

# Exploring the Classic Perturbation Method for Obtaining Single and Multiple Solutions of Nonlinear Algebraic Problems with Application to Microelectronic Circuits

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**Abstract**— It is well-known in electronics that establishing symbolic analytic expressions for the operating point and transient analysis of circuits including diodes is not possible due to the fact that these devices are modelled using an exponential function. In general, to solve this problem, in this work, the Hybrid-Newton-Raphson Perturbation technique is proposed, which allows obtaining highly accurate and handy analytical approximate solution obtained by the use of the perturbation method. In turn, these solutions are used as a starting point for the Newton-Raphson method in order to improve its accuracy in few iterations. To verify this method, two case studies are solved. The first case is the solution for single operating point for direct current (DC) analysis of a circuit with two rectifier diodes. For the second case, the solution for multiple operating points in DC analysis of a tunnel diode, voltage source, and a resistor is carried out. Therefore, for a wide range of values for the voltage source and resistor, the proposed method is able to obtain valid and highly accurate solutions. To validate the results of the method we propose, we compared the results of the first case study against commercial simulators, while in the second case study, our results were compared against a homotopy continuation method, obtaining results in good agreement for both study cases.

**Keywords**— Perturbation method; Nonlinear circuits; Multiple operating points; Homotopy continuation method.

## I. INTRODUCTION

Many physical phenomena in science and engineering are modelled by nonlinear equation systems (NAES) because is the best and direct way to describe its behaviour, considering the laws that rule them. These NAES pose a problem because it is not possible to obtain, analytically, exact results [1] because the equations complexity is high. Therefore, to solve this kind of equations, numerical methods are employed; these methods allows us to formulate mathematical

problems in a way that can be solved using arithmetic operations.

Numerical methods allows solving equations and systems of equations by means of approaching, mathematically, the solutions of a problem expressed in mathematical terms. [2] details known numerical methods to solve differential equations like Euler's Method or Runge-Kutta Method; as for solving nonlinear algebraic equations, available methods are fixed point iteration [2], Secant Method [2], Brent's Method [2], Newton-Raphson Method (NR) [2, 3], Homotopy Continuation [3-9], among others. Generally speaking, algebraic and differential problems are usually solved by the previous mentioned methods. Nevertheless, those techniques generally exhibit numerical instabilities, oscillations, false equilibrium states, among other problems [2]. Other aspect to be taken into account of numerical methods is that only provide solutions in the form of a set of numerical data. This data helps to visualize phenomena behaviour, leaving aside the possibility of performing analytical analysis that can only be performed when exact solution is able to be calculated. Nonlinear algebraic or differential problems commonly employed in engineering or applied physics usually do not possess known exact solution, however. As a response to this problem, approximative methods emerge, capable to provide analytic approximative solutions able to imitate the behaviour of the exact solution with a certain degree of accuracy and convergence domain.

As for the solution of differential equations, approximative methods reported in literature are: iterative variational method (VIM) [10, 11], modified Taylor series method (MTSM) [12], Adomian [13, 14], power series [15], classic perturbation [16, 17], homotopy perturbation method (HPM) [18],

[19], series expansion [20], rational homotopy perturbation method (RHPM), [21], homotopy fixed point [15], among others. Among all these methods, classic perturbation method [16] is the oldest and well-founded, but is restricted to problems containing, at least, one perturbative parameter  $\varepsilon \ll 1$ . Therefore, new derived methods from the classic method previously mentioned, like homotopy perturbation method (HPM) that avoids the restriction of the perturbative parameter and achieves more general solutions. Nevertheless, even this method may fail due to the high nonlinearity of diverse phenomena. [22] proposes the generalized homotopy method (GHM) as a generalization of the homotopy perturbation method (HPM), which is capable to provide precise solutions. Nevertheless, when a parameter of the perturbative type is available, it is always desirable to use the classic perturbation method. In addition, the classic perturbation method has the advantage that it can be applied to algebraic equations and differential equations, unlike other methods that can only be applied to differential equations.

In the absence of an exact solution, an approximative solution could provide valuable information about the general behaviour of the dynamic system in terms of the parameters that make it up; information that would be impossible to obtain otherwise. For the case of electronic circuits, the transient behaviour is modelled using differential equations; usually, commercial simulators just provide a numerical solution like a curve or table of values. In fact, circuit designers usually resort to empirical rules based on try and failure tests to optimize or characterize their designs. A tangible benefit for applying approximative methods in electronics [16, 23] is the possibility to analyze the transient behaviour of the circuits, that is, it is possible to identify which parameters have the most influence on the establishing time on the circuit or which parameter combination may extend or shorten the transient, potentially allowing the creation of analytical schemes to reduce power consumption, control setting times, among other aspects. In general, find the exact solution on direct current (DC) circuits that include diodes is not possible given the exponential characteristic of these devices. Nevertheless, it is possible to model the behaviour of a diode in DC circuits using approximative methods [24]. Therefore, the perturbation method can be employed to propose accurate analytical solutions and avoid the use of Newton-Raphson.

An approximative solution is introduced for the transient and steady state (DC analysis) of a simple series circuit, which has one independent voltage source, an inductor, a resistor, and a diode [16]. The provided solutions have good accuracy, but require previous knowledge of the steady state of the circuit in order to select an adequate expansion point. On the other hand, [25] proposes an exact expression for the DC analysis of the same circuit. Nevertheless, the obtained expression is expressed in terms of the Lambert W function, which is little known and not included in standard libraries of many mathematical programs. To this problem is added the fact that the methodology proposed by [25] cannot be generalized to circuits containing multiple diodes given the high nonlinearities. Likewise, [23] provides an approximative solution for the transient and DC analysis for the same circuit, which was obtained using the perturbation method [26–31].

The perturbation method from [23] was applied to a series circuit containing resistor, inductor, and diode obtaining an algebraic expression for the current in the circuit. Two case studies are proposed. The first case will obtain an approximative analytical expression by the perturbation method for the electrical current in a circuit containing a voltage source, a resistor, and two different diodes avoiding the application of NR. This circuit, but with a single diode, have been analysed in analogue electronic books, obtaining the operating point in a graphical and numerical way [32, 33]. For the second case, the goal is to obtain an approximative analytical expression for multiple operating points of a circuit with a tunnel diode, a resistor, and a voltage source. The operating points will be located using hybrid technique that includes the classic perturbation method and the Newton-Raphson method.

The hybrid perturbation Newton-Raphson technique is explored as an alternative to the homotopy continuation method [4, 6, 8, 34, 35] because it requires deep understanding of several specialized numerical methods regularly adapted to the problem under study [36]. Continuation methods also exhibit numerical problems, especially with tracing methods since the corrective step may show convergence issues as the number of equations increases. As it is well-known, continuation methods provide only numerical solutions, while perturbation methods provide approximative analytical solutions simple to evaluate in terms of the parameters under study.

This work is organized as follows. Section 2 introduces the classical perturbation method. In Section 3 provides a brief description of the Newton-Raphson numerical method. A case study that proposes an approximative solution for single and multiple operating points of the current in a circuit with two diodes is presented in Section 5. In Section 6, a case study featuring the use of the hybrid homotopy perturbation method to calculate multiple operating points for a tunnel diode is presented. Finally, Section 7 provides conclusions of this work and suggests possible future work.

## II. CLASSIC PERTURBATION METHOD

Nonlinear equations that model phenomena, whether they are physical, chemical, among others, usually do not possess exact solutions. Exact solutions [37] to these problems are not always available, thus the need of using approximative methods to obtain a solution. Approximative analytic solutions for nonlinear equations obtained using classic perturbation should be compared against the numerical solution of the problem through the use of known numerical methods [2, 3]. In the absence of an exact solution, the approximative solution can provide valuable information about the general behaviour of the static or dynamic system in terms of their parameters; this information would be impossible to obtain by any other way. Procedure of the perturbation method is provided by [24, 38]; following the guidelines, now consider solving the following nonlinear equation

$$F(u) = v, \quad (1)$$

also, consider that

$$L(u) = v, \quad (2)$$

is an auxiliary equation that possess an explicit solution known in the form

$$u = T(v), \quad (3)$$

is the unique solution of (2).  $L(u)$  is the linear operator, that is, complies with the principles of superposition and homogeneity; this is

$$L(c_1 u_1 + c_2 u_2) = c_1 L(u_1) + c_2 L(u_2), \quad (4)$$

for the case of two functions  $u_1$  and  $u_2$ , scalars  $c_1$  and  $c_2$ , we can then rewrite (1) in the following form

$$L(u) + (F(u) - L(u)) = v. \quad (5)$$

Introducing function  $N(u) = L(u) - F(u)$ , (5) becomes

$$L(u) = v + N(u). \quad (6)$$

To ease the discussion of particular solutions for (6), the introduction of a small parameter  $\varepsilon$  is considered, where  $\varepsilon \ll 1$ ; therefore, the equation is written as

$$L(u) = v + \varepsilon N(u). \quad (7)$$

considering the power series in  $\varepsilon$ , with independent coefficients of  $\varepsilon$ , becoming

$$u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \dots, \quad (8)$$

by doing this can be seen that term  $u_0$ , obtained when  $\varepsilon = 0$ , is a linear solution of

$$L(u_0) = v. \quad (9)$$

This solution can be written as  $u_0 = T(v)$ .

In order to obtain the following terms  $u_1$ ,  $u_2$ ,  $\dots$ , systematically, we substitute  $u$  given by (8) in (7) and equate terms, obtaining

$$L(u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \dots) = v + \varepsilon N(u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \dots). \quad (10)$$

Because the term  $L$  is a linear operator, the left side of the equality can be written as

$$L(u_0) + \varepsilon L(u_1) + \varepsilon^2 L(u_2) + \dots. \quad (11)$$

Assuming that  $N(u)$  is analytical in  $u$ , we can expand the right side of (10) in power series of  $\varepsilon$ , achieving the expression

$$N(u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \dots) = N(u_0) + \varepsilon N_1(u_0, u_1) + \varepsilon^2 N_2(u_0, u_1, u_2) + \dots, \quad (12)$$

where each coefficient  $\varepsilon^k$  only depends of the quantities  $u_0, u_1, u_2, \dots, u_k$ . Combining the expansions of (11) and (12), equalling coefficients of  $\varepsilon$ , equation (7) gives rise to an infinite system of equations.

$$\begin{aligned} L(u_0) &= v \\ L(u_1) &= N(u_0) \\ L(u_2) &= N_1(u_0, u_1) \\ &\vdots \\ L(u_{k+1}) &= N_k(u_0, u_1, \dots, u_k), \end{aligned}$$

and so on. It is important to note that is an infinite system and can be solved recursively. Obtaining  $u_k$  requires to know  $u_n$  when  $0 \leq n \leq k-1$ . Therefore, the first equation is given as follows

$$u_0 = T(v). \quad (13)$$

For the second, we obtain

$$u_1 = T(N(u_0)) = T(N(T(v))). \quad (14)$$

Continuing, we can express each  $u_k$  in terms of  $v$ . By doing this way we can state that when a perturbation parameter is available, it is always convenient consider the use of the classic perturbation method. Besides, this method has the advantage that can be applied to algebraic equations and differential equations as well [24].

### III. THE NEWTON-RAPHSON METHOD

The Newton-Raphson method exhibits local convergence, thus, is necessary to propose an initial value close enough to the desired root. The relative closeness of the starting point to the root is closely related to the nature of the function itself. Once this step is performed, the method linearise the function to a tangent line at the initial point. The abscissa at the origin of the line will be, accordingly to the method, a better approximation to the root than the previous value. Successive iterations are performed until the method converges close enough to the desired root (see Figure 1).

Be  $f(x)$  a real domain continuous function in the interval  $[a, b]$ , with defined derivative in the interval  $[a, b]$ . Starting with an initial value  $x_j$ , an approximation  $x_{j+1}$  is obtained closer to the root, given by the formula

$$x_{j+1} = x_j - \frac{f(x_j)}{f'(x_j)}, j = 0, 1, \dots \quad (15)$$

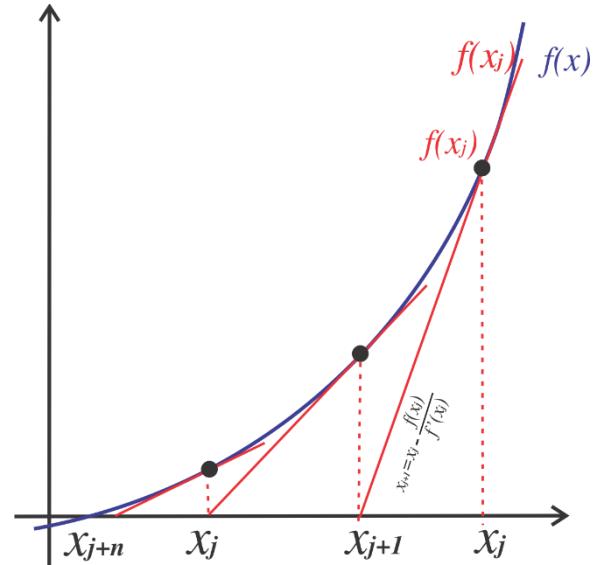


Figure 1: Geometric interpretation of the Newton-Raphson method.

The iterative process is repeated until the  $x$ -intercept of the tangent line matches the equation root or when the difference between two successive approximations fulfils an already defined tolerance error. The Newton-Raphson method exhibits quadratic convergence given by the fact that

significant numbers double on each iteration [6]. Although the NR method is efficient, if the equation possess multiple roots, multiple turning points or a steep slope in the vicinity of the root, then, the probabilities of the algorithm to diverge or oscillate increase [3, 39].

#### IV. HOMOTOPY CONTINUATION METHOD.

Obtaining multiple solutions using local convergence methods is complicated since they can obtain unique solutions based on the initial condition established at the beginning of the process. Besides, these methods do not provide information about the existence of additional solutions. Nevertheless, homotopy continuation methods allows solving a system of nonlinear equations like this:

$$ccf(x) = 0, f: \in R^n \rightarrow R^n, \quad (16)$$

for multiple solutions. The original system can be expressed as:

$$H(x, \lambda) = \lambda f(x) + (1 - \lambda) G(x), \quad (17)$$

where  $H(F(x)) = H(x, \lambda): R^{n+1} \rightarrow R^n, x \in R^n, \lambda \in [0,1]$ . Function  $G(x)$  is a auxiliar function that has known, or trivial solution. If  $\lambda = 0$ , then a known, or trivial, solution for  $G(x)$  is obtained. If  $\lambda = 1$ , solutions of the original system  $f(x)$  are found. By this method, homotopy continuation generates a set of solutions  $H^{-1}(0)$  for the system  $f(x) = 0$ . All the available solutions to solve the system during the continuous deformation are found from  $\lambda = 0$  to  $\lambda = 1$ . Figure 2 shows an schematic diagram of the Homotopy continuation method procedure. This is described step by step as follows:

**Step 1.** Define the nonlinear equation that models the problem to solve. It has the form (16), where  $x$  represents the  $n$  variable(s) of the problem. Then, we define  $x_0$ , the point where the homotopy begins.

**Step 2.** The starting point of the algorithm.  $I_{max}$  is the maximum number of roots to search in the system,  $\lambda$  is the homotopic parameter and  $i$  is the counter loop for algorithm. Therefore, we assign  $x_0 = 0, \lambda = 0, i = 0, \lambda = 0, i = 0$ ,

**Step 3.** This step in the algorithm involves two stages. First, Euler method is applied in order to perform a prediction; starting from  $(x_j, \lambda_j)$  an Euler based prediction is made under a tangential scheme to obtain  $(\bar{x}_j, \bar{\lambda}_j)$ . The second part of this step consists in applying an iterative method to correct the estimation obtained from Euler method and thus to obtain a point over  $(x_j + 1, \lambda_{j+1})$  trajectory, see Fig. 2.

**Step 4.** In case a crossing at  $\lambda = 1$  is detected, then, a jump to step 5 is performed. Otherwise, step 3 is repeated.

**Step 5.** Once a crossing at  $\lambda = 1$  is detected, points  $P_1$  y  $P_2$  are utilized to interpolate  $P_3$  over  $\lambda = 1$  [35].

**Step 6.** The interpolated  $P_3$  point is used as the initial point of NR method; this procedure is done to improve the root precision [23].

**Step 7.** If the maximum number of roots  $I_{max}$  is achieved, then it is proceeded to store the found roots and the trajectory is plotted; otherwise, the cycle is repeated at step 3.

#### V. SINGLE OPERATION POINT CASE.

The computer setup to perform analysis on this paper is as follows: Intel i3 CPU, speed of 2.40GHz, 3GB of RAM, Windows 7 Ultimate 64 bits operating system, and the software environment (Maple 17 mathematical and simulation software).

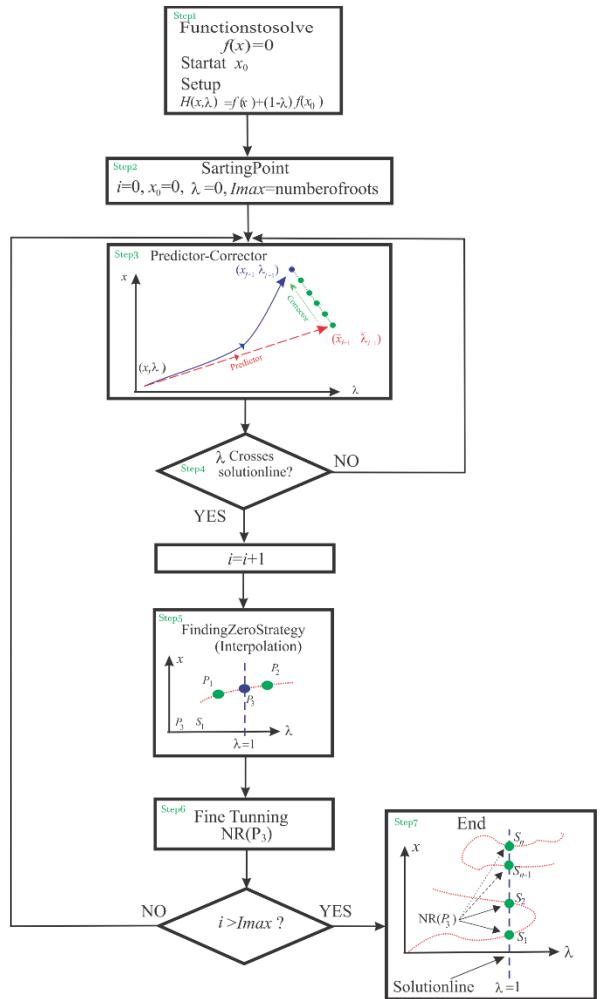


Figure 2: Homotopy continuation method procedure.

On electronics, the rectifier diode is one of the most important fundamental semiconductor devices; one of their main applications is to perform half-wave rectification of a single-phase supply. This process allows converting alternating current (AC) to direct current (DC) [32, 40]. Now it is proposed, as case study, a series circuit containing a single independent voltage source, one resistor and two diodes; assuming that each diode has different saturation current  $I_S$ . The circuit is depicted in Figure 3.

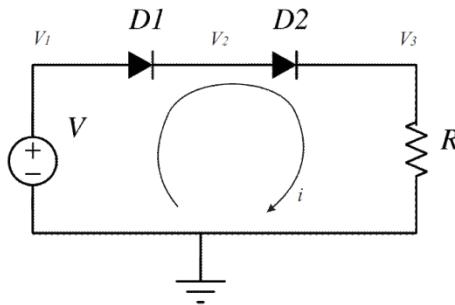


Figure 3: Series circuit with two diodes.

Using Shockley diode equation [32, 33], that approximates the behaviour of the diode as follows

$$i_d = I_S \left( \exp \left( \frac{qV_D}{nkT} \right) - 1 \right), \quad (18)$$

where  $i_d$  is the diode current, in Amperes;  $V_D$  is the voltage across the diode, in Volts;  $I_S$  is the reverse bias saturation current (or scale current);  $q$  is the magnitude of the charge of an electron, its value is  $1.602 \times 10^{-19}$  Coulomb;  $T$  is the absolute temperature of the  $p-n$  junction;  $k$  is the Boltzmann constant, its value is  $1.380649 \times 10^{-23} JK^{-1}$ ;  $n$  is the ideality factor, also known as the quality factor or sometimes emission coefficient, depends on the fabrication process, its value is 1 for germanium and 2 for silicon. The quotient  $V_T = kT/q$  is the thermal voltage, is approximatively  $25.85mV$  at  $300^\circ C$ . Therefore, solving for  $V_D$ , the voltage drop for both diodes, considering  $n = 1$ , the expression becomes

$$V_D = V_T \ln \left( \frac{i_d}{I_S} + 1 \right), i > 0, \quad (19)$$

where  $I_S$  is the reverse saturation current;  $i$  is the diode current, and  $V_T$  is the thermal voltage. Applying Kirchhoff laws, the algebraic equation to describe the operating point is

$$Ri + V_{D1} + V_{D2} - V = 0. \quad (20)$$

Substituting (19) in (20)

$$Ri + V_T \ln \left( 1 + \frac{i}{I_{S2}} \right) - V = 0. \quad (21)$$

Considering that  $\frac{i}{I_S} \gg 1$ , the complexity of (19) can be reduced by approximating the logarithmic term as follows

$$V_D = V_T \ln \left( \frac{i}{I_S} \right) = V_T \ln(i) - V_T \ln(I_S). \quad (22)$$

Therefore, for each diode we obtain

$$V_{D1} = V_T \ln \left( \frac{i}{I_{S1}} \right) = V_T \ln(i) - V_T \ln(I_{S1}), \quad (23)$$

$$V_{D2} = V_T \ln \left( \frac{i}{I_{S2}} \right) = V_T \ln(i) - V_T \ln(I_{S2}). \quad (24)$$

Rewriting (21) using (23) and (24), the mesh equation is obtained

$$Ri + 2V_T \ln(i) - V_T \ln(I_{S1}) - V_T \ln(I_{S2}) - V = 0. \quad (25)$$

The operating point of the circuit, modelled by (20), (23), and (24), is the intersection of three surfaces (see Figure 4). For this example, the proposed values are  $V = 3V$ ,  $R = 5\Omega$ ,  $I_{S1} = 1E - 12A$ , and  $I_{S2} = 1E - 9A$ . Solving the system of

three equations using NR, the circuit current is  $0.3605448941A$ .

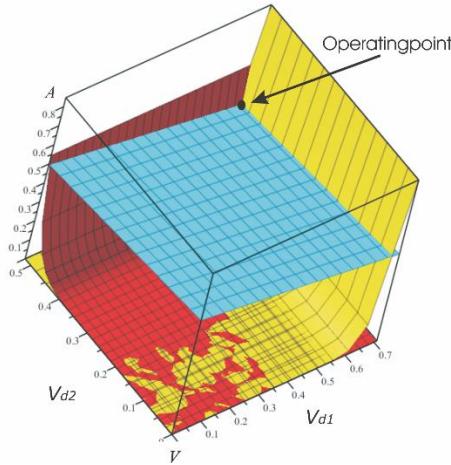


Figure 4: Circuit operating point with two diodes.

## VI. SOLUTION USING THE PERTURBATION METHOD.

To solve (25), the current is expressed in terms of a power series for  $V_T$  [17, 41], given that  $V_T \ll 1$  and in agreement to the classic perturbation [24, 38]

$$i = i_0 + i_1 V_T + i_2 V_T^2 + i_3 V_T^3 + \dots \quad (26)$$

Taking the first four terms of (26), an approximation of level 3 is obtained; substituting in (25), we obtain

$$cR(i_0 + i_1 V_T + i_2 V_T^2 + i_3 V_T^3) + 2V_T \ln(i_0 + i_1 V_T + i_2 V_T^2 + i_3 V_T^3) - V_T \ln(I_{S2}) - V_T \ln(I_{S1}) - V = 0. \quad (27)$$

Rearranging, (27) can be expressed as

$$cR(i_0 + i_1 V_T + i_2 V_T^2 + i_3 V_T^3) + 2V_T \ln(i_0) + 2V_T \ln \left( 1 + \frac{i_1 V_T + i_2 V_T^2 + i_3 V_T^3}{i_0} \right) - V_T \ln(I_{S2}) - V_T \ln(I_{S1}) - V = 0. \quad (28)$$

Replacing the first term of the Taylor expansion of the natural logarithm, we obtain

$$cR(i_0 + i_1 V_T + i_2 V_T^2 + i_3 V_T^3) + 2V_T \left( \ln(i_0) + \frac{i_1 V_T + i_2 V_T^2 + i_3 V_T^3}{i_0} \right) - V_T \ln(I_{S2}) - V_T \ln(I_{S1}) - V = 0. \quad (29)$$

Regrouping terms based on  $V_T$  and equalling to zero each coefficient, we create the following system of equations

$$\begin{aligned}
 V_T^0: \quad & Ri_0 - V = 0, \\
 V_T^1: \quad & Ri_1 + 2\ln(i_0) - \ln(I_{S2}) - \ln(I_{S2}) = 0, \\
 V_T^2: \quad & Ri_2 + 2\left(\frac{i_1}{i_0}\right) = 0, \\
 V_T^3: \quad & Ri_3 + 2\left(\frac{i_2}{i_0}\right) = 0.
 \end{aligned} \tag{30}$$

Solving (30), it is possible to obtain approximative analytic expressions for the operating point current

$$i = \frac{1}{R} \left( V + V_T \ln \left( \frac{I_{S1} I_{S2} R^2}{V^2} \right) \right), R > 0, V > 0, \tag{31}$$

$$i = \frac{V}{R} + \frac{1}{R} \left( V_T - \frac{2V_T^2}{V} \right) \ln \left( \frac{I_{S1} I_{S2} R^2}{V^2} \right), R > 0, V > 0, \tag{32}$$

$$i = \frac{V}{R} + \frac{1}{R} \left( V_T - \frac{2V_T^2}{V} + \frac{4V_T^3}{V^2} \right) \ln \left( \frac{I_{S1} I_{S2} R^2}{V^2} \right), R > 0, V > 0. \tag{33}$$

The approximative solution of the circuit operating point in Figure 3 obtained using classic perturbation uses two

diodes with different saturation currents. Table 1 and Table 2 show results obtained using the following saturation currents  $I_{S1} = 1E - 12A$  and  $I_{S2} = 1E - 9A$ . As can be seen, the results show that our method is general, whether saturation currents are different or equal. Also, it can be noticed the circuit current was calculated for different supply voltages and resistor values; the results are in good agreement between the different approximation orders and the exclusively numerical solutions given by Orcad or LTspice (commercial simulators).

It is important to notice that the order 1 of the analytic approximation is a compact analytic expression, simple to evaluate, accurate compared to orders 2 and 3 which provide higher accuracy, closer to the results obtained from electronic circuit simulators. The downside of these higher order expressions is the large number of terms.

As have seen in the tables, we tested different supply voltages and different resistor values, this allowed to empirically conclude that the valid region of the bias voltage where approximations have good accuracy is about  $V > 1.355$  Volts. In particular, it would be interesting to broaden the proposed method to nonlinear circuits that include MOS transistors and/or bipolar transistors; using the modified nodal analysis as a tool to obtain the equilibrium equation of the circuit.

Current $i$	E <sub>1</sub> [3V, 5Ω]	E <sub>2</sub> [4V, 10Ω]	E <sub>3</sub> [5V, 15Ω]	E <sub>4</sub> [6V, 20Ω]	E <sub>5</sub> [12V, 20Ω]	E <sub>6</sub> [20V, 20Ω]
Order 1 (31)	0.3195771786	0.2618848439	0.2418849642	0.2316860801	0.5298942946	0.9285738105
Iterations PNR	2	2	2	2	2	1
Order 2 (32)	0.3244097985	0.2636699823	0.2428305403	0.2322747184	0.5301963334	0.9287584471
Iterations PNR	2	2	2	2	1	1
Order 3 (33)	0.3243265163	0.2636469094	0.2428207632	0.2322696463	0.5301950321	0.9287579698
Iterations PNR	2	2	2	1	1	1
Orcad 16.6	0.3258754939	0.2640259201	0.2429704051	0.2323434187	0.5302105407	0.9287613770
LTspice IV	0.3257340000	0.2639550000	0.2429230000	0.2323080000	0.5301740000	0.9287240000

Table 1: Equations (31), (32), and (33) with  $I_{S1} = I_{S2} = 1E - 12A$ ,  $E = [V(\text{Volts}), R(\Omega)]$ , versus commercial simulators.

Current $i$	E <sub>1</sub> [3V, 5Ω]	E <sub>2</sub> [4V, 10Ω]	E <sub>3</sub> [5V, 15Ω]	E <sub>4</sub> [6V, 20Ω]	E <sub>5</sub> [12V, 20Ω]	E <sub>6</sub> [20V, 20Ω]
Order 1 (31)	0.3910033682	0.2975979387	0.2656936941	0.2495426275	0.5477508420	0.9464303578
Iterations PNR	2	2	2	2	2	1
Order 2 (32)	0.3946050768	0.2989214854	0.2663930879	0.2499774019	0.5479759489	0.9465688353
Iterations PNR	2	2	2	2	1	1
Order 3 (33)	0.3945430073	0.2989043786	0.2663858562	0.2499736556	0.5479749791	0.9465684773
Iterations PNR	2	2	1	2	1	1
Orcad 16.6	0.3953079029	0.299095941	0.2664621786	0.2500113734	0.5479827285	0.9465696966
LTspice IV	0.3952020000	0.2990430000	0.2664270000	0.2499850000	0.5479550000	0.9465410000

Table 2: Equations (31), (32), and (33) with  $I_{S1} = I_{S2} = 1E - 9A$ ,  $E = [V(\text{Volts}), R(\Omega)]$ , versus commercial simulators.

Current $i$	E <sub>1</sub> [3V, 5Ω]	E <sub>2</sub> [4V, 10Ω]	E <sub>3</sub> [5V, 15Ω]	E <sub>4</sub> [6V, 20Ω]	E <sub>5</sub> [12V, 20Ω]	E <sub>6</sub> [20V, 20Ω]
Order 1 (31)	0.3552902734	0.2797413913	0.2537893292	0.2406143538	0.5388225683	0.9375020841
Iterations PNR	2	2	2	2	2	1
Order 2 (32)	0.3595074377	0.2812957338	0.2546118141	0.2411260601	0.5390861412	0.9376636412
Iterations PNR	2	2	2	1	1	1
Order 3 (33)	0.3594347618	0.2812756439	0.2546033097	0.2411216509	0.5390850056	0.9376632236
Iterations PNR	2	2	1	1	1	1
Orcad 16.6	0.3605448941	0.2815510671	0.2547126722	0.2411756791	0.5390962851	0.9376654206
LTspice IV	0.3604220000	0.28114890000	0.2546720000	0.2411450000	0.5390640000	0.9376330000

Table 3: Equations (31), (32), and (33) with  $I_{S1} = 1E - 12A$ ,  $I_{S2} = 1E - 9A$ ,  $E = [V(\text{Volts}), R(\Omega)]$ , versus commercial simulators.

## VII. MULTIPLE OPERATION POINT CASE.

It is known, based on semiconductor literature, that the characteristic curve of tunnel diode, even in the simplest circuit, may have multiple operating points (see Figure 5a). For this study case, multiple operating points are calculated in a circuit with such device. One of the common applications of a tunnel diode is in high frequency oscillators [32]. Figure 5b depicts the circuit under study, include voltage source, one resistor, and a tunnel diode all connected in series.

[6, 42] proposes a model for tunnel diode, given as

$$i_d = \frac{I_p V_D}{V_p} \exp\left(1 - \frac{V_D}{V_p}\right) + I \exp\left(\frac{q}{kT} V_D\right), \quad (34)$$

where  $V_D$  is the voltage between terminals, given in Volts;  $V_p$  peak tension, given in Volts;  $I_p$  peak current, given in Amperes;  $i_d$  is the current flowing through the diode, in Amperes. The quotient  $\frac{q}{kT}$  is the reciprocal of the thermal voltage defined by the elemental charge, given in Volts  $^{-1}$ ;  $k$

is Boltzmann constant, and  $T$  is temperature given in Kelvin. The current  $I$  is a function of the intrinsic constants of diode [42].

Applying Kirchhoff voltage law, the mesh equation of the circuit is

$$V_D = V - iR. \quad (35)$$

Substituting (35) in (34), we obtain

$$-i + \frac{I_p}{V_p} (V - iR) \exp\left(1 - \frac{V - iR}{V_p}\right) + I \exp\left(\frac{q}{kT} (V - iR)\right) = 0, \quad (36)$$

equation (36) of the circuit can not be solved using conventional algebraic methods [43].

## VIII. SOLVING METHOD BY COMPUTER.

From the classic perturbation theory [17, 24, 38, 41], when a nonlinear equation has a parameter much less than 1, it is possible to apply the methodology mentioned above to obtain an approximative solution. For this case, given that  $I \ll 1$ , is

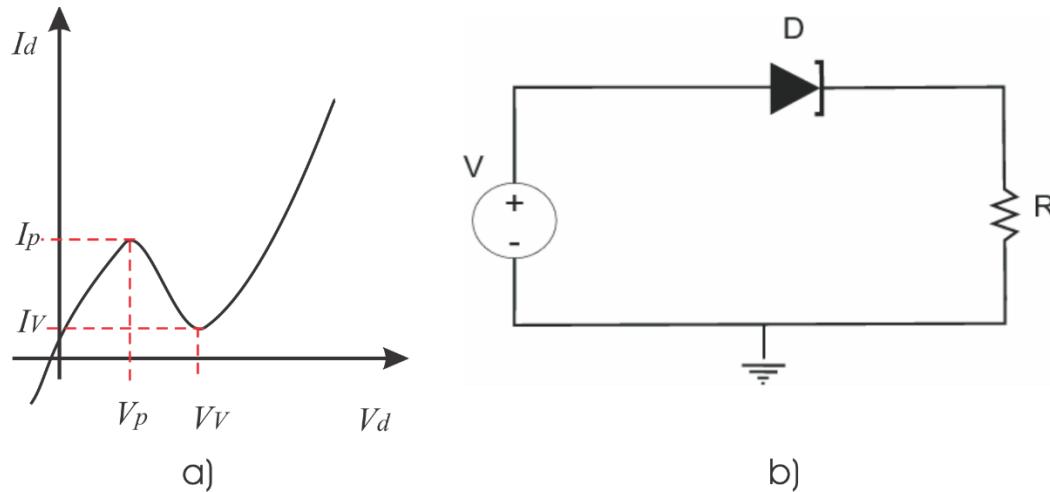


Figure 5: Tunnel diode circuit.

feasible to apply classic perturbation. Thus, the circuit current may be expressed as power series. For this study case

$$i = i_0 + i_1 I + i_2 I^2 + i_3 I^3 + \dots \quad (37)$$

Substituting (37) in (36) and expanding using Taylor series of order 2, we obtain

$$c - i + \frac{I_p(V - iR)}{V_p} \exp\left(1 - \frac{V - aR}{V_p}\right) + \frac{\exp\left(1 - \frac{V - aR}{V_p}\right) R(i - a)}{V_p} + I \left( \exp\left(\frac{q}{kT}(V - aR)\right) - \exp\left(\frac{q}{kT}(V - aR)\right) \frac{q}{kT} R(i - a) \right) = 0. \quad (38)$$

Grouping terms of order zero from (38), we have

$$cci = \left( -\left( \frac{I_P R^2 \exp\left(\frac{aR}{V_P} + 1\right)}{V_P^2 \exp\left(\frac{V}{V_P}\right)} \right) i_0^2 + \left( \frac{I_P R^2 a + I_P R V - I_P R V_p}{V_P^2 \exp\left(\frac{V}{V_P}\right)} \exp\left(\frac{aR}{V_P} + 1\right) - 1 \right) i_0 - \left( \frac{I_P V \exp\left(\frac{aR}{V_P} + 1\right) a R}{V_P^2 \exp\left(\frac{V}{V_P}\right)} + \frac{I_P V \exp\left(\frac{aR}{V_P} + 1\right)}{V_P \exp\left(\frac{V}{V_P}\right)} \right) + \dots \right) \mathbb{I}_1 + \dots \mathbb{I}_2^2 + \dots \quad (39)$$

Successive terms of (39) are not shown. Taking the coefficient of order zero from perturbative sum (39), which is a second degree trinomial in terms of  $i_0$ , we are able to solve it using elemental algebra

$$ci = \left( \frac{-1}{2(I_P R^2 \exp\left(\frac{aR}{V_P} + 1\right))} \right) \sqrt{(I_P^2 R^2 \exp(2)(aR - V_p - V)^2 (\exp(\frac{aR}{V_p}))^2 - 2RV_p^2 I_p \exp(-V_p + V + \frac{aR}{V_p}(V + 1)) + V_p^4 (\exp(\frac{V}{V_p}))^2 + (aR^2 + (-V_p + V)R)I_p \exp(\frac{aR}{V_p} + 1) - V_p^2 \exp(\frac{V}{V_p}))} \quad (40)$$

and

$$ci = \left( \frac{1}{2(I_P R^2 \exp\left(\frac{aR}{V_P} + 1\right))} \right) \sqrt{(I_P^2 R^2 \exp(2)(aR - V_p - V)^2 (\exp(\frac{aR}{V_p}))^2 - 2RV_p^2 I_p \exp(-V_p + V + \frac{aR}{V_p}(V + 1)) + V_p^4 (\exp(\frac{V}{V_p}))^2 + (aR^2 + (-V_p + V)R)I_p \exp(\frac{aR}{V_p} + 1) - V_p^2 \exp(\frac{V}{V_p}))} \quad (41)$$

where  $a$  is the expansion point for the Taylor series. In consequence, (40) and (41) are expressed in terms of the circuit parameters. Values for the voltage source, resistor, and saturation current of the tunnel diode are proposed as follows:  $V_p = 50E - 3V$ ,  $V = 0.3V$ ,  $R = 4\Omega$ ,  $I_p = 100E - 3A$ ,  $\mathbb{I} = 1E - 9A$ ,  $\frac{q}{kT} = 40V^{-1}$ . Figure 6 shows the operating points for the circuit in Figure 5b using the values mentioned above.

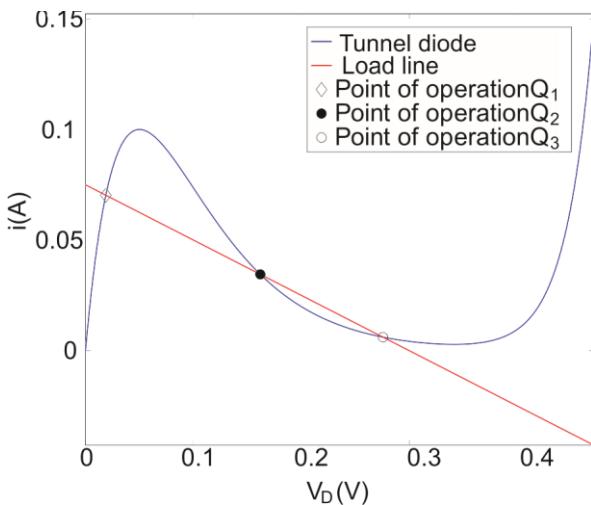


Figure 6: Operating point with loading line and tunnel diode curve.

In this way, the hybrid technique PNR applies classic perturbation to (36) proposing a loop current given by (37), the result are equations (40) and (41). Arbitrary expansion points employed to find the operating points of the circuit are  $a = 0.05$  and  $a = 0.00$ . Using the proposed values in the circuit and performing the algebraic procedures, allows obtaining  $i_{p01}$ ,  $i_{p02}$ ,  $i_{p03}$ ,  $i_{p04}$ ; where suffix  $p$  refers to perturbation. Therefore, to calculate the bias point  $Q_1$  in Figure 6,  $i_{p01}$  is selected as initial point for NR the estimation of  $i_{01}$ . In consequence, fast converge of NR is guaranteed to find every operating point (36). The first two columns to the

left of Table 4 show all the approximative roots calculated using the perturbation method. Notice that solution  $i_{p04}$  has been discarded because is not within the operating point region, as it can be seen in Figure 6.

Applying the hybrid scheme PNR on the NR process can be seen that using current values from the first two columns of Table 4 as starting points, the number of iterations are 7, 5, and 3 while taking 0.261s, 0.218s, and 0.140s, respectively; the error criteria is about  $1E - 10$  in Maple 17.

The advantage of this hybrid NNR technique lies on being a symbolic method, simple to evaluate avoiding the use of homotopy continuation. From the literature about homotopy continuation methods is known that this algorithm is built from fundamental blocks, these are: predictive step; NR corrective step, which is the most expensive block in computational terms;  $\lambda = 1$  crossing detection; interpolation and fine-tuning [23, 35], among the most important. Using homotopy continuation took 362 iterations to reach  $\lambda = 1$  and 16091 iterations of the NR corrector, taking a total of 115.924s to obtain all operating points. As already shown for the case of multiple operating points, the number of NR iterations exceeded by 16076, being this the main cause for the computation time difference between these two techniques. In consequence, the advantage of using the Hybrid PNR technique, in computational terms, to calculate operating points of the circuit in Figure 6 is noticeable since took just 15 NR iterations and 0.619s total time.

Figure 7 and Figure 8 depict the homotopy trajectories [23] of solution to determine the different equilibrium states of the circuit with the three operating points when  $\lambda = 1$ . The operating points calculated using continuation homotopy are located in the penultimate column in Table 4.

Perturbation a = 0.05	Perturbation a = 0.00	PNR	PNR Iteration	Homotopy	Iteration
$i_{p01} = 0.06495043272A$	—	$i_1 = 0.07028795173A$	7	$i_{01} = 0.070208925515A$	335
$i_{p02} = 0.04330225198A$	—	$i_2 = 0.03450748128A$	5	$i_{02} = 0.034456395350A$	347
—	$i_{p03} = 0.005364498003A$	$i_3 = 0.006118858757A$	3	$i_{03} = 0.006053597072A$	362
—	$i_{p04} = -0.1747600592A$	—	—	—	—

Table 4: Comparison between operating points obtained by different solving methods.

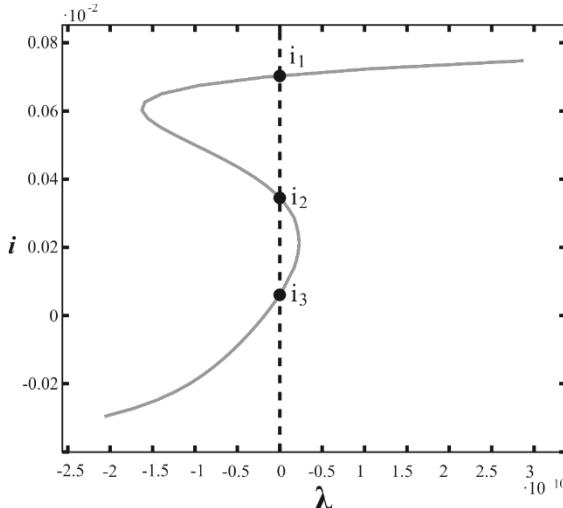


Figure 7: Homotopy trajectory of the circuit current for multiple operating points.

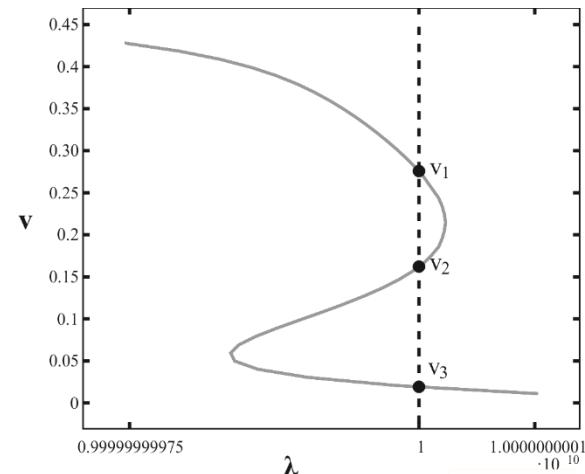


Figure 8: Homotopy trajectory of  $V_a$  voltage for multiple operating points.

## IX. CONCLUSIONS.

The first case study proposed an approximative solution using classic perturbation of the operating point for a circuit with two diodes, independent voltage source, and resistor. The solution is an analytical expression in terms of the parameters in the circuit, such as bias value, thermal voltage, saturation current, and resistance. The analytical solution mentioned above has the advantage of being derivable and integrable in terms of all its parameters. For the second case study, the proposal included the use of a hybrid PNR scheme as a mathematical strategy to obtain an approximative result for the multiple solutions that a circuit with nonlinear elements, afterwards the solution will be used as initial point for the NR convergence method in order to increase the accuracy of the solutions for all operating points. Another aspect of the hybrid PNR technique is that it avoids the use of random start points on the NR algorithm. The comparison between the hybrid PNR method and homotopy methods showed that the proposed scheme improves the easiness to implement because its fast convergence in a few iterations, thus, computation time is significantly reduced.

Given the success of the methodology, further work includes to extend it in circuits with higher number of components, that includes more diodes and decide if the resultant expressions can provide analytical information. It is possible to improve the understanding about the behaviour of such circuits. A potential application of the analytical expressions is to apply them on stability or frequency analysis [40, 44] in circuits with nonlinear elements.

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