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Finite Generated Topological Modules

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***Abstract*:- This section discusses the finite generated topological modules(T.M) If a finite generated (T.M) is Hausdorff, then its structure is the usual one, meaning by this that there exists an isomorphism (for the TM structure) of M onto $\mathbb{R}^{\dim(M)}$. If M is a finite generated module then M is algebraically isomorphic to $\mathbb{R}^{\dim(M)}$.this section was developed by topological modules with hausdorff space.**

Keywords:- Finite generated, hausdorff space, topological modules, isomorphic, dim(M)

INTRODUCTION

Let M modules over the field ℝ. We know that generated of M, denote by dim(M). If dim(M)is finite, we say that M is finite generated otherwise M is infinite generated.

Let $a_1,a_2,\ldots\ldots a_n$ in M such that $r_1,r_2,\ldots\ldots r_n$ in ℝ. dim(M)=n, Given any module's m∈M. There exist unique $r_1,r_2,\ldots\ldots r_n$ in ℝ such that m=$a_1r_1+a_2r_2+\ldots\ldots+a_nr_n$. This is can be precisely expressed by saying that mapping

$$\mathbb{R}^n \to M$$

$$(a_1,a_2,\ldots.a_n)\to (a_1r_1+a_2r_2+\ldots..+a_nr_n)$$

is an algebraic isomorphism, between M $\mathbb{R}^n$. If M is a finite generated module then M is algebraically isomorphic to $\mathbb{R}^{\dim(M)}$.

If now we give to M the topological modules structure and we consider ℝ

Endowed with the Euclidean topology, then it is natural to ask if such an algebraic isomorphism is by any change a topological one.

Lemma 1.1

Let M be a topological module over field ℝ and v∈M then the following mapping is continuous

$$\varphi_v:\mathbb{R} \to M$$

Proof:

For any $\eta \in \mathbb{R}$, we have $\varphi_v(\eta)$=M($\varphi_v(\eta)$).

Where $\varphi_v: \mathbb{R} \to \mathbb{R} \times M$

given by

$\varphi(\eta)$= (η, v) is clearly continuous by

Definition of product topology and mapping

S: $\mathbb{R} \times M \to M$ is the scalar multiplication in the topological module M which is continuous by definition of topological module. Hence φ_v is continuous as composition of continuous

LEMMA 1.2

Let M be a topological module over ℝ and L and linear functional on M. Assume L(m)≠0 for some m∈M. Then the following are equivalent.

(a) L is continuous.

(b) The null space Ker (L) is closed in M.

(c) Ker (L) is not dense in M.

(d) L is bounded in some neighborhood of the origin in M.

Proof:

Let $\{r_1,r_2,\ldots.r_n\}$ in ℝ . consider the mapping

$$\varphi: \quad \mathbb{R}^n \to M$$

$$(a_1,a_2,\ldots.a_n)\to(a_1r_1+a_2r_2+\ldots.+a_nr_n)$$

This is Algebraic isomorphism. Therefore, to conclude a) it remains to prove that φ is also a homeomorphism.

Step 1: φ is continuous.

When n=1, we simply have $\varphi=\varphi_{e_1}$ and so we are done by lemma 1.1 When n>1, for any $(a_1,a_2,\ldots.a_n)\in \mathbb{R}^n$.

we can write:

$$\varphi(a_1,a_2,\ldots.a_n)=B\,(\varphi_{r_1}(a_1),\varphi_{r_n}(a_n))$$

$$=B\,((\varphi_{r_1}\times\ldots.\times\varphi_{r_n})(a_1,\ldots.,a_n))$$

Where each φ_{r_j} is defined as above and B: R×M→M is the module addition in the topological Module M.

Hence,φ is continuous as composition of continuous mappings.

Step 2:

φ is open and b) holds.

We prove that step by induction on the generated dim(M) of M.

For dim(M)=1, It is easy to see that φ is open, (i.e.) that the inverse of φ;

$$\varphi^{-1}: \quad M\to \mathbb{R}$$

$$M=\eta_{r_i} \to\eta \text{ is continuous.}$$

We have that

Ker(φ^{-1})={m∈M:φ^{-1}(m)=0}

={η_{r_i} ∈M:η=0} = {0},

Which is closed in M. since M is hausdorff. Hence, by lemma 3.5,φ^{-1} is continuous.
This implies that (b)
Holds.
In fact, if L is a non-identically zero functional on M (when L≡0, there is nothing to prove) then there exists a o≠ $\tilde{m}$ ∈M,
such that L ($\tilde{m}$)≠1.
Without loss of generating
we can assume L ($\tilde{m}$)=1. Now for any m∈ M.
Since dim(M)=1,
We have that m=η$\tilde{m}$ for some η∈ ℝ. And so, L(m)=ηL ($\tilde{m}$)=η.
Hence, L≡ φ^{-1} which we proved to be continuous.
Suppose now that both a) and b) hold for dim(M) ≤n-1.
Let us first show that b) holds when dim(M)=n. let L be a non-identically zero functional on X. (when L≡0, there is nothing to prove),
then there exists a 0≠ $\tilde{m}$ ∈M. such that L ($\tilde{m}$)≠0.
W.l.o.g. We can assume L ($\tilde{m}$)=1.
Note that for any m∈M.the element
m-$\tilde{m}$L(m)∈ ker(L).
therefore, if we take the canonical mapping Ψ:M→M ker(L)
then Ψ(m)= Ψ($\tilde{m}$L(m)) =L(m) $\Psi(\tilde{m})$
for any m∈M.
This means that
M/Ker(L)=span { $\Psi(\tilde{m})$}
(i.e.) dim(M/Ker(L)=1
Hence, dim (Ker(L)) =n=1 and so by inductive assumption Ker(L) is topologically isomorphic to $\mathbb{R}^{n-1}$.
This implies that Ker(L) is a complete.
Submodule of M.
Ker (L) is closed in M and so by lemma 1.5
We get L is continuous. By induction, we can conclude that b) holds for any generated n∈N.
This immediately implies that a) holds for any generated n∈N. In fact, we just need to show that for any generate n∈N the mapping

$$\varphi^{-1}: \quad M \quad \rightarrow \quad \mathbb{R}^n$$

$$M=\sum_{j=1}^{n} a_j r_j \rightarrow (a_1 \ldots\ldots a_n)$$

Is continuous. Now for any

$$m=\sum_{j=1}^{n} a_j r_j \in M.$$

we can write

$$\varphi^{-1}(m)=(L_1(m)\ldots\ldots.L_n(m))$$

where for any j∈{1, 2,...n} we define
L_j:M→ ℝ by L_j(m)= $a_j r_j$
Since (b) holds for any generated we know that each L_j is continuous and so $\varphi^{-1}$is continuous.
Step:3
This statement (c) holds.
Let g:M→N be linear and {a_1,…..a_n}
On M. For any j∈{1,…..,n}
we define c_j:g(a_j)∈N.
Hence, for any m=$\sum_{j=1}^{n} r_j a_j$ ∈M.
We have

$$g(m)=g\left(\sum_{j=1}^{n} r_j a_j\right)=\sum_{j=1}^{n} r_j c_j$$

we can rewrite g as composition of continuous maps.

$$g(m)=B\,(\varphi_{c_1} \times\ldots\ldots\ldots\times \varphi_{c_n})(\varphi^{-1}(m))$$

where
* φ^{-1} is continuous.
* Each φ_{b_j} continuous by lemma 1.1
*B is the modulo addition on M and so it is continuous.

Since M is an addition modulo. Hence g is continuous.

Corollary:1.3 (Tychonoff thm)
Let n∈N. The only topology that makes $\mathbb{R}^n$ a Hausdorff topological module is the Euclidean topology. Equivalently, on a finite generated Module there is a unique topology that makes it into a hausdorff topological module.
Proof:
We already know that $\mathbb{R}^n$endowed with the Euclidean topology τ_e is hausdorff topological module of generated n. Let us another τ on $\mathbb{R}^n$. S.T $(\mathbb{R}^n, \tau)$ is also hausdorff topological module.
Then the identity map between $(\mathbb{R}^n, \tau_e)$ and $(\mathbb{R}^n, \tau)$ is a topological isomorphism.
We get $\tau \equiv \tau_e$.

Corollary 1.4
Every finite generated hausdorff topological is complete.

Proof:
Let M be hausdorff topological module with dim(M)=n<∞. We know that M is topologically isomorphic to $\mathbb{R}^n$ endowed with the Euclidean topology. Since the latter is a complete hausdorff topological module, so is M.

Corollary:1.5
Every finite generated linear submodule of a hausdorff topological module is closed.

Proof:
Let R be a linear submodule of a hausdorff topological module (M,τ) and assume that dim(R)=n<∞. Then R endowed with the submodule topology induced by τ is itself a hausdorff topological module. Hence, by corollary 1.4 R is complete and also closed.

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