Existence of Solutions for Sturm-Liouville Boundary Value Problems with Impulses

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Abstract—In the paper, we investigate the Sturm-Liouville boundary value problem. By using fixed point methods, we establish sufficient conditions to guarantee the existence of solutions. At the end of the paper, two examples are given to illustrate our main results.

Keywords—Sturm-Liouville; Impulses; Existence

1. INTRODUCTION
Consider the impulsive differential equation with Sturm-Liouville boundary conditions
\[
\begin{align*}
(p(t)u'(t))' + f(t, u, Tu, Su) &= 0, \quad t \in J, \quad t \neq t_j \\
\Delta p(t_j)u'(t_j) &= I_j(u(t_j)), \\
-\Delta p(t_j)u'(t_j) &= J_j(u(t_j)), \quad j=1,2,\ldots,n \\
\alpha u(0) - \beta p(0)u'(0) &= 0, \\
\gamma u(1) + \delta p(1)u'(1) &= 0.
\end{align*}
\]

Where \( p \in C[0,1], p(t) > 0, J = [0,1], \\
Tu = \int_0^t k(t,s)u(s)ds, \quad Su = \int_0^t k(t,s)u(s)ds, \quad k \geq 0, \\
k_0 \geq 0, \quad \alpha, \beta, \gamma, \delta \geq 0, \quad \Gamma = \beta \gamma + \alpha \delta + \alpha \gamma \int_0^1 \frac{dt}{p(t)} > 0, \\
f, I, J, J', \text{ are continuous, } \Delta(t_j) = z(t_j^+) - z(t_j^-). \\

Recently, the research of impulsive initial and boundary value problems is extensive and there is increasing interest on the existence of impulsive differential equations. Numerous papers have been published on this class of equations and good results were obtained [1,3,6,7,10]. For instance, in 2008, Kaufmann [4] studied a second-order nonlinear differential equation subject to Sturm-Liouville type boundary conditions and impulsive conditions. The authors used Krasnoselskii’s fixed point theorem to obtain the existence of solutions. In 2012, Wang [8] studied impulsive fractional differential equation. Some sufficient conditions for existence of the solutions are obtained by using fixed point methods.

This paper is organized as follows. In Section 2, we shall offer some basic definitions, preliminary results and Lemmas. In section 3, we prove the main results. To illustrate our results, two examples are given in Section 4.

2. PRELIMINARIES
In order to prove our Theorems, we need the following definition and Lemmas.

Let \( PC^1(J,R) = \{ u : J \rightarrow R, u|_{[t_{k-1}, t_k)} \in C(t_{k-1}, t_k) \}, \)

\( u(t_k^-) = u(t_k), \exists u'(t_k^+), u'(t_k^-), u'(t_k^+) \} \) with the norm \( \| u \| = \max{\sup_{t \in [0,1]} |u(t)|, \sup_{t \in [0,1]} |u(t)|} \). A function \( u \) is called a solution of Eq.(1.1) if \( u \in PC^1(J,R) \) satisfies Eq.(1.1).

It is easy to know that \( u \) is the solution of Eq.(1.1) if and only if \( u \) satisfies the integral equation
\[
u(t) = \int_0^t G(s,t) f(s,u(Tu,Su)ds + \sum_{j=1}^n G(t,t_j) \int_{t_j}^{t_{j+1}} H(t,t_j) I_j(u(t_j)) dt_j,
\]

Where
\[
\begin{align*}
G(t,s) &= \int_0^1 \left( d \gamma + \gamma \int_0^1 \frac{dx}{p(\tau)} \right) (\beta \gamma + \alpha \delta + \alpha \gamma \int_0^1 \frac{dt}{p(\tau)}), \quad 0 \leq s \leq t \leq 1, \\
H(t,s) &= \int_0^1 \frac{d \gamma}{p(\tau)} (\beta \gamma + \alpha \delta + \alpha \gamma \int_0^1 \frac{dx}{p(\tau)}), \quad 0 \leq t \leq s \leq 1.
\end{align*}
\]

and
\[
\begin{align*}
\omega & = \int_0^1 \frac{d \gamma}{p(\tau)} (\beta \gamma + \alpha \delta + \alpha \gamma \int_0^1 \frac{dx}{p(\tau)}).
\end{align*}
\]

The following Lemmas are needed.

**Lemma 1.** [2] Let \( X \) be a Banach space with \( \Omega \subset X \) be closed and convex. Assume \( U \) is a relatively open subset of.
\(\Omega\) with \(0 \in U\), and let \(S : \overline{U} \rightarrow \Omega\) be a compact, continuous map. Then either

(i) \(S\) has a fixed point in \(\overline{U}\), or

(ii) there exists \(u \in \partial U\) and \(v \in (0,1)\\) with \(u = v S u\).

**LEMMA 2.** ([5]) Let \(M\) be a closed convex and nonempty subset of a Banach space \(X\). Let \(A, B\) be two operators such that

(i) \(Ax + By \in M\) whenever \(x, y \in M\).

(ii) \(A\) is a compact and continuous.

(iii) \(B\) is a contraction mapping.

Then there exists \(a \in M\) such that \(z = Az + Bz\).

**LEMMA 3.** ([9]) Let \(X\) be a Banach space and \(W \subset PC(J, X)\). If the following conditions are satisfied:

(i) \(W\) is uniformly bounded subset of \(PC(J, X)\).

(ii) \(W\) is equicontinuous in \((t_0, t_{n+1})\) for some \(t_0 \in (0,1)\\)

Then \(\exists (t) = 0, \quad x(t) \geq 0, \quad x(t) < 0\).

**THEOREM 1.** Assume (H1) hold and \(J_i = 0, 0 \leq J_i(x) \leq M_j\). There exist \(R_0, a_j > 0, j = 1, 2, \ldots, n\) such that

\[
\int_0^{R_0} G(t, s) f (s, u, T(u, v), S(u, v)) + g(s) ds > 0, \quad t \in [0,1], 0 \leq u \leq R_0, \quad \Gamma R_0 > \max_{0 \leq u \leq R_0} f (s, u, T(u, v), S(u, v)) + g(s) ds
\]

where

\[
x(t) = \begin{cases} x(t), & x(t) \geq 0, \\ 0, & x(t) < 0. \end{cases}
\]

In the following Theorem 1, we shall prove (3.2) has a solution \(u(t) \geq v(t)\) and \(u(t) - v(t)\) is a nonnegative solution of Eq.(1.1).

\[
\begin{align*}
0 & = \int_0^1 G(t, s) f (s, u, T(u, v), S(u, v)) + g(s) ds \\
& + \sum_{j=1}^n G(t, t_j) a_j + \sum_{j=1}^n k_j(t_j) a_j \\
& \leq \max_{t \in [0,1]} \left| G(t, s) f (s, u, T(u, v), S(u, v)) + g(s) ds \right|
\end{align*}
\]

**LEMMA 2.** ([5]) Let \(X\) be a Banach space and \(F : X \times X\) be a completely continuous operator. If the set \(E(F) = \{ y \in X : y = \lambda F y \text{ for some } \lambda \in [0,1]\}\) is bounded, then \(F\) has at least a fixed point.

**3. MAIN RESULTS**

We make the following assumptions:

(H1) There exists a positive function \(g \in C[0,1]\) such that

\[
f(t, u, T(u, v), S(u, v)) \geq -g(t), \quad t \in (0,1), \quad u \in [0, +\infty).
\]

(H2) \(|f(t, u, v, w) - f(t, u, \bar{v}, \bar{w})| \leq m(t)(|u - \bar{u}| + |v - \bar{v}| + |w - \bar{w}|)\), where \(m(t) \in C[0,1]\).

(H3) There exist \(L_1, L_2, M_1, M_2\) such that

\[
\|I_s(u) - I_s(v)\| \leq L_1 \|u - v\|, \quad \|I_s(u)\| \leq M_1, \\
\|J_s(u) - J_s(v)\| \leq L_2 \|u - v\|, \quad \|J_s(u)\| \leq M_2.
\]

(H4) There exists a \(L_3\) such that

\[
|f(t, u, T(u, v), S(u, v))| \leq L_3(1 + |u| + |T(u)| + |S(u)|).
\]

Assume \(v(t)\) is the solution of the following equation

\[
\begin{align*}
(p(t)v'(t))' + f(t, u, v, w, T(u, v), S(u, v)) + g(t) &= 0, \\
\Delta p(t)v'(t_j) &= a_j, \\
\alpha v(0) - \beta p(0)v'(0) &= 0, \\
\gamma v(1) + \delta p(1)v'(1) &= 0,
\end{align*}
\]

then \(v(t)\) satisfies

\[
v(t) = \int_0^t G(t, s) f (s, u, T(u, v), S(u, v)) + g(s) ds + \sum_{j=1}^n G(t, t_j) a_j,
\]

where \(a_j \) will be defined in Theorem 1, \(G(t, s)\) is defined as \((2.1)\).

Consider
\[ \leq \frac{A}{\Gamma} \int_0^1 k_2(s)ds + nMM_5 + M \sum_{j=1}^n a_j. \]

Therefore \( F(D) \) is bounded.

It is easy to know \( F : P \rightarrow P \) is equicontinuous. Hence \( F \) is completely continuous.

Let \( U = \{ u \in P \mid \| u \| < R_0 \} \), notice that \( F : U \rightarrow PC^1(J, R) \) is continuous and completely continuous. Choose \( u \in U \) and \( \lambda \in (0, 1) \) such that \( \lambda u = Fu \).

We claim that \( \| u \| \neq R_0 \). If not, then \( \| u \| = R_0 \).

\[ \| u \| = \| Fu \| \leq \frac{1}{\Gamma} \left( \int_0^1 k_2(s) \max_{0 \leq c \leq d \leq c_k} \{ f(s, z, Tz, S(z)) + g(s) \} ds \right. \]
\[ + M \sum_{j=1}^n k_2(t_j) + \sum_{j=1}^n k_2(t_j) a_j \]

which implies \( \| u \| \neq R_0 \). By the nonlinear alternative theorem of Leray-Schauder type, \( F \) has a fixed point \( u \in U \).

\[ u(t) = \int_0^t G(t,s)g(s)ds + \sum_{j=1}^n G(t,t_j)a_j \]
\[ + \sum_{j=1}^n G(t,t_j)J_j(u(t_j) - v(t_j)) \]
\[ + \int_0^t G(t,s)f(s,(u-v)^+ , T(u-v)^+, S(u-v)^+)ds \geq v(t). \]

Let \( x(t) = u(t) - v(t) \geq 0 \). For \( t \neq t_j \), we get

\[ p(t)x(t) = -f(t, x(t), x(Tx(t), x(Sx(t)), \lambda t, (x(t))). \]

Furthermore we obtain \( \alpha x(0) - \beta p(0)x(0) = 0 \), and \( \gamma x(t) + \delta p(t)x(t) = 1 \). So Eq.(1.1) has a positive solution.

The proof is complete.

**THEOREM 2.** Assume that (H2), (H3) hold,
\[ L = \sup_{(t,s) \in [0,1] \times [0,1]} |H(t,s)|, \]
\[ \sup_{(t,s) \in [0,1] \times [0,1]} |(Fu)(t)| + \sup_{(t,s) \in [0,1] \times [0,1]} |(Fv)(t)| \leq M \leq 1, \]
where
\[ M = \inf \{ a > 0 : \int_0^1 G(t,s)m(s)(k_2(s) + \int_0^t k_2(s, \tau)k_2(\tau)d\tau)ds + L_2 \sum_{j=1}^n G(t,t_j)k_2(t_j) \}
\[ \leq ak_2(t), \quad (3.4) \]
then Eq.(1.1) has a unique continuous solution.

**PROOF.** Define \( F : PC^1(J, R) \rightarrow PC^1(J, R) \) as follows:

\[ (Fu)(t) = \int_0^t G(t,s)f(s,u,Tu,Su)ds + \sum_{j=1}^n G(t,t_j)J_j(u(t_j)). \]

For \( u \in PC^1(J, R) \), we have

\[ \| (Fu)(t) \| \leq \int_0^t G(t,s) \| f(s,u,Tu,Su) - f(s,0,0,0) \| ds \]
\[ + \int_0^t G(t,s) \| f(s,0,0,0) \| ds + \sum_{j=1}^n G(t,t_j) \| J_j(u(t_j)) \| \]
\[ \leq \frac{k_2(t)}{\Gamma} \left( \int_0^1 m(s) \| u \| + |Tu| + |Su| \right)ds \]
\[ + \int_0^1 f(s,0,0,0) \| ds + nM \sum_{j=1}^n a_j. \]

So, \( F \) maps \( PC^1[0,1] \) into the following
\[ PC^1[0,1] = \{ x \in PC^1[0,1] : \exists \alpha > 0 \text{ such that } -\alpha k_2(t) \leq x(t) \leq \alpha k_2(t) \}. \]

We know \( PC^1[0,1] \) is a subspace of \( PC^1[0,1] \) and \( PC^1[0,1] \) is a Banach space with the norm \( \| x \| \).

Let \( u, v \in PC^1[0,1] \),
\[ \| (Fu)(t) - (Fv)(t) \| \leq \| u - v \| \| \int_0^t G(t,s)m(s)(k_2(s) + \int_0^t k_2(s, \tau)k_2(\tau)d\tau)ds \]
\[ + \int_0^t k_2(s, \tau)k_2(\tau)d\tau \| ds + L_2 \sum_{j=1}^n G(t,t_j)k_2(t_j) \]
\[ \leq M_k(t) \| u - v \| \| . \]

So \( \| Fu - Fv \| \leq M \| u - v \| \). From \( M < 1 \), the operator \( F \) is a contraction. By Banach’s contraction principle, \( F \) has a unique fixed point. The proof is complete.

**THEOREM 3.** Assume that (H2), (H3) hold.
\[ L = \sup_{(t,s) \in [0,1] \times [0,1]} |H(t,s)|, \]
\[ \sup_{(t,s) \in [0,1] \times [0,1]} |(Fu)(t)| + \sup_{(t,s) \in [0,1] \times [0,1]} |(Fv)(t)| \leq M \leq 1, \]
\[ L_2 \left( \sum_{j=1}^n k_2(t_j) \right) \|
\[ + nL_2, \quad (3.6) \]
then the Eq.(1.1) has at least one solution.

**PROOF.** Choose \( B = \{ u \in PC^1(J, R) : \| u \| \leq r \} \), where
\[ r \geq \frac{\eta_2}{1 - \eta_1}, \]
\[ \eta_2 = \frac{1}{\Gamma} \left( \int_0^1 k_2(s) \| f(s,0,0,0) \| ds + \frac{M}{L_2} \sum_{j=1}^n k_2(t_j) + nLM \right), \]
and define on \( B \), the operator \( \Phi, \Psi \) by
\[ (\Phi u)(t) = \int_0^1 G(t,s)f(s,u,Tu,Su)ds \]
\[ + \sum_{j=1}^n G(t,t_j)J_j(u(t_j)), \]
\[ (\Psi u)(t) = \sum_{j=1}^n G(t,t_j)J_j(u(t_j)) + \sum_{j=1}^n H(t,t_j)I_j(u(t_j)). \]

Let us observe that if \( u, v \in B \), then \( \Phi u + \Psi v \in B \). Indeed
\[ \| (\Phi u) + (\Psi v) \| \]
\[ \leq \int_0^1 G(t,s) \| f(s,u,Tu,Su) - f(s,0,0,0) \| ds \]
\[ + \int_0^1 G(t,s) \| f(s,0,0,0) \| ds + \sum_{j=1}^n G(t,t_j) \| J_j(u(t_j)) \| \]
\[ + \sum_{j=1}^n H(t,t_j) \| I_j(u(t_j)) \| \]
\[ \leq \frac{1}{\Gamma} \left( \int_0^1 k_2(s) \| f(r + Tr + Sr) \| ds + \eta_2 \right. \]
\[ \leq \eta_1 \| u \| + \eta_2 \leq r. \]
It is easy to see that \( \Psi \) is a contraction mapping. Since \( f \) is continuous, we get \( \Phi \) is continuous and \( \| \Phi u \| < r \).

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It is easy to see that $\Phi$ is equicontinuous on interval $(t_*, t_{i+1}]$. So $\Phi$ is relatively compact on $B_r$. Hence by PC-
type Arzela-Ascoli Theorem, $\Phi$ is compact on $B_r$. By Lemma 2, Eq.(1.1) has at least one solution on $J$. The proof
is complete.

**THEOREM 4.** Assume (H3), (H4) hold and

$$B = \frac{L_1}{1} \int_0^1 k_2(s) \left(1 + \int_0^s k_1(s, \tau) d\tau \right) ds + \frac{M_1}{1} \sum_{j=1}^{n} k_j(t_j) + nLM_1,$$

then the Eq.(1.1) has at least one solution.

**PROOF.** Define

$$F(u(t)) = \int G(t, s)f(s, u, Tu, Su)ds + \sum_{j=1}^{n} G(t, t_j)J_j(u(t_j)) + \sum_{j=1}^{n} H(t, t_j)J_j(u(t_j)).$$

We first prove $F$ is continuous. Let $\{u_n\}$ be a sequence such that $u_n \rightarrow u$ in $PC^\circ(J, R)$. For $t \in J$,\n
$$\| (F(u_n)(t) - (F(u)(t)) \| \leq \frac{1}{1} \int_0^1 [k_2(s)ds \| f(\cdot, u_n, Tu_n, Su_n) - f(\cdot, u, Tu, Su) \| ]$$

$$+ \frac{1}{1} L_1 \int_0^1 k_1(t_j) \| u_n - u \| + LLM_1 \sum_{j=1}^{n} \| u_n - u \| ,$$

where $L := \sup_{(t, s) \in [0,1]} \| H(t, s) \|$. By the continuous of $f$, we get $F$ is continuous.

Second, we prove $F$ maps bounded sets into bounded set in $PC^\circ(J, R)$.

For any $\eta > 0$, $B_\eta = \{ u \in PC^\circ(J, R) : \| u \| \leq \eta \}$.

$$\| (F(u)(t)) \| \leq \frac{L_1}{1} \int_0^1 k_2(s)ds \| u \| + \| Tu \| + \| Su \| ds$$

$$+ \frac{M_1}{1} \sum_{j=1}^{n} k_j(t_j) + LnM_1,$$

$$\leq \frac{L_1}{1} \int_0^1 (1 + \eta + \eta \int_0^1 k(s, \tau) d\tau + \eta \int_0^1 k_1(s, \tau) d\tau) k(s)ds$$

$$+ \frac{M_1}{1} \sum_{j=1}^{n} k_j(t_j) + LnM_1 \Rightarrow l.$$\n
Which implies that $\| F \| \leq l$.

It is easy to know that $F$ is equicontinuous on the interval $(t_j, t_{j+1})$. By Lemma 3, $F$ is continuous and complete continuous.

Let $E(F) = \{ u \in PC^\circ(J, R), u = \lambda F(u), \lambda \in (0, 1) \}$.

$$\| u(t) \| \leq \| F(u(t)) \| \leq \frac{L_1}{1} \int_0^1 k_2(s)ds + \frac{L_1}{1} \| u \| \int_0^1 k_1(s)1 + \int_0^1 k_1(s, \tau) d\tau$$

$$+ \frac{M_1}{1} \int_0^1 k_1(s, \tau) d\tau ds + \frac{M_1}{1} \sum_{j=1}^{n} k_j(t_j) + nLM_1$$

$$\Rightarrow A = \frac{L_1}{1} \int_0^1 k_2(s)ds + \frac{M_1}{1} \sum_{j=1}^{n} k_j(t_j) + nLM_1.$$\n
We obtain $\| u \| \leq A$. By Lemma 4, we deduce $F$ has a fixed point. The proof is complete.

**4 EXAMPLES.**

In this section we give two examples to illustrate our main results.

**EXAMPLE 1.** Consider the following equation

$$\begin{cases}
\ddot{u}(t) + f(t, u, Tu, Su) = 0, t \in [0,1], t \neq \frac{1}{2} \\
-\Delta u(0) = \frac{1}{10} |\sin u(\frac{1}{2})|, \\
u(0) = u(0) = 0, \\
u(1) = 0, \end{cases}$$

(4.1)

Then the Eq.(4.1) has at least one solution.

**EXAMPLE 2.** Consider the following equation

$$\begin{cases}
\ddot{u}(t) = -u - \int_0^t \sin s u(s) ds - \int_0^t \sin s u(s) ds, t \in [0,1], t \neq \frac{1}{2} \\
\Delta u(0) = \frac{1}{2} \sin u(\frac{1}{2}), \\
-\Delta u(1) = \frac{1}{3} \cos u(\frac{1}{2}), \\
u(0) = u(0) = 0, \\
u(1) = 0. \end{cases}$$

(4.2)

where $f(t, u, v, w) = v + w, Tu = \int_0^t \sin s u(s) ds$, Su = $\int_0^t \sin s u(s) ds$. It is easy to know that (H2), (H3) hold, and $L = 1, L_1 = \frac{1}{12}, L_2 = \frac{1}{3}$.

$$\int_0^1 k_2(s)ds(1 + T1 + S1)ds \leq \frac{5 - 3\cos 1}{6} < 1,$$

and (3.6) holds. By Theorem 3, the Eq. (4.2) has at least one solution.
REFERENCES


