X,d) is $T-orbitall$ complete, then T has a unique fixed point in X .

Corollary 4.2. For $f(t_1,t_2,t_3,t_4,t_5) = t_1$ we have the Banach's Contraction principle in b-generalized metric space.

Corollary 4.3. For $f(t_1,t_2,t_3,t_4,t_5) = \frac{t_2+t_3}{2}$ we have the Kannan's Contraction principle in b-generalized metric space.

Corollary 4.4. For $f(t_1,t_2,t_3,t_4,t_5) = \max\{t_2,t_3\}$ we have an extension and generalization of Bianchini's Contraction principle in b-generalized metric space.

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