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Existence of Solutions for Sturm-Liouville Boundary Value Problems with Impulses

Qiaoluan Li, Shuang Zhang, Lina Zhou
College of Mathematics & Information Science
Hebei Normal University,
Shijiazhuang, 050024, China

Weihua Jiang
(corresponding author) *
School of Science,
Hebei University of Science and Technology,
Shijiazhuang, 050018,China

***Abstract*—-In the paper, we investigate the Sturm-Liouville boundary value problem. By using fixed point methods, we establish sufficient conditions to guarantee the existence of solutions. At the end of the paper, two examples are given to illustrate our main results.**

Keywords—Sturm-Liouville; Impulses; Existence

1. INTRODUCTION

Consider the impulsive differential equation with Sturm-Liouville boundary conditions

$$\begin{cases} (p(t)u'(t))'+f(t,u,Tu,Su)=0,\ t\in J, t\neq t_j \\ \Delta p(t_j)u(t_j)=I_j(u(t_j)), \\ -\Delta p(t_j)u'(t_j)=J_j(u(t_j)), \qquad j=1,2,\cdots n, \\ \alpha u(0)-\beta p(0)u'(0)=0, \\ \gamma u(1)+\delta p(1)u'(1)=0. \end{cases} \tag{1.1}$$

Where $p\in C[0,1], p(t)>0, J=[0,1]$,

$Tu=\int_0^t k(t,s)u(s)ds,\ Su=\int_0^1 k_1(t,s)u(s)ds,\ k\geq 0$,

$k_1\geq 0$, $\alpha,\beta,\gamma,\delta\geq 0$, $\Gamma=\beta\gamma+\alpha\delta+\alpha\gamma\int_0^1\frac{d\tau}{p(\tau)}>0$,

f, I_j, J_j are continuous, $\Delta z(t_j)=z(t_j^+)-z(t_j^-)$.

Recently, the research of impulsive initial and boundary value problems is extensive and there is increasing interest on the existence of impulsive differential equations. Numerous papers have been published on this class of equations and good results were obtained [1,3,6,7,10]. For instance, in 2008, Kaufmann [4] studied a second-order nonlinear differential equation subject to Sturm-Liouville type boundary conditions and impulsive conditions. The authors used Krasnoselskii's fixed point theorem to obtain the existence of solutions. In 2012, Wang [8] studied impulsive fractional differential equation. Some sufficient conditions for existence of the solutions are obtained by using fixed point methods.

This paper is organized as follows. In Section 2, we shall offer some basic definitions, preliminary results and Lemmas.
In section 3, we prove the main results. To illustrate our results, two examples are given in Section 4.

2. PRELIMINARIES

In order to prove our Theorems, we need the following definition and Lemmas.

Let $PC^1(J,R)=\{u:J\to R, u|_{(t_k,t_{k+1})}, u'|_{(t_k,t_{k+1})}\in C(t_k,t_{k+1}),$ $u(t_k^-)=u(t_k), \exists u'(t_k^-), u(t_k^+), u'(t_k^+)\}$ with the norm $\|u\|=\max\{\sup_{t\in[0,1]}|u(t)|, \sup_{t\in[0,1]}|u'(t)|\}$. A function u is called a solution of Eq.(1.1) if $u\in PC^1(J,R)$ satisfies Eq.(1.1).

It is easy to know that u is the solution of Eq.(1.1) if and only if u satisfies the integral equation

$$u(t)=\int_0^1 G(t,s)f(s,u,Tu,Su)ds+\sum_{j=1}^n G(t,t_j)J_j(u(t_j))$$

$$+\sum_{j=1}^n H(t,t_j)I_j(u(t_j)),$$

Where

$$G(t,s)=\frac{1}{\Gamma}\begin{cases}(\delta+\gamma\int_t^1\frac{d\tau}{p(\tau)})(\beta+\alpha\int_0^s\frac{d\tau}{p(\tau)}), 0\leq s\leq t\leq 1,\\ (\beta+\alpha\int_0^t\frac{d\tau}{p(\tau)})(\delta+\gamma\int_s^1\frac{d\tau}{p(\tau)}), 0\leq t\leq s\leq 1,\end{cases}$$

(2.1)
and

$$H(t,s)=\frac{1}{\Gamma}\begin{cases}\frac{\alpha}{p(s)}(\delta+\gamma\int_t^1\frac{d\tau}{p(\tau)}), 0\leq s<t\leq 1,\\ -\frac{\gamma}{p(s)}(\beta+\alpha\int_0^t\frac{d\tau}{p(\tau)}), 0\leq t\leq s\leq 1.\end{cases}$$

Further by [11] we know that

$$\omega k_2(t)k_2(s)\leq G(t,s)\leq\frac{1}{\Gamma}k_2(s)\ (or\ k_2(t)),$$

where $k_2(t)=(\beta+\alpha\int_0^t\frac{d\tau}{p(\tau)})(\delta+\gamma\int_t^1\frac{d\tau}{p(\tau)})$,

$$\omega=\frac{\Gamma}{(\beta+\alpha\int_0^1\frac{d\tau}{p(\tau)})(\delta+\gamma\int_0^1\frac{d\tau}{p(\tau)})}.$$

The following Lemmas are needed.

LEMMA 1.([2]) Let X be a Banach space with $\Omega\subset X$ be closed and convex. Assume U is a relatively open subset of

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Ω with $0 \in U$, and let $S : \overline{U} \to \Omega$ be a compact, continuous maps. Then either

(i) S has a fixed point in $\overline{U}$, or

(ii) there exists $u \in \partial U$ and $v \in (0,1)$ with $u = vSu$.

LEMMA 2. ([5]) Let M be a closed convex and nonempty subset of a Banach space X . Let A, B be two operators such that

(i) $Ax + By \in M$ whenever $x, y \in M$.

(ii) A is a compact and continuous.

(iii) B is a contraction mapping.

Then there exits a $z \in M$ such that $z = Az + Bz$.

LEMMA 3.([9]) Let X be a Banach space and $W \subset PC(J, X)$. If the following conditions are satisfied:

(i) W is uniformly bounded subset of $PC(J, X)$,

(ii) W is equicontinuous in (t_k, t_{k+1}), $k = 0,1,2\cdots m$ where $t_0 = 0, t_{m+1} = 1$,

(iii) W(t)={u(t): $u \in W$, $t \in J \setminus \{t_1,...,t_m\}$},

$W(t_k^+) = \{u(t_k^+) : u \in W\}$ *and* $W(t_k^-) = \{u(t_k^-) : u \in W\}$

are relatively compact set of X . Then W is a relatively compact subset of $PC(J, X)$.

LEMMA 4.([8]) Let X be a Banach spaces and $F : X \to X$ be a completely continuous operator. If the set

$$E(F) = \{y \in X : y = \lambda Fy \text{ for some } \lambda \in [0,1]\}$$

is bounded, then F has at least a fixed point.

3. MAIN RESULTS

We make the following assumptions:

(H1) There exists a positive function $g \in C[0,1]$ such that $f(t,u,Tu,Su) \geq -g(t)$, $t \in (0,1)$, $u \in [0,+\infty)$.

(H2) $| f(t,u,v,w) - f(t,\bar{u},\bar{v},\bar{w}) | \leq m(t)(| u - \bar{u} | + | v - \bar{v} | + | w - \bar{w} |)$, where $m(t) \in C[0,1]$.

(H3) There exist L_1, L_2, M_1, M_2 such that

$\| I_k(u) - I_k(v) \| \leq L_1 \| u - v \|$, $\| I_k(u) \| \leq M_1$,

$\| J_k(u) - J_k(v) \| \leq L_2 \| u - v \|$, $\| J_k(u) \| \leq M_2$.

(H4) There exists a constant L_3 such that $| f(t,u,Tu,Su) | \leq L_3(1+| u | + | Tu | + | Su |)$.

Assume $v(t)$ is the solution of the following equation

$$\begin{cases} (p(t)v'(t))' + g(t) = 0, \ t \in J, t \neq t_j \\ -\Delta p(t_j)v'(t_j) = a_j, \qquad j=1,2,\cdots n, \\ \alpha v(0) - \beta p(0)v'(0) = 0, \\ \gamma v(1) + \delta p(1)v'(1) = 0, \end{cases} \tag{3.1}$$

then v satisfies

$$v(t) = \int_0^1 G(t,s)g(s)ds + \sum_{j=1}^n G(t,t_j)a_j,$$

where a_j will be defined in Theorem 1, $G(t,s)$ is defined as (2.1).

Consider

$$\begin{cases} (p(t)u'(t))' + f(t,(u-v)^*, T(u-v)^*, S(u-v)^*) \\ +g(t) = 0, \qquad t \in J, \ t \neq t_j \\ -\Delta p(t_j)u'(t_j) = J_j(u(t_j) - v(t_j)) + a_j, \quad j=1,2,\cdots n, \\ \alpha u(0) - \beta p(0)u'(0) = 0, \\ \gamma u(1) + \delta p(1)u'(1) = 0. \end{cases} \tag{3.2}$$

where

$$x^*(t) = \begin{cases} x(t), & x(t) \geq 0, \\ 0, & x(t) < 0. \end{cases}$$

In the following Theorem 1, we shall prove (3.2) has a solution $u(t) \geq v(t)$ and $u(t) - v(t)$ is a nonnegative solution of Eq.(1.1).

THEOREM 1. Assume (H1) hold and $I_j = 0$, $0 \leq J_j(x) \leq M_2$. There exist $R_0, a_j > 0, j = 1,2,\cdots,n$ such that $\int_0^1 G(t,s) f(s,u,Tu,Su)ds \geq 0$, for $t \in [0,1], 0 \leq u \leq R_0$, and

$$\Gamma R_0 > \int_0^1 k_2(s) \max_{0\leq s\leq 1, 0\leq z\leq R_0} (f(s,z,Tz,Sz) + g(s))ds + M_2 \sum_{j=1}^n k_2(t_j) + \sum_{j=1}^n k_2(t_j)a_j. \tag{3.3}$$

Then Eq. (1.1) has at least one positive solution.

PROOF. Let

$$Fu = \int_0^1 G(t,s)(f(s,(u-v)^*, T(u-v)^*, S(u-v)^*) + g(s))ds + \sum_{j=1}^n G(t,t_j)J_j(u(t_j) - v(t_j)) + \sum_{j=1}^n G(t,t_j)a_j.$$

We define $P = \{x \in PC^1(J,R) : x(t) \geq \Gamma \omega k_2(t) \| x \|\}$.

It is easy to know $Fu \geq 0$ for $u \in P$ and

$$Fu \geq \omega k_2(t) \Big(\int_0^1 k_2(s)(f(s,(u-v)^*, T(u-v)^*, S(u-v)^*) + g(s))ds + \sum_{j=1}^n k_2(t_j)J_j(u(t_j) - v(t_j)) + \sum_{j=1}^n k_2(t_j)a_j \Big).$$

Furthermore

$$Fu \leq \frac{1}{\Gamma} \Big(\int_0^1 k_2(s)(f(s,(u-v)^*, T(u-v)^*, S(u-v)^* + g(s))ds) + \sum_{j=1}^n k_2(t_j)J_j(u(t_j) - v(t_j)) + \sum_{j=1}^n k_2(t_j)a_j \Big),$$

then $Fu \geq \omega k_2(t)\Gamma \| Fu \|$. Hence $F(P) \subset P$.

Since f and J_j are continuous, we get F is continuous. We will prove F is uniformly bounded.

Let $D \subset P$ be bounded, i.e. there exists $L > 0$ such that $\| u \| \leq L$ for $u \in D$.

Let $A = \max_{t\in[0,1], y\in[0,L]} | f(t,y,Ty,Sy) + g(t) |$,

$M = \max_{(t,s)\in[0,1]\times[0,1]} G(t,s)$,

$$| Fu | \leq \frac{1}{\Gamma} \int_0^1 k_2(s)(f(s,(u-v)^*, T(u-v)^*, S(u-v)^*) + g(s))ds + M_2 \sum_{j=1}^n G(t,t_j) + \sum_{j=1}^n G(t,t_j)a_j$$

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$$\leq \frac{A}{\Gamma}\int_0^1 k_2(s)ds + nMM_2 + M\sum_{j=1}^n a_j.$$

Therefore $F(D)$ is bounded.

It is easy to know $F: P \to P$ is equicontinuous. Hence F is completely continuous.

Let $U = \{u \in P \text{ and } \|u\| < R_0\}$, notice that $F: \bar{U} \to PC^1(J,R)$ is continuous and completely continuous. Choose $u \in \bar{U}$ and $\lambda \in (0,1)$ such that $u = \lambda Fu$. We claim that $\|u\| \neq R_0$. If not, then $\|u\| = R_0$.

$$\|u\| \leq \|FU\|$$

$$\leq \frac{1}{\Gamma}\Big(\int_0^1 k_2(s) \max_{0\leq s\leq 1, 0\leq z\leq R_0}\{f(s,z,Tz,Sz)+g(s)\}ds$$

$$+M_2\sum_{j=1}^n k_2(t_j) + \sum_{j=1}^n k_2(t_j)a_j\Big)$$

which implies $\|u\| \neq R_0$. By the nonlinear alternative theorem of Leray-Schauder type, F has a fixed point $u \in \bar{U}$.

$$u(t) = \int_0^1 G(t,s)g(s)ds + \sum_{j=1}^n G(t,t_j)a_j$$

$$+\sum_{j=1}^n G(t,t_j)J_j(u(t_j)-v(t_j))$$

$$+\int_0^1 G(t,s)f(s,(u-v)^*,T(u-v)^*,S(u-v)^*)ds$$

$$\geq v(t).$$

Let $x(t) = u(t) - v(t) \geq 0$. For $t \neq t_j$ we get

$(p(t)x'(t))' = -f(t,x(t),Tx(t),Sx(t))$,

and $\Delta p(t_j)x'(t_j) = -J_j(x(t_j))$.

Furthermore we obtain $\alpha x(0) - \beta p(0)x'(0) = 0$, and $\gamma x(1) + \delta p(1)x'(1) = 0$. So Eq.(1.1) has a positive solution. The proof is complete.

THEOREM 2. Assume that (H2) holds, $I_j = 0$,

$|J_j(u) - J_j(v)| \leq L_4|u-v|$, $\|J_j(u)\| \leq M_2$, M<1,

where

$$M = \inf\{a>0: \int_0^1 G(t,s)m(s)\Big(k_2(s)+\int_0^s k(s,\tau)k_2(\tau)d\tau$$

$$+\int_0^1 k_1(s,\tau)k_2(\tau)d\tau\Big)ds + L_4\sum_{j=1}^n G(t,t_j)k_2(t_j)$$

$$\leq ak_2(t)\}, \quad (3.4)$$

then Eq.(1.1) has a unique continuous solution.

PROOF. Define $F: PC^1(J,R) \to PC^1(J,R)$ as follows:

$$(Fu)(t) = \int_0^1 G(t,s)f(s,u,Tu,Su)ds + \sum_{j=1}^n G(t,t_j)J_j(u(t_j)).$$

For $u \in PC^1(J,R)$, we have

$$|(Fu)(t)| \leq \int_0^1 G(t,s)|f(s,u,Tu,Su) - f(s,0,0,0)|ds$$

$$+\int_0^1 G(t,s)|f(s,0,0,0)|ds + \sum_{j=1}^n G(t,t_j)|J_j(u(t_j))|$$

$$\leq \frac{k_2(t)}{\Gamma}\{\int_0^1 m(s)\big(|u|+|Tu|+|Su|\big)ds$$

$$+\int_0^1 |f(s,0,0,0)|ds + nM_2\}.$$

So, F maps $PC^1[0,1]$ into the following

$PC^1_{k_2}[0,1] = \{x \in PC^1[0,1]: \exists \alpha > 0$ such that $-\alpha k_2(t) \leq x(t) \leq \alpha k_2(t)\}$.

Define $\|x\|_{k_2} = \inf\{\alpha > 0: -\alpha k_2(t) \leq x(t) \leq \alpha k_2(t)\}$.

We know $PC^1_{k_2}[0,1]$ is a subspace of $PC^1[0,1]$ and $PC^1_{k_2}[0,1]$ is an Banach space with the norm $\|x\|_{k_2}$. Let $u,v \in PC^1_{k_2}[0,1]$,

$$|(Fu)(t) - (Fv)(t)|$$

$$\leq \|u-v\|_{k_2}\Big(\int_0^1 G(t,s)m(s)\big(k_2(s)+\int_0^s k(s,\tau)k_2(\tau)d\tau$$

$$+\int_0^1 k_1(s,\tau)k_2(\tau)d\tau\big)ds + L_4\sum_{j=1}^n G(t,t_j)k_2(t_j)\Big)$$

$$\leq Mk_2(t)\|u-v\|_{k_2}.$$

So $\|Fu - Fv\|_{k_2} \leq M\|u-v\|_{k_2}$. From $M<1$, the operator F is a contraction. By Banach's contraction principle, F has a unique fixed point. The proof is complete.

THEOREM 3. Assume that (H2), (H3) hold, $L := \sup_{(t,s)\in[0,1]\times[0,1]}|H(t,s)|$,

$$\sup_{t\in[0,1]}\frac{1}{\Gamma}\int_0^1 k_2(s)m(s)(1+T1+S1)ds = \eta_1 < 1, \quad (3.5)$$

$$\frac{L_2}{\Gamma}\sum_{j=1}^n k_2(t_j) + nLL_1 < 1, \quad (3.6)$$

then the Eq.(1.1) has at least one solution.

PROOF. Choose $B_r = \{u \in PC^1(J,R), \|u\| \leq r\}$, where

$$r \geq \frac{\eta_2}{1-\eta_1},$$

$$\eta_2 = \frac{1}{\Gamma}\int_0^1 k_2(s)|f(s,0,0,0)|ds + \frac{M_2}{\Gamma}\sum_{j=1}^n k_2(t_j) + nLM_1,$$

and define on B_r the operator Φ, Ψ by

$$(\Phi u)(t) = \int_0^1 G(t,s)f(s,u,Tu,Su)ds$$

and

$$(\Psi u)(t) = \sum_{j=1}^n G(t,t_j)J_j(u(t_j)) + \sum_{j=1}^n H(t,t_j)I_j(u(t_j)).$$

Let us observe that if $u,v \in B_r$ then $\Phi u + \Psi v \in B_r$. Indeed

$$|(\Phi u) + (\Psi v)|$$

$$\leq \int_0^1 G(t,s)|f(s,u,Tu,Su) - f(s,0,0,0)|ds$$

$$+\int_0^1 G(t,s)|f(s,0,0,0)|ds + \sum_{j=1}^n G(t,t_j)|J_j(u(t_j))|$$

$$+\sum_{j=1}^n |H(t,t_j)||I_j(u(t_j))|$$

$$\leq \frac{1}{\Gamma}\int_0^1 k_2(s)m(s)(r+Tr+Sr)ds + \eta_2$$

$$\leq r\eta_1 + \eta_2 \leq r.$$

It is easy to see that Ψ is a contraction mapping. Since f is continuous, we get Φ is continuous and $\|\Phi u\| < r$.

It is easy to see that Φ is equicontinuous on interval $(t_k, t_{k+1}]$. So Φ is relatively compact on B_r. Hence by PC-type Arzela-Ascoli Theorem, Φ is compact on B_r. By Lemma 2, Eq.(1.1) has at least one solution on J. The proof is complete.

THEOREM 4. Assume (H3), (H4) hold and

$$B = \frac{L_3}{\Gamma}\int_0^1 k_2(s)(1+\int_0^s k(s,\tau)d\tau + \int_0^1 k_1(s,\tau)d\tau)ds < 1, \qquad (3.7)$$

then the Eq.(1.1) has at least one solution.

PROOF. Define

$$Fu(t) = \int_0^1 G(t,s) f(s,u,Tu,Su)ds + \sum_{j=1}^n G(t,t_j)J_j(u(t_j))$$

$$+\sum_{j=1}^n H(t,t_j)I_j(u(t_j)).$$

We first prove F is continuous. Let $\{u_m\}$ be a sequence such that $u_m \to u$ in $PC^1(J,R)$. For $t \in J$,

$$|(Fu_m)(t)-(Fu)(t)|$$

$$\le \frac{1}{\Gamma}\int_0^1 k_2(s)ds \| f(.,u_m,Tu_m,Su_m) - f(.,u,Tu,Su)\|$$

$$+\frac{1}{\Gamma}L_2\sum_{j=1}^n k_2(t_j)\|u_m-u\| + LL_1\sum_{j=1}^n \|u_m-u\|,$$

where $L := \sup_{(t,s)\in[0,1]\times[0,1]} |H(t,s)|$. By the continuous of f, we get F is continuous.

Second, we prove F maps bounded sets into bounded set in $PC^1(J,R)$.

For any $\eta > 0$, $B_\eta = \{u \in PC^1(J,R) : \|u\| \le \eta\}$,

$$|(Fu)(t)|$$

$$\le \frac{L_3}{\Gamma}\int_0^1 k_2(s)(1+|u|+|Tu|+|Su|)ds$$

$$+\frac{M_2}{\Gamma}\sum_{j=1}^n k_2(t_j) + LnM_1$$

$$\le \frac{L_3}{\Gamma}\int_0^1 (1+\eta+\eta\int_0^s k(s,\tau)d\tau + \eta\int_0^1 k_1(s,\tau)d\tau)k(s)ds$$

$$+\frac{M_2}{\Gamma}\sum_{j=1}^n k_2(t_j) + LnM_1 := l.$$

Which implies that $\|Fu\| \le l$.

It is easy to know that F is equicontinuous on the interval $(t_j, t_{j+1}]$. By Lemma 3, F is continuous and complete continuous.

Let $E(F) = \{u \in PC^1(J,R), u = \lambda F(u), \lambda \in (0,1)\}$.

$$|u(t)| \le |Fu(t)|$$

$$\le \frac{L_3}{\Gamma}\int_0^1 k_2(s)ds + \frac{L_3\|u\|}{\Gamma}\int_0^1 k_2(s)(1+\int_0^s k(s,\tau)d\tau$$

$$+\int_0^1 k_1(s,\tau)d\tau)ds + \frac{M_2}{\Gamma}\sum_{j=1}^n k_2(t_j) + nLM_1$$

$$:= A + \frac{L_3}{\Gamma}\int_0^1 k_2(s)(1+\int_0^s k(s,\tau)d\tau + \int_0^1 k_1(s,\tau)d\tau)ds\|u\|,$$

where $A = \frac{L_3}{\Gamma}\int_0^1 k_2(s)ds + \frac{M_2}{\Gamma}\sum_{j=1}^n k_2(t_j) + nLM_1$. We obtain $\|u\| \le \frac{A}{1-B}$. By Lemma 4, we deduce F has a fixed point. The proof is complete.

4.EXAMPLES

In this section we give two examples to illustrate our main results.

EXAMPLE 1. Consider the following equation

$$\begin{cases} u''(t) + f(t,u,Tu,Su) = 0,\ t\in[0,1],\ t \ne \frac{1}{2} \\ -\Delta u'(\frac{1}{2}) = \frac{1}{10}|\sin u(\frac{1}{2})|, \\ u(0)-u'(0)=0, \\ u(1)=0, \end{cases} \qquad (4.1)$$

where $f(t,u,Tu,Su) = u^2(t) - 2u(t) + \frac{7}{8} + \int_0^t su(s)ds$ $+\frac{1}{2}(\int_0^1 su(s)ds)^2$. Choose $g(t) = \frac{3}{16}$ for $u \ge 0$,

$$f(t,u,Tu,Su) + g(t) \ge (u-1)^2 + \frac{1}{16}$$

$$+\int_0^t su(s)ds + \frac{1}{2}(\int_0^1 su(s)ds)^2 > 0.$$

We choose $R_0 = \frac{2}{3}, a_1 = \frac{10}{27}$, it is easy to see that (3.3) holds.

By Theorem 1, Eq.(4.1) has at least one positive solution.

EXAMPLE 2. Consider the following equation

$$\begin{cases} u''(t) = -u - \int_0^t tsu(s)ds - \int_0^1 \sin s\ u(s)ds, \quad t\in[0,1], t\ne\frac{1}{2} \\ \Delta u(\frac{1}{2}) = \frac{1}{2}\sin u(\frac{1}{2}), \\ -\Delta u'(\frac{1}{2}) = \frac{1}{3}\cos u(\frac{1}{2}), \\ u(0)-u'(0)=0, \\ u(1)=0. \end{cases} \qquad (4.2)$$

where $f(t,u,v,w) = u+v+w$, $Tu = \int_0^t tsu(s)ds$, $Su = \int_0^1 \sin s\, u(s)ds$. It is easy to know that (H2), (H3) hold, and $L=1, L_1 = \frac{1}{2}, L_2 = \frac{1}{3}$,

$$\frac{1}{\Gamma}\int_0^1 k_2(s)m(s)(1+T1+S1)ds \le \frac{5-3\cos 1}{6} < 1,$$

and (3.6) holds. By Theorem 3, the Eq. (4.2) has at least one solution.

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