Fixed Point Results for Single-Valued Mappings on a Set with Two Metrics using a Jaggi-Type Bilateral Contraction

Muhammad Umair, Asad Hassan, Ali Hassan, Adan Qureshi, Arfatul Mamun, Ajabul Islam,
Usman Ahmed, Nabeel Iftikhar, Aneela Bashir
Jiangsu Normal University

***Abstract*:- This is a research paper exploring some new fixed point results using a bilateral contraction. First, we recall the work on fixed-point results in different research papers. Then we made the new results by combining the results of two papers, one is of Rus, in which different fixed point results on a set with two metrics were discussed, and the second is of Chen, in which different fixed point results were proved using bilateral contractions. The principal aim of this paper is to make new results for single-valued mappings on a single set with two different metrics. All this has been done using the idea of a Jaggi-type bilateral contraction.**

Keywords: Bilateral contractions, Jaggi-type bilateral contraction, fixed point results on a set with two metrics

INTRODUCTION AND PRELIMINARIES

Fixed point theory gives important aspects to solve the problems of different branches of mathematics. In the last five decades, fixed point theory has been a growing area of mathematics [1]. A metric space is a non-empty set with metric (or distance function) defined on it. There is much use of metric spaces in different fields and applications, so it is expanded in many ways [2] [3] [5] [9] [16]. In [6] Zhang and Huang explained cone metric spaces. They briefly explained Banach's fixed point theorem for such spaces. Banach's fixed point theorem explains the conditions for the uniqueness of fixed points.

In 1968, Maia investigated the famous result of the Banach contraction principle using two metrics on a non-empty set [10]. In 1975, Iseki described a fixed point theorem in a metric space [7]. In 1977, Rus proved a fixed point theorem in a set containing two metrics [12]. In 1981, Sigh and Pant proved a fixed point theorem in two metrics [14]. In 1989, Kaneko and Sessa established an idea about a fixed point theorem for contractive single and multivalued mappings [8]. In 1996, Takahashi introduced a fixed point of the multivalued mappings in convex metric spaces [13]. In 2007, Muresan gave some results about the fixed point theorem of Maia and expressed how to use these results in the sets with two metrics [11]. In 2019, Joonaghany and Karapinar enhanced the composition by combining the execution of results of two bilateral contractions; Jaggi-type bilateral contraction and Dass Gupta-type bilateral contraction [4]. In 2020, Stinson, Almuthaybiri and Tisdell described a notation about the development of fixed point theorems in a set containing two metrics with the help of iterated method [15].

MAIN BODY

At the start of this section, we define a Jaggi-type bilateral contraction, which is cited in a well-known paper on bilateral contractions by Chen [4].

Definition 1. Let $(S\,,\rho)$ be a non-empty set. The function $F : S \to S$ is called Jaggi-type bilateral contraction, if there is a $\phi : S \to [0\,,\infty)$ such that for all distinct $u, v \in S$

$$\rho(u\,,Fu) > 0$$

implies

$$\rho(Fu\,,Fv) \le [\phi(u) - \phi(Fu)] \cdot \max\{\rho(u\,,v)\,,\rho(v\,,Fv)\}$$

Firstly, suppose that $\max\{\rho(u\,,v)\,,\rho(v\,,Fv)\} = \rho(u\,,v)$ then take a set with two metrics, and we make the new result, which is:

Theorem 1. Let S be a non-empty set. Suppose ρ_1 and ρ_2 be two metrics on S and $F : (S\,,\rho_1) \to (S\,,\rho_1)$ be a function. If there is a $\phi : S \to [0\,,\infty)$ and for all $u, v \in S$

(a) $\rho_1(Fu\,,Fv) \le [\phi(u) - \phi(Fu)] \cdot \rho_2(u\,,v)$
(b) $(S\,,\rho_1)$ is a complete metric space
(c) $F : (S\,,\rho_1) \to (S\,,\rho_1)$ is continuous
(d) $\exists\ \mu \in (0\,,1)$ we have $\rho_2(Fu\,,Fv) \le \mu \cdot \rho_2(u\,,v)$

Then F has a unique fixed point.

Proof. We prove the theorem by the iterative method. For any $u \in S$, let

$$u_0 = u$$
$$u_1 = Fu_0$$

International Journal of Engineering Research & Technology (IJERT)
ISSN: 2278-0181
Vol. 11 Issue 11, November-2022

$$u_2 = Fu_1$$
$$\dots\dots\dots$$
$$\dots\dots\dots$$
$$\dots\dots\dots$$
$$u_p = Fu_{p-1}$$

where $p \in \mathbb{N}$.

This implies that $\{u_p\}$ converges in S.

If $u_p = Fu_p$ then our theorem has been proved.

Suppose $u_p \neq Fu_p$. Then for any distinct $u_{p-1}, u_p \in S$, let $\tau_p = \rho_1(u_{p-1}, u_p)$ then by the given condition

$$\begin{aligned}\tau_{p+1} &= \rho_1(u_p, u_{p+1}) \\ &= \rho_1(Fu_{p-1}, Fu_p) \\ &\leq [\phi(u_{p-1}) - \phi(Fu_{p-1})] \cdot \rho_2(u_{p-1}, u_p) \\ &= [\phi(u_p - 1) - \phi(u_p)] \cdot \rho_2(u_{p-1}, u_p)\end{aligned}$$

It follows

$$\frac{\rho_1(u_1, u_{p+1})}{\rho_2(u_{p-1}, u_p)} \leq \phi(u_{p-1}) - \phi(u_p)$$
$$0 < \frac{\rho_1(u_p, u_{p+1})}{\rho_2(u_{p-1}, u_p)} \leq \phi(u_{p-1}) - \phi(u_p)$$
$$0 < \phi(u_{p-1}) - \phi(u_p)$$
$$\phi(u_{p-1}) > \phi(u_p)$$

We conclude that the sequence $\{\phi(u_p)\}$ is not only strictly decreasing but also necessarily positive. So $\{\phi(u_p)\}$ converges to some limit $l \geq 0$.

Now for each $p \in \mathbb{N}$ we have

$$\begin{aligned}\sum_{i=1}^{p} \frac{\rho_1(u_i, u_{i+1})}{\rho_2(u_{i-1}, u_i)} &\leq \sum_{i=1}^{p} [\phi(u_{i-1}) - \phi(u_i)] \\ &\leq [\phi(u_0) - \phi(u_1)] + [\phi(u_1) - \phi(u_2)] + \cdots + [\phi(u_{p-1}) - \phi(u_p)] \\ &\leq \phi(u_0) - \phi(u_1) + \phi(u_1) - \phi(u_2) + \phi(u_2) + \cdots - \phi(u_{p-1}) + \phi(u_{p-1}) - \phi(u_p) \\ &\leq \phi(u_0) - \phi(u_p)\end{aligned}$$

If $p \to \infty$ then $\phi(u_p) \to l$

$$\sum_{i=1}^{p} \frac{\rho_1(u_i, u_{i+1})}{\rho_2(u_{i-1}, u_i)} \leq \phi(u_0) - l < \infty$$

In other words, we can say $\sum_{i=1}^{\infty} \frac{\rho_1(u_p, u_{p+1})}{\rho_2(u_{p-1}, u_p)}$ is a finite positive number.

By induction, $\frac{\rho_1(u_p, u_{p+1})}{\rho_2(u_{p-1}, u_p)}$ is bounded in $(0, 1)$, then there exists some $\mu \in (0, 1)$ we have

$$\frac{\rho_1(u_p, u_{p+1})}{\rho_2(u_{p-1}, u_p)} \leq \mu$$

$$\begin{aligned}\rho_1(u_p, u_{p+1}) &\leq \mu \cdot \rho_2(u_{p-1}, u_p) \\ &\leq \mu^2 \cdot \rho_2(u_{p-2}, u_{p-1}) \\ &\leq \mu^3 \cdot \rho_2(u_{p-3}, u_{p-2}) \\ &\quad \dots\dots\dots \\ &\quad \dots\dots\dots \\ &\quad \dots\dots\dots \\ &\leq \mu^p \cdot \rho_2(u_0, u_1)\end{aligned}$$

Now, for each $p, q \in \mathbb{N}$ with $p < q$ such that

$$\begin{aligned}\rho_1(u_p, u_q) &\leq [\phi(u_p) - \phi(Fu_p)] \cdot \rho_2(u_{p-1}, u_{q-1}) \\ &\leq [\phi(u_p) - \phi(u_{p+1})]\mu \cdot \rho_2(u_{p-2}, u_{q-2}) \\ &\leq [\phi(u_p) - \phi(u_{p+1})]\mu^2 \cdot \rho_2(u_{p-3}, u_{q-3}) \\ &\quad \dots\dots\dots \\ &\quad \dots\dots\dots \\ &\quad \dots\dots\dots \\ &\leq [\phi(u_p) - \phi(u_{p+1})]\mu^{p-1} \cdot \rho_2(u_0, u_{q-p})\end{aligned}$$

International Journal of Engineering Research & Technology (IJERT)
ISSN: 2278-0181
Vol. 11 Issue 11, November-2022

Since, $\phi(u_p)$ is strictly decreasing, then $[\phi(u_p) - \phi(u_{p+1})]$ is very small and $\mu \in (0,1)$ then we can conclude that $[\phi(u_p) - \phi(u_{p+1})]\mu^{p-1} < \epsilon$ then

$$\begin{aligned}\rho_1(u_p, u_q) &< \epsilon \cdot \rho_2(u_0, u_{q-p}) \\ &< \epsilon\end{aligned}$$

This implies that $\{u_p\}$ is the Cauchy sequence.

Since S is complete. By the continuity of : $(S, \rho_1) \to (S, \rho_1)$, for any $u_0 \in S$

$$\begin{aligned}u_0 &= \lim_{p\to\infty} [F^p(u_0)] \\ &= \lim_{p\to\infty} [F.F^{p-1}(u_0)] \\ &= F\left(\lim_{p\to\infty} [F^{p-1}(u_0)]\right) \\ &= F(u_0)\end{aligned}$$

Thus, $u_0 \in S$ is a fixed point of F.

Suppose $v_0 \in S$ is another fixed point of F, then

$$\begin{aligned}\rho_2(u_0, v_0) &= \rho(Fu_0, Fv_0) \\ &\le u \cdot \rho_2(u_0, v_0) \\ (1-\mu)\cdot\rho_2(u_0, v_0) &\le 0 \\ \rho_2(u_0, v_0) &= 0 \\ u_0 &= v_0\end{aligned}$$

Hence, u_0 is a unique fixed point of F.

By applying some more conditions to the above theorem, we make a new result. More conditions were taken from the paper by Rus [12].

Theorem 2. Let S be a non-empty set. Suppose ρ_1 and ρ_2 be two metrics on S and $F, F_p : S \to S$ be the functions. If for all $u, v \in S$ such that

(a) $(S, \rho_1), (S, \rho_2)$ and F satisfy the hypothesis of Theorem 1

(b) The sequence F_p uniformly converges on (S, ρ_1) to F

(c) $\exists\ \lambda > 0$ we have $\rho_2(u, v) \le \lambda \cdot \rho_1(u, v)$

Then for every $u_p \in S$, sequence $\{u_p\}$ converges to a unique fixed point u_0 of F.

Proof. We prove that every sequence $\{u_p\} \subseteq S$ converges to a unique fixed point $u_0 \in S$. Since for some $p \in \mathbb{N}$

$$F^p(u_p) = u_p$$

Now,

$$\begin{aligned}\rho_1(u_p, u_0) &= \rho_1\left(F_p^2(u_p), F^2(u_0)\right) \\ &\le \rho_1\left(F_p^2(u_p), F^2(u_p)\right) + \rho_1\left(F^2(u_p), F^2(u_0)\right) \\ &\le \rho_1\left(F_p^2(u_p), F^2(u_p)\right) + [\phi(u_p) - \phi(Fu_p)] \cdot \rho_2\left(F^2(u_p), F^2(u_0)\right) \\ &\le \rho_1\left(F_p^2(u_p), F^2(u_p)\right) + [\phi(u_p) - \phi(Fu_p)]\mu \cdot \rho_2(u_p, u_0) \\ &\le \rho_1\left(F_p^2(u_p), F^2(u_p)\right) + [\phi(u_p) - \phi(Fu_p)]\mu\lambda \cdot \rho_1(u_p, u_0)\end{aligned}$$

Since

$$\begin{aligned}&[\phi(u_p) - \phi(Fu_p)]\mu\lambda < 1 \\ &=> [\phi(u_p) - \phi(Fu_p)]\mu\lambda \to 0 \\ &=> [\phi(u_p) - \phi(Fu_p)]\mu\lambda < \epsilon\end{aligned}$$

Then

$$\begin{aligned}\rho_1(u_p, u_0) &\le \rho_1\left(F_p^2(u_p), F^2(u_p)\right) + \epsilon \cdot \rho_1(u_p, u_0) \\ (1-\epsilon)\cdot\rho_1(u_p, u_0) &\le \rho_1\left(F_p^2(u_p), F^2(u_p)\right) \\ \rho_1(u_p, u_0) &\le (1-\epsilon)^{-1} \cdot \rho_1\left(F_p^2(u_p), F^2(u_p)\right) \\ &\le (1-\epsilon)^{-1} \cdot \left[\rho_1\left(F_p^2(u_p), F.F_p(u_p)\right) + \rho_1\left(F.F_p(u_p), F^2(u_p)\right)\right] \\ &\le (1-\epsilon)^{-1} \cdot \left[\rho_1\left(F_p^2(u_p), F.F_p(u_p)\right) + \epsilon_1 \cdot \rho_1\left(F_p(u_p), F(u_p)\right)\right]\end{aligned}$$

It is given that F_p uniformly converges to F in metric ρ_1 , then $\rho_1\left(F_p^2(u_p), F\cdot F_p(u_p)\right) \to 0$ and $\rho_1\left(F_p(u_p), F(u_p)\right) \to 0$ as $p \to \infty$. Thus

$$\rho_1(u_p, u_0) \le (1-\epsilon)^{-1} \cdot \left[\rho_1\left(F_p^2(u_p), F\cdot F_p(u_p)\right) + \epsilon_1 \cdot \rho_1\left(F_p(u_p), F(u_p)\right)\right] \to 0$$

It means

$$\rho_1(u_p , u_0) \to 0$$

as $p \to \infty$.

Hence, $\{u_p\}$ converges in (S , ρ_1) to a unique fixed point u_0 of F.

Now, suppose in the definition-1, if $\max\{\rho(u , v) , \rho(v , Fv)\} = \rho(v , Fv)$ then one more new result is generated.

Theorem 3 Let S be a non-empty set. Suppose ρ_1 and ρ_2 be two metrics on S and $F : (S , \rho_1) \to (S , \rho_1)$ be a function. If there is a $\phi : S \to [0 , \infty)$ and for all $u, v \in S$ such that

(a) $\rho_1(Fu , Fv) \leq [\phi(u) - \phi(Fu)] \cdot \rho_2(v , Fv)$

(b) (S , ρ_1) is a complete metric space

(c) $F : (S , \rho_1) \to (S , \rho_1)$ is continuous

(d) $\exists\ \mu \in (0 , 1)$ we have $\rho_2(Fu , Fv) \leq \mu \cdot \rho_2(u , v)$

Then F has a unique fixed point.

Proof. We prove the theorem by the iterative method. For any $u \in S$, let

$$u_0 = u$$
$$u_1 = Fu_0$$
$$u_2 = Fu_1$$
$$\dots\dots\dots$$
$$\dots\dots\dots$$
$$\dots\dots\dots$$
$$u_p = Fu_{p-1}$$

where $p \in \mathbb{N}$.

This implies that $\{u_p\}$ converges in S.

If $u_p = Fu_p$ then our theorem has been proved.

Suppose $u_p \neq Fu_p$. Then for any distinct $u_{p-1} , u_p \in S$, let $\tau_p = \rho_1(u_{p-1} , u_p)$ then by the given condition

$$\begin{aligned}\tau_{p+1} &= \rho_1(u_p , u_{p+1}) \\ &= \rho_1(Fu_{p-1} , Fu_p) \\ &\leq [\phi(u_{p-1} - \phi(Fu_{p-1})] \cdot \frac{\rho_2(u_{p-1} , Fu_{p-1}) \cdot \rho_2(u_p , Fu_p)}{\rho_2(u_{p-1} , u_p)} \\ &\leq [\phi(u_{p-1}) - \phi(u_p)] \cdot \frac{\rho_2(u_{p-1} , u_p) \cdot \rho_2(u_p , u_{p+1})}{\rho_2(u_{p-1} , u_p)} \\ &\leq [\phi(u_{p-1}) - \phi(u_p)] \cdot \rho_2(u_p , u_{p+1})\end{aligned}$$

It follows

$$\frac{\rho_1(u_p , u_{p+1})}{\rho_2(u_p , u_{p+1})} \leq \phi(u_{p-1}) - \phi(u_p)$$
$$0 < \frac{\rho_1(u_p , u_{p+1})}{\rho_2(u_p , u_{p+1})} \leq \phi(u_{p-1}) - \phi(u_p)$$
$$0 < \phi(u_{p-1}) - \phi(u_p)$$
$$\phi(u_{p-1}) > \phi(u_p)$$

We conclude that the sequence $\{\phi(u_p)\}$ is not only strictly decreasing but also necessarily positive. So $\{\phi(u_p)\}$ converges to some limit $l \geq 0$.

Now for each $p \in \mathbb{N}$ we have

$$\begin{aligned}\sum_{i=1}^{p} \frac{\rho_1(u_i , u_{i+1})}{\rho_2(u_i , u_{i+1})} &\leq \sum_{i=1}^{p} [\phi(u_{i-1}) - \phi(u_i)] \\ &\leq [\phi(u_0) - \phi(u_1)] + [\phi(u_1) - \phi(u_2)] + \cdots + [\phi(u_{p-1}) - \phi(u_p)] \\ &\leq \phi(u_0) - \phi(u_1) + \phi(u_1) - \phi(u_2) + \phi(u_2) + \cdots - \phi(u_{p-1}) + \phi(u_{p-1}) - \phi(u_p) \\ &\leq \phi(u_0) - \phi(u_p)\end{aligned}$$

If $p \to \infty$ then $\phi(u_p) \to l$

$$\sum_{i=1}^{p} \frac{\rho_1(u_i , u_{i+1})}{\rho_2(u_i , u_{i+1})} \leq \phi(u_0) - l < \infty$$

In other words, we can say $\sum_{i=1}^{\infty} \frac{\rho_1(u_p , u_{p+1})}{\rho_2(u_p , u_{p+1})}$ is a finite positive number.

International Journal of Engineering Research & Technology (IJERT)
ISSN: 2278-0181
Vol. 11 Issue 11, November-2022

By induction, $\frac{\rho_1(u_p, u_{p+1})}{\rho_2(u_p, u_{p+1})}$ is bounded in $(0, 1)$, then there exists some $\mu \in (0, 1)$ we have

$$\frac{\rho_1(u_p, u_{p+1})}{\rho_2(u_p, u_{p+1})} \le \mu$$

$$\begin{aligned}\rho_1(u_p, u_{p+1}) &\le \mu \cdot \rho_2(u_p, u_{p+1}) \\ &\le \mu^2 \cdot \rho_2(u_{p-1}, u_p) \\ &\le \mu^3 \cdot \rho_2(u_{p-2}, u_{p-1}) \\ &\quad \dots \dots \dots \\ &\quad \dots \dots \dots \\ &\quad \dots \dots \dots \\ &\le \mu^{p+1} \cdot \rho_2(u_0, u_1)\end{aligned}$$

Now, for each $p, q \in \mathbb{N}$ with $p < q$ such that

$$\begin{aligned}\rho_1(u_p, u_q) &\le [\phi(u_p) - \phi(Fu_p)] \cdot \rho_2(u_p, u_q) \\ &\le [\phi(u_p) - \phi(u_{p+1})]\mu \cdot \rho_2(u_{p-1}, u_{q-1}) \\ &\le [\phi(u_p) - \phi(u_{p+1})]\mu^2 \cdot \rho_2(u_{p-2}, u_{q-2}) \\ &\quad \dots \dots \dots \\ &\quad \dots \dots \dots \\ &\quad \dots \dots \dots \\ &\le [\phi(u_p) - \phi(u_{p+1})]\mu^p \cdot \rho_2(u_0, u_{q-p})\end{aligned}$$

Since, $\phi(u_p)$ is strictly decreasing, then $[\phi(u_p) - \phi(u_{p+1})]$ is very small and $\mu \in (0, 1)$ then we can conclude that $[\phi(u_p) - \phi(u_{p+1})]\mu^p < \epsilon$ then

$$\begin{aligned}\rho_1(u_p, u_q) &< \epsilon \cdot \rho_2(u_0, u_{q-p}) \\ &< \epsilon\end{aligned}$$

This implies that $\{u_p\}$ is the Cauchy sequence.
Since S is complete. By the continuity of $: (S, \rho_1) \to (S, \rho_1)$, for any $u_0 \in S$

$$\begin{aligned}u_0 &= \lim_{p\to\infty}[F^p(u_0)] \\ &= \lim_{p\to\infty}[F.F^{p-1}(u_0)] \\ &= F\left(\lim_{p\to\infty}[F^{p-1}(u_0)]\right) \\ &= F(u_0)\end{aligned}$$

Thus, $u_0 \in S$ is a fixed point of F.
Suppose $v_0 \in S$ is another fixed point of F, then

$$\begin{aligned}\rho_2(u_0, v_0) &= \rho(Fu_0, Fv_0) \\ &\le u \cdot \rho_2(u_0, v_0) \\ (1-\mu) \cdot \rho_2(u_0, v_0) &\le 0 \\ \rho_2(u_0, v_0) &= 0 \\ u_0 &= v_0\end{aligned}$$

Hence, u_0 is a unique fixed point of F.

Similarly, by applying some more conditions to the above result, we make a new result.

Theorem 4. Let S be a non-empty set. Suppose ρ_1 and ρ_2 be two metrics on S and $F, F_p : S \to S$ be the functions. If for all $u, v \in S$ such that

(a) $(S, \rho_1), (S, \rho_2)$ and F satisfy the hypothesis of Theorem 3
(b) The sequence F_p uniformly converges on (S, ρ_1) to F
(c) $\exists\ \lambda > 0$ we have $\rho_2(Fu, Fv) \le \lambda \cdot \rho_1(u, v)$

Then for every $u_p \in S$, sequence $\{u_p\}$ converges to a unique fixed point u_0 of F.

Proof. We prove that every sequence $\{u_p\} \subseteq S$ converges to a unique fixed point $u_0 \in S$. Since for some $p \in \mathbb{N}$

$$F^p(u_p) = u_p$$

Now,

$$\begin{aligned}\rho_1(u_p, u_0) &= \rho_1\left(F_p^2(u_p), F^2(u_0)\right) \\ &\le \rho_1\left(F_p^2(u_p), F^2(u_p)\right) + \rho_1\left(F^2(u_p), F^2(u_0)\right) \\ &\le \rho_1\left(F_p^2(u_p), F^2(u_p)\right) + [\phi(u_p) - \phi(u_{p+1})] \cdot \rho_2\left(F^2(u_p), F^2(u_0)\right)\end{aligned}$$

International Journal of Engineering Research & Technology (IJERT)
ISSN: 2278-0181
Vol. 11 Issue 11, November-2022

$$\leq \rho_1\left(F_p^2(u_p)\,,F^2(u_p)\right) + [\phi(u_p) - \phi(u_{p+1})]\mu \cdot \rho_2(Fu_p\,,Fu_0)$$
$$\leq \rho_1\left(F_p^2(u_p)\,,F^2(u_p)\right) + [\phi(u_p) - \phi(u_{p+1})]\mu\lambda \cdot \rho_1(u_p\,,u_0)$$

Since $\phi(u_p)$ is strictly decreasing then $[\phi(u_p) - \phi(u_{p+1})]$ is very small and $\mu \in (0\,,1)$ then we conclude that $[\phi(u_p) - \phi(u_{p+1})]\mu\lambda < \epsilon$ then

$$\rho_1(u_p\,,u_0) \leq \rho_1\left(F_p^2(u_p)\,,F^2(u_p)\right) + \epsilon \cdot \rho_1(u_p\,,u_0)$$
$$(1-\epsilon) \cdot \rho_1(u_p\,,u_0) \leq \rho_1\left(F_p^2(u_p)\,,F^2(u_p)\right)$$
$$\rho_1(u_p\,,u_0) \leq (1-\epsilon)^{-1} \cdot \rho_1\left(F_p^2(u_p)\,,F^2(u_p)\right)$$
$$\leq (1-\epsilon)^{-1} \cdot \left[\rho_1\left(F_p^2(u_p)\,,F.F_p(u_p)\right) + \rho_1\left(F.F_p(u_p)\,,F^2(u_p)\right)\right]$$
$$\leq (1-\epsilon)^{-1} \cdot \left[\rho_1\left(F_p^2(u_p)\,,F.F_p(u_p)\right) + \epsilon_1 \cdot \rho_1\left(F_p(u_p)\,,F(u_p)\right)\right]$$

It is given that F_p uniformly converges to F in metric ρ_1 , then $\rho_1\left(F_p^2(u_p)\,,F \cdot F_p(u_p)\right) \to 0$ and $\rho_1\left(F_p(u_p)\,,F(u_p)\right) \to 0$ as $p \to \infty$. This implies that

$$\rho_1(u_p\,,u_0) \leq (1-\epsilon)^{-1} \cdot \left[\rho_1\left(F_p^2(u_p)\,,F \cdot F_p(u_p)\right) + \epsilon_1 \cdot \rho_1\left(F_p(u_p)\,,F(u_p)\right)\right] \to 0$$

It means

$$\rho_1(u_p\,,u_0) \to 0$$

as $p \to \infty$.

Hence, $\{u_p\}$ converges in $(S\,,\rho_1)$ to a unique fixed point u_0 of F.

By the above theorems, we proved the new fixed point results on a set with two metrics using the idea of a bilateral contraction. Now, we will take an example, which helps us to prove the inequalities, which we used in the above results and disprove the other contraction inequalities.

Example 1. Let $S = \{0\,,1\,,2\}$ endowed with the metric ρ_1 and ρ_2 defined for all $u, v \in S$

$$\rho_1(u\,,v) = \begin{cases} 0 & if \quad u = v \\ 1 & if \quad u \neq v \end{cases} \qquad \text{and} \qquad \rho_2(u\,,v) = |u - v|$$

Let $F : S \to S$ defined by

$$F(0) = 0\,, F(1) = 2 \; and \; F(2) = 0$$

Define $\phi : S \to [0\,,\infty)$ as

$$\phi(0) = 0\,, \phi(1) = 4 \; and \; \phi(2) = 2$$

Prove that for all , $v \in S$, F satisfies

$$\rho_1(Fu\,,Fv) \leq [\phi(u) - \phi(Fu)] \cdot \rho_2(u\,,v)$$

We prove for all $u, v \in S$

(i) For $(u\,,v) = (0\,,0)$:

$$\rho_1(F0\,,F0) \leq [\phi(0) - \phi(F0)] \cdot \rho_2(0\,,0)$$
$$\rho_1(0\,,0) \leq [\phi(0) - \phi1(0)] \cdot \rho_2(0\,,0)$$
$$0 \leq [4-2] \cdot |1-0|$$
$$1 \leq 2$$

(iii) For $(u\,,v) = (1\,,1)$:

$$\rho_1(F1\,,F1) \leq [\phi(1) - \phi(F1)] \cdot \rho_2(1\,,1)$$
$$\rho_1(2\,,2) \leq [\phi(1) - \phi(2)] \cdot \rho_2(1\,,1)$$
$$0 \leq [4-2] \cdot |1-1|$$
$$0 \leq 0$$

(iv) For $(u\,,v) = (1\,,2)$:

$$\rho_1(F1\,,F2) \leq [\phi(1) - \phi(F1)] \cdot \rho_2(1\,,2)$$
$$\rho_1(2\,,0) \leq [\phi(1) - \phi(2)] \cdot \rho_2(1\,,2)$$
$$1 \leq [4-2] \cdot |1-2|$$
$$1 \leq 2$$

(v) For $(u\,,v) = (2\,,0)$:

$$\rho_1(F2\,,F0) \leq [\phi(2) - \phi(F2)] \cdot \rho_2(2\,,0)$$
$$\rho_1(0\,,0) \leq [\phi(2) - \phi(0)] \cdot \rho_2(2\,,0)$$
$$0 \leq [2-0] \cdot |2-0|$$
$$0 \leq 4$$

(vi) For $(u\,,v) = (2\,,2)$:

$$\rho_1(F2\,,F2) \leq [\phi(2) - \phi(F2)] \cdot \rho_2(2\,,2)$$

$\rho_1(0,0) \leq [\phi(2) - \phi(0)] \cdot \rho_2(2,2)$
$0 \leq [2-0] \cdot |2-2|$
$0 \leq 0$

Hence, for all $, v \in S$, F satisfied the given inequality.
Now, we check F doesn't satisfy other contraction inequalities. Suppose the contraction inequality on two metrics:

$$\rho_1(Fu, Fv) \leq \lambda \cdot \rho_2(u, v)$$

for some $\lambda > 0$.
For $(u, v) = (1, 2)$:

$\rho_1(F1, F2) \leq \lambda \cdot \rho_2(1,2)$
$1 \leq \lambda \cdot |1-2|$
$1 \leq \lambda \cdot |-1|$
$1 \leq \lambda$

This is false for $0 < \lambda < 1$. So

$1 \nleq \lambda$

Hence, F doesn't satisfy $\rho_1(Fu, Fv) \leq \lambda \cdot \rho_2(u, v)$.

REFERENCES:

[1] Almezel, S., Ansari, Q. H., & Khamsi, M. A. (Eds.). (2014). Topics in fixed point theory (Vol. 5). Cham: Springer.
[2] Alqahtani, O., & Karapınar, E. (2019). A bilateral contraction via simulation function. Filomat, 33(15), 4837-4843.
[3] Chávez, E., Navarro, G., Baeza-Yates, R., & Marroquín, J. L. (2001). Searching in metric spaces. ACM computing surveys (CSUR), 33(3), 273-321.
[4] Chen, C. M., Joonaghany, G. H., Karapınar, E., & Khojasteh, F. (2019). On bilateral contractions. Mathematics, 7(6), 538.
[5] Hjaltason, G. R., & Samet, H. (2003). Properties of embedding methods for similarity searching in metric spaces. IEEE Transactions on Pattern Analysis and machine intelligence, 25(5), 530-549.
[6] Huang, L. G., & Zhang, X. (2007). Cone metric spaces and fixed point theorems of contractive mappings. Journal of mathematical Analysis and Applications, 332(2), 1468-1476.
[7] Iseki, K. (1975). A common fixed point theorem. Rendiconti del Seminario Matematico della Università di Padova, 53, 13-14.
[8] Kaneko, H., & Sessa, S. (1989). Fixed point theorems for compatible multi-valued and single-valued mappings. International Journal of Mathematics and Mathematical Sciences, 12(2), 257-262.
[9] Kreyszig, E. (1978). Introductory functional analysis with applications. John Wiley & Sons Canada.
[10] Maia, M. G. (1968). Un'osservazione sulle contrazioni metriche. Rendiconti del Seminario Matematico della Universita di Padova, 40, 139-143.
[11] MURESan, A. S. (2007). From Maia fixed point theorem to the fixed point theory in a set with two metrics. Carpathian Journal of Mathematics, 133-140.
[12] Rus, I. A. (1977). On a fixed point theorem in a set with two metrics. Mathematica-Revue d'analyse numérique et de théorie de l'approximation. L'analyse numérique et la théorie de l'approximation, 6(2), 197-201.
[13] Shimizu, T., & Takahashi, W. (1996). Fixed points of multivalued mappings in certain convex metric spaces. Topological Methods in Nonlinear Analysis, 8(1), 197-203.
[14] Sigh, S. P., & Pant, R. P. (1981). A common fixed point theorem in a metric space with two metrics. Pure appl. Math. Sci, 14(1-2), 35-37.
[15] Stinson, C. P., Almuthaybiri, S. S., & Tisdell, C. C. (2019). A note regarding extensions of fixed point theorems involving two metrics via an analysis of iterated functions. ANZIAM Journal, 61, C15-C30.
[16] Zezula, P., Amato, G., Dohnal, V., & Batko, M. (2006). Similarity search: the metric space approach (Vol. 32). Springer Science & Business Media.