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Existence of Solutions for Front-Type Forced Waves of Leslie-Gower Predator-Prey Model in Shifting Habitats

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***Abstract* - In this paper, the definitions of front type and a pair of generalized upper and lower solutions are firstly described. Then, Leslie-Gower predator-prey model and the properties of kernel and growth rate function are expressed. The suitable upper and lower solutions combined with the Schauder's fixed-point theorem, fatou's lemma is utilized to solve the existence of nonnegative solution. Additionally, under the appropriate parameter assumptions, the existence of front type forced wave is verified.**

Keywords - Schauder's fixed-point theorem; fatou's lemma; upper and lower solutions; compact; precompact; integration

I. INTRODUCTION

Wave propagation is an important phenomenon in many scientific areas such as physics, biology and ecology. Reaction-diffusion models are commonly used to describe how a quantity, such as a biological population or chemical substance, spreads in space over time. In classical theory, traveling waves move with a constant speed that is determined only by the internal properties of the system. However, in real world situation, wave propagation is often influenced by the external effects. These effects may include environmental forcing, boundary conditions. Front-type forced waves describe wave solution that appear as moving fronts connecting two different stable states under the influence of forcing.

II. PRELIMINARIES

We begin our article by giving the definition of front- type, a pair of generalized upper and lower solutions.

1. Definition

First, For a scalar wave profile $\phi(x-st)$, if $\phi(-\infty)\neq\phi(+\infty)$, it is called a **front type**.

2. Definition

The continuous functions $(\bar{\phi}_1,\bar{\phi}_2)$ and $(\underline{\phi}_1,\underline{\phi}_2)$ are called a pair of generalized **upper and lower solutions** if $\bar{\phi}_i,\underline{\phi}_i,\ i=1,2$ are bounded functions and satisfy the following inequalities

$$\left.\begin{aligned}
&U_1(z):=d_1^*\left(J_1*\bar{\phi}_1(z)-\bar{\phi}_1(z)\right)-s\bar{\phi}_1{}'(z)+\bar{\phi}_1(z)\left[r(-z)-\bar{\phi}_1(z)-k\underline{\phi}_2(z)\right]\le 0,\\
&U_2(z):=d_2^*\left(J_2*\bar{\phi}_2(z)-\bar{\phi}_2(z)\right)-s\bar{\phi}_2{}'(z)+\bar{\phi}_2(z)\left[r(-z)-\frac{\bar{\phi}_2(z)}{\bar{\phi}_1(z)+h}\right]\le 0,\\
&L_1(z):=d_1^*\left(J_1*\underline{\phi}_1(z)-\underline{\phi}_1(z)\right)-s\underline{\phi}_1'(z)+\underline{\phi}_1(z)\left[r(-z)-\underline{\phi}_1(z)-k\bar{\phi}_2(z)\right]\ge 0,\\
&L_2(z):=d_2^*\left(J_2*\underline{\phi}_2(z)-\underline{\phi}_2(z)\right)-s\underline{\phi}_2'(z)+\underline{\phi}_2(z)\left[r(-z)-\frac{\underline{\phi}_2(z)}{\underline{\phi}_1(z)+h}\right]\ge 0,
\end{aligned}\right\}\quad (1)$$

for all $z\in\mathbb{R}\setminus E$ for some finite subset E of $\mathbb{R}$.

III. LISLIE-GOWER PREDATOR-PREY MODEL

In this section, the forced waves of Leslie-Gower predator-prey model in shifting habitats with nonlocal dispersal is focused. First, we consider the following diffusive predator-prey model with one prey and one predator:

$$\left.\begin{aligned}
&u_t(x,t)=d_1^*(J_1*u-u)(x,t)+u\left[r(x-st)-u(x,t)-kv(x,t)\right],\ x\in\mathbb{R},\ t>0,\\
&v_t(x,t)=d_2^*(J_2*v-v)(x,t)+v\left[r(x-st)-\frac{v(x,t)}{u(x,t)+h}\right],\ x\in\mathbb{R},\ t>0,
\end{aligned}\right\}(2)$$

where the unknown functions u denote the population density of the prey and v denote the population density of the predator at position x and time t. All parameters $d_1^*,d_2^*,r,\ h,\ k$ are positive. Parameters d_1^*,d_2^* represent diffusion rates for prey and predator, the function r, represents the growth rate, k denote the per capita capturing rate of the prey by a predator per unit of time, and $\frac{v}{u+h}$ represents Leslie-Gower terms, which means that the carrying capacity of the predator is proportional to the population size of the prey.

The parameters h and k satisfy the conditions

$$0<h<1,\quad k>0,\quad hk<1. \qquad (3)$$

A. Properties of Kernel and Growth Rate Function

We always assume that the kernel functions $J_i(\cdot)\ (i=1,2)$ satisfy the following properties:

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(J) $J_i(\cdot) \in C(\mathbb{R}, \mathbb{R}^+)$, $J_i(-x) = J_i(x)$, $\int_{\mathbb{R}} J_i(x)dx = 1$ and $\int_{\mathbb{R}} J_i(x)e^{-\lambda x}dx < \infty$ for any $\lambda > 0$, $i = 1, 2$.

The growth rate function $r(\cdot)$ satisfies the following two properties:

(H) $r(\cdot)$ is continuous in $\mathbb{R}$, $\lim_{z \to \pm\infty} r(z)$ exists satisfying $-\infty < r(-\infty) < 0 < r(+\infty) < \infty$ and $r(z) \le r(+\infty)$ for all $z \in \mathbb{R}$. Without loss of generality (up to a rescaling), we choose $r(+\infty) = 1$;

(H^*) there exists $C > 0$ and $\rho > 0$ such that

$$\lim_{z \to +\infty} \frac{r(+\infty) - r(z)}{e^{-\rho z}} = C.$$

We are interested in the propagation phenomena for system (2). We study the special C^1 solution of form

$$\left(u(x,t), v(x,t)\right) = \left(\phi_1(st - x), \phi_2(st - x)\right)$$

where parameter s being the shifting speed of the climatic condition, which is called the forced wave.

Let $z = st - x$, and the corresponding wave profile system to system (2) is as follows

$$\left.\begin{aligned} s\phi_1'(z) &= d_1^*\left(J_1 * \phi_1(z) - \phi_1(z)\right) + \phi_1(z)\left[r(-z) - \phi_1(z) - k\phi_2(z)\right], z \in \mathbb{R}, \\ s\phi_2(z) &= d_2^*\left(J_2 * \phi_2(z) - \phi_2(z)\right) + \phi_2(z)\left[r(-z) - \frac{\phi_2(z)}{\phi_1(z) + h}\right], z \in \mathbb{R}. \end{aligned}\right\} \quad (4)$$

From assumption (H), the environment is favourable to the prey ahead of the climate change and then gradually deteriorates until it becomes hostile to the species. This is equivalent to the boundary condition $(\phi_1, \phi_2)(+\infty) = (0, 0)$.

We shall consider the constant unique coexistence state of system (2) such as $E_* = (v^*, \omega^*)$ where $v^* = \frac{1 - hk}{1 + k}$ and $\omega^* = \frac{1 + h}{1 + k}$.

We define

$$\Phi(\lambda) = \frac{d_2^*\left(\int_{\mathbb{R}} J_2(y)e^{-\lambda y}dy - 1\right) + 1}{\lambda}.$$

B. Existence of Nonnegative Solution

In this section, we state the theorems for the existence of nonnegative solution.

1 Theorem

Suppose that $s > 0$. If $(\bar{\phi}_1, \bar{\phi}_2)$ and $(\underline{\phi}_1, \underline{\phi}_2)$ are a pair of upper and lower solutions of (4) satisfying

$$0 \le \underline{\phi}_1 \le \bar{\phi}_1 \le 1,\ 0 \le \underline{\phi}_2 \le \bar{\phi}_2 \le 1 + h \ \text{ in } \mathbb{R},$$

then (4) admits a solution (ϕ_1, ϕ_2) such that $\underline{\phi}_i(z) \le \phi_i(z) \le \bar{\phi}_i(z)$ for all $z \in \mathbb{R}$, $i = 1, 2$.

Proof:

Let $X = BUC(\mathbb{R}, \mathbb{R})$ be the space of all uniformly continuous and bounded functions defined in $\mathbb{R}$. Then, X is a Banach space equipped with the sup-norm.

Let $Y = \left\{(\phi_1, \phi_2) \in X^2 : 1 \ge \phi_1(z) \ge 0, 1 + h \ge \phi_2(z) \ge 0 \text{ for all } z \in \mathbb{R}\right\}$.

For $\Phi = (\phi_1, \phi_2) \in Y$, we consider the nonlinear operators $F_i\ (i = 1, 2)$ defined on Y by

$$F_1[\Phi](z) = \theta\phi_1(z) + d_1^*\left(J_1 * \phi_1 - \phi_1\right)(z) + \phi_1(z)\left[r(-z) - \phi_1(z) - k\phi_2(z)\right], z \in \mathbb{R},$$
$$F_2[\Phi](z) = \theta\phi_2(z) + d_2^*\left(J_2 * \phi_2 - \phi_2\right)(z) + \phi_2(z)\left[r(-z) - \frac{\phi_2(z)}{\phi_1(z) + h}\right], z \in \mathbb{R},$$

where $\theta = \max\{\sigma_1, \sigma_2\} > 0$ with

$$\left.\begin{aligned} \sigma_1 &= d_1^* + \|r\|_{L^\infty(\mathbb{R})} + k(1 + h) + 2, \\ \sigma_2 &= d_2^* + \|r\|_{L^\infty(\mathbb{R})} + \frac{2(1 + h)}{h}. \end{aligned}\right\} \quad (5)$$

We define the following operator

$$P_i[\Phi](z) = \frac{1}{s}\int_{-\infty}^{z} e^{\frac{\theta(y - z)}{s}} F_i[\Phi](y)dy, z \in \mathbb{R}, i = 1, 2.$$

Let $P[\Phi] = \left(P_1[\Phi], P_2[\Phi]\right)$, $(\tilde{\phi}_1, \tilde{\phi}_2) = P(\phi_1, \phi_2)$.

Then,

$$P : Y \to X^2 \text{ and } s\tilde{\phi}_i'(z) = -\theta\tilde{\phi}_i(z) + F_i[\Phi](z), z \in \mathbb{R}.$$

Thus, a fixed point of P is a solution of (4).

Let $\mu \in \left(0, \frac{\theta}{s}\right)$ be a constant and we define the norm

$$|\Phi|_\mu = \sup_{z \in \mathbb{R}}\left\{\max\left(|\phi_1(z)|, |\phi_2(z)|\right)e^{-\mu|z|}\right\}, \Phi \in Y.$$

Moreover, the set

$$A = \left\{(\phi_1, \phi_2) \in Y : \bar{\phi}_i \ge \phi_i \ge \underline{\phi}_i \ge 0, i = 1, 2\right\}$$

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is a non-empty convex, closed, and bounded set in $\left(Y, |\cdot|_\mu\right)$.

Then, we show that P maps A into A. Let $\Phi \in A$.
By using (5), we can get

$$F_1[\Phi](z) \ge F_1\left[\left(\underline{\phi}_1, \bar{\phi}_2\right)\right](z),$$

and hence, $P_1[\Phi](z) \ge P_1\left[\left(\underline{\phi}_1, \bar{\phi}_2\right)\right](z)$ for all $z \in \mathbb{R}$.

From (1), we have

$$P_1\left[\left(\underline{\phi}_1, \bar{\phi}_2\right)\right](z) = \frac{1}{s}\int_{-\infty}^{z} e^{\frac{\theta(y-z)}{s}} F_1\left[\left(\underline{\phi}_1, \bar{\phi}_2\right)\right](y)\, dy$$

$$= \lim_{\varepsilon \to 0} \int_{[-\infty, z)\setminus U_{z_i} \in E(z_i - \varepsilon, z_i + \varepsilon)} e^{\frac{\theta(y-z)}{s}} \frac{F_1\left[\left(\underline{\phi}_1, \bar{\phi}_2\right)\right](y)}{s} dy$$

$$\ge \liminf_{\varepsilon \to 0} \int_{[-\infty, z)\setminus U_{z_i} \in E(z_i - \varepsilon,\, z_i + \varepsilon)} e^{\frac{\theta(y-z)}{s}} \frac{c\underline{\phi}_1{}'(y) + \theta \underline{\phi}_1(y)}{s} dy$$

$$= \underline{\phi}_1(z), \forall z \in \mathbb{R}.$$

Thus, $P_1\left[\left(\underline{\phi}_1, \bar{\phi}_2\right)\right](z) \ge \underline{\phi}_1(z)$. Similarly, we have

$$\bar{\phi}_1(z) \ge P_1\left[\left(\bar{\phi}_1, \underline{\phi}_2\right)\right](z).$$

By using the choice of θ and the definition of super and sub-solutions, we can calculate

$$\bar{\phi}_1 \ge P_1\left[\left(\bar{\phi}_1, \underline{\phi}_2\right)\right] \ge P_1[\Phi] \ge P_1\left[\left(\underline{\phi}_1, \bar{\phi}_2\right)\right] \ge \underline{\phi}_1,$$

$$\bar{\phi}_2 \ge P_2\left[\left(\bar{\phi}_1, \bar{\phi}_2\right)\right] \ge P_2[\Phi] \ge P_2\left[\left(\underline{\phi}_1, \underline{\phi}_2\right)\right] \ge \underline{\phi}_2.$$

Therefore, $P(A) \subset A$.

Next, we show that the mapping $P: A \to A$ is completely continuous with respect to the norm $|\cdot|_\mu$. We give some details and show the continuity of P on A.

For any $\Phi_1 = (\psi_1, \psi_2) \in A$ and $\Phi_2 = (\varphi_1, \varphi_2) \in A$, we have

$$\left|P_1(\psi_1, \psi_2)(z) - P_1(\varphi_1, \varphi_2)(z)\right|_\mu$$

$$= \left|P_1(\psi_1, \psi_2)(z) - P_1(\varphi_1, \varphi_2)(z)\right| e^{-\mu|z|}$$

$$\left|P_1(\psi_1, \psi_2)(z) - P_1(\varphi_1, \varphi_2)(z)\right|_\mu$$

$$\le \frac{d_1^*}{s}\left|\int_{-\infty}^{z}\left[J_1 * \psi_1(y) - J_1 * \varphi_1(y)\right] e^{\frac{-\theta(z-y)}{s}} dy\right| e^{-\mu|z|}$$

$$+ \frac{\theta + d_1^* + \|r\|_{L^\infty(\mathbb{R})} + 2 + k(1+h)}{s}\left|\int_{-\infty}^{z}\left[\psi_1(y) - \varphi_1(y)\right] e^{\frac{-\theta(z-y)}{s}} dy\right| e^{-\mu|z|}$$

$$+ \frac{k}{s}\left|\int_{-\infty}^{z}\left[\psi_2(y) - \varphi_2(y)\right] e^{\frac{-\theta(z-y)}{s}} dy\right| e^{-\mu|z|}.$$

We note that

$$\frac{d_1^*}{s}\left|\int_{-\infty}^{z}\left[J_1 * \psi_1(y) - J_1 * \varphi_1(y)\right] e^{\frac{-\theta(z-y)}{s}} dy\right| e^{-\mu|z|}$$

$$\le \frac{d_1^*}{s}\int_{-\infty}^{z}\int_{\mathbb{R}} J_1(y-\xi)\left|\psi_1(\xi) - \varphi_1(\xi)\right| e^{-\mu|\xi|} e^{\mu|\xi|} d\xi\, e^{\frac{-\theta(z-y)}{s}} dy$$

$$\le \frac{d_1^*}{s}\int_{\mathbb{R}} J_1(y) e^{\mu|y|} dy \int_{-\infty}^{z} e^{\mu|y|} e^{\frac{-\theta(z-y)}{s}} dy\, e^{-\mu|z|}\left|\psi_1 - \varphi_1\right|_\mu$$

and

$$\frac{1}{s}\int_{-\infty}^{z} e^{\mu|y|} e^{\frac{-\theta(z-y)}{s}} dy\, e^{-\mu|z|} \le \frac{2}{\theta - \mu s}.$$

Thus, we have

$$\left|P_1(\psi_1, \psi_2)(z) - P_1(\varphi_1, \varphi_2)(z)\right|_\mu \le L_1\left|\psi_1 - \varphi_1\right|_\mu + L_2\left|\psi_2 - \varphi_2\right|_\mu$$

where

$$L_1 = \frac{2}{\theta - \mu s}\left[d_1^*\int_{\mathbb{R}} J_1(y) e^{\mu|y|} dy + \theta + d_1^* + \|r\|_{L^\infty(\mathbb{R})} + 2 + k(1+h)\right],$$

$$L_2 = \frac{2k}{\theta - \mu s}.$$

Similarly, we have

$$\left|P_2(\psi_1, \psi_2)(z) - P_2(\varphi_1, \varphi_2)(z)\right|_\mu$$

$$= \left|P_2(\psi_1, \psi_2)(z) - P_2(\varphi_1, \varphi_2)(z)\right| e^{-\mu|z|}$$

$$\le \frac{d_2^*}{s}\left|\int_{-\infty}^{z}\left[J_2 * \psi_2(y) - J_2 * \varphi_2(y)\right] e^{\frac{-\theta(z-y)}{s}} dy\right| e^{-\mu|z|}$$

$$+ \frac{\theta + d_2^* + \|r\|_{L^\infty(\mathbb{R})} + 2\frac{(1+h)}{h}}{s}\left|\int_{-\infty}^{z}\left[\psi_2(y) - \varphi_2(y)\right] e^{\frac{-\theta(z-y)}{s}} dy\right| e^{-\mu|z|}$$

$$+ \frac{\frac{(1+h)^2}{h^2}}{s}\left|\int_{-\infty}^{z}\left[\psi_1(y) - \varphi_1(y)\right] e^{\frac{-\theta(z-y)}{s}} dy\right| e^{-\mu|z|}$$

$$\le M_1\left|\psi_1 - \varphi_1\right|_\mu + M_2\left|\psi_2 - \varphi_2\right|_\mu,$$

where

$$M_1 = \frac{\frac{(1+h)^2}{h^2}}{\theta - \mu s},$$

$$M_2 = \frac{2}{\theta - \mu s}\left[d_2^*\int_{\mathbb{R}} J_2(y) e^{\mu|y|} dy + \theta + d_2^* + \|r\|_{L^\infty(\mathbb{R})} + 2\frac{(1+h)}{h}\right].$$

Therefore, there exists a positive constant $\tilde{C}$ such that

$$\left|P\Phi_1 - P\Phi_2\right|_\mu \le \tilde{C}\left|\Phi_1 - \Phi_2\right|_\mu.$$

Hence, P is continuous with respect to the norm $|\cdot|_\mu$.

Now, we will prove P in A is compact with respect to the norm $|\cdot|_\mu$. For any $(\phi_1, \phi_2) \in A$ and $n \in \mathbb{N}$, we define

$$P^n[\phi_1, \phi_2](z) = \begin{cases} P[\phi_1, \phi_2](-n), z \in (-\infty, -n), \\ P[\phi_1, \phi_2](z), z \in [-n, n], \\ P[\phi_1, \phi_2](n), z \in (n, \infty). \end{cases}$$

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$P^n[\phi_1,\phi_2](z)$ is equicontinuous and uniform bounded in $(Y,|\cdot|_\mu)$.

Thus, there exists a constant N such that

$$\left|P^n[\phi_1,\phi_2](z)-P[\phi_1,\phi_2](z)\right|e^{-\mu|z|}\le Ne^{-\mu n}.$$

Then, $P^n[\phi_1,\phi_2](z)$ converges to $P[\phi_1,\phi_2](z)$ as $n\to\infty$. Thus, $P[\phi_1,\phi_2](z)\big|_{[-n,n]}$ is compact and then $P^n[\phi_1,\phi_2](z)$ is compact. We verify that $P[\phi_1,\phi_2](z)$ is precompact. Hence, by Schauder□s fixed-point theorem, the proof completes. □

2. Theorem

Assume that $(\bar\phi_1,\bar\phi_2)$ and $(\underline{\phi}_1,\underline{\phi}_2)$ are a pair of upper and lower solutions of (4) satisfying

$$0\le\underline{\phi}_1\le\bar\phi_1\le 1,\ 0\le\underline{\phi}_2\le\bar\phi_2\le 1+h \ \text{ in } □,$$

and admits a solution (ϕ_1,ϕ_2) such that $\underline{\phi}_i(z)\le\phi_i(z)\le\bar\phi_i(z)$ for all $z\in□$ for $i=1,2,$ then $(\phi_1,\phi_2)(+\infty)=(0,0)$ for any nonnegative solution (ϕ_1,ϕ_2) of (4).

Proof:

For contradiction, we assume that $\phi_1^+=\limsup_{z\to+\infty}\phi_1(z)>0.$

When ϕ_1 is oscillatory near $z=+\infty$, there is a maximal sequence $\{z_n\}$ of ϕ_1 such that $z_n\to+\infty$ and $\phi_1(z_n)\to\phi_1^+$ as $n\to+\infty$.

From the first equation of (4), we get

$$0=s\phi_1'(z)=d_1^*\left(\int_{□}J_1(y)\phi_1(z_n-y)dy-\phi_1(z_n)\right)+\phi_1(z_n)\left[r(-z_n)-\phi_1(z_n)-k\phi_2(z_n)\right]. \quad (6)$$

By letting $n\to+\infty$, it follows from Fatou□s lemma that

$$\limsup_{n\to+\infty} d_1^*\int_{□}J_1(y)\left(\phi_1(z_n-y)-\phi_1(z_n)\right)dy\le 0 \quad \text{and}$$

$$\limsup_{n\to+\infty}\left[\phi_1(z_n)\left(r(-z_n)-\phi_1(z_n)-k\phi_2(z_n)\right)\right]<\phi_1^+\left(r(-\infty)-\phi_1^+-k\liminf_{n\to+\infty}\phi_2(z_n)\right)<0.$$

From $r(-\infty)<0,$ it contradicts the equation (6). On the other hand, suppose that ϕ_1 is monotone ultimately at $z=+\infty$, then $\phi_1(+\infty)=\phi_1^+$.

By integrating the first equation of (4) from 0 to $n,$ we get

$$s[\phi_1(n)-\phi_1(0)]=d_1^*\int_0^n(J_1*\phi_1-\phi_1)(z)\,dz+\int_0^n\phi_1(z)\left(r(-z)-\phi_1(z)-k\phi_2(z)\right)dz$$
$$=d_1^*\int_{-\infty}^{+\infty}J_1(y)\int_0^n\int_0^1\phi_1'(z-ry)(-y)dr\,dz\,dy+\int_0^n\phi_1(z)\left(r(-z)-\phi_1(z)-k\phi_2(z)\right)dz.$$

Hence,

$$s[\phi_1(n)-\phi_1(0)]+d_1^*\int_{□}J_1(y)y\int_0^1\left(\phi_1(n-ry)-\phi_1(-ry)\right)dr\,dy \quad (7)$$

$$=\int_0^n\left(\phi_1(z)(r(-z)-\phi_1(z)-k\phi_2(z)\right)dz.$$

By taking a sufficiently large positive constant M and $\alpha\in(0,1)$ so that $\phi_1(z)\ge\alpha\phi_1^+$ and $r(-z)\le 0$ for all $z>M,$ we get

$$\phi_1(z)\left(r(-z)-\phi_1(z)-k\phi_2(z)\right)\le-(\alpha\phi_1^+)^2<0.$$

Thus,

$$\lim_{n\to+\infty}\int_0^n\phi_1(z)\left(r(-z)-\phi_1(z)-k\phi_2(z)\right)dz=-\infty,$$

which contradicts the boundedness of (7).

Thus, $\phi_1(+\infty)=0.$

For the contradiction, we set that $\phi_2^+=\limsup_{z\to+\infty}\phi_2(z)>0.$

When ϕ_2 is oscillatory near $z=+\infty$, we have a maximal sequence $\{z_n\}$ of ϕ_2 such that $z_n\to\infty$ and $\phi_2(z_n)\to\phi_2^+$ as $n\to\infty.$

From the ϕ_2 equation of (4), we get

$$0=\limsup_{n\to\infty}\left[d_2^*\left(\int_{□}J_2(y)\phi_2(z_n-y)dy-\phi_2(z_n)\right)+\phi_2(z_n)\left(r(-z_n)-\frac{\phi_2(z_n)}{\phi_1(z_n)+h}\right)\right]$$
$$\le\phi_2^+\left(r(-\infty)-\frac{\phi_2^+}{\phi_1^++h}\right)<0.$$

This is a contradiction. When ϕ_2 is monotone ultimately at $z=+\infty$ the process is similarly to the proof of ϕ_1.

Hence, $\phi_2(+\infty)=0.$

Thus, $(\phi_1,\phi_2)(+\infty)=(0,0,0)$ for any nonnegative solution (ϕ_1,ϕ_2) of (4).

IV EXISTENCE OF FRONT-TYPE FORCED WAVES

In this section, the existence of front type forced waves connecting E_* to $(0,0)$ is shown. Since r is non-monotonic, we consider the following problem

$$\left.\begin{array}{ll} c\phi'(z) = d(J*\phi-\phi)(z)+\phi(z)\left[r(-z)-\phi(z)\right], z\in\mathbb{R}, \\ \phi(-\infty)=r(\infty), \qquad \phi(+\infty)=0. \end{array}\right\} \quad (8)$$

A positive function $\underline{\phi}_1$ satisfying

$$\begin{cases} s\underline{\phi}_1{}' = d_1^*(J_1*\underline{\phi}_1(z)-\underline{\phi}_1(z))+\underline{\phi}_1(z)\left[r(-z)-k(1+h)-\underline{\phi}_1(z)\right], z\in\mathbb{R}, \\ \underline{\phi}_1(-\infty)=r(\infty)-k(1+h), \qquad \underline{\phi}_1(+\infty)=0. \end{cases}$$

Similarly, there exists a function $\underline{\phi}_2$ satisfying

$$s\underline{\phi}_2{}' = d_2^*(J_2*\underline{\phi}_2(z)-\underline{\phi}_2(z))+\underline{\phi}_2(z)\left[r(-z)-\frac{\underline{\phi}_2(z)}{\underline{\phi}_2(z)+h}\right], z\in\mathbb{R}$$

and

$$\lim_{z\to-\infty}\underline{\phi}_2(z)=r(\infty)\left[r(\infty)-k(1+h)+h\right]>0, \ \lim_{z\to\infty}\underline{\phi}_2(z)=0.$$

3. *Theorem*

Suppose that $k<\frac{1}{1+h}$. Then, for each $s>0$ there exists a positive solution (ϕ_1,ϕ_2) of (4) such that $\underline{\phi}_1\le\phi_1\le 1$ and $\underline{\phi}_2\le\phi_2\le 1+h$ in $\mathbb{R}$.

Proof:

We denote $(\bar{\phi}_1,\bar{\phi}_2)=(1,1+h)$. By the definition of $(\underline{\phi}_1,\underline{\phi}_2)$, we get $L_i(z)\ge 0,\ i=1,2$.

Since $r(-z)\le 1$ for $z\in\mathbb{R}$, there are

$$d_1^*(J_1*\bar{\phi}_1(z)-\bar{\phi}_1(z))-s\bar{\phi}_1{}'(z)+\bar{\phi}_1(z)\left[r(-z)-\bar{\phi}_1(z)-k\underline{\phi}_2(z)\right] \le\left[r(-z)-1\right]\le 0,$$

and

$$d_2^*(J_2*\bar{\phi}_2(z)-\bar{\phi}_2(z))-s\bar{\phi}_2{}'(z)+\bar{\phi}_2(z)\left[r(-z)-\frac{\bar{\phi}_2(z)}{\bar{\phi}_1(z)+h}\right]\le 0.$$

Therefore, $(\bar{\phi}_1,\bar{\phi}_2)$ and $(\underline{\phi}_1,\underline{\phi}_2)$ are a pair of upper and lower solutions of system (4). Hence, by Theorem 1, the proof completes.

To proceed further, we set

$$\phi_i^+=\limsup_{z\to-\infty}\phi_i(z),\quad \phi_i^-=\liminf_{z\to-\infty}\phi_i(z),\ i=1,2.$$

Since $\phi_i\ge\underline{\phi}_i$, we have

$\phi_i^-\ge\beta_i$ for $i=1,2,$ where $\beta_1=1-k(1+h),\ \beta_2=\beta_1+h$.

4. *Theorem*

Assume that $k<\frac{1}{1+h}$. Let (ϕ_1,ϕ_2) be a solution of (4) obtained from Theorem 3. Then, $(\phi_1,\phi_2)(-\infty)=E_*=(v^*,\omega^*)$.

Proof:

For $\theta\in[0,1]$, we define the following functions

$$m_1(\theta)=\theta v^*+(1-\theta)(k-\varepsilon),$$
$$m_2(\theta)=\theta\omega^*+(1-\theta)(\beta_2-r_1\varepsilon),$$
$$M_1(\theta)=\theta v^*+(1-\theta)(1+\varepsilon),$$
$$M_2(\theta)=\theta\omega^*+(1-\theta)(1+h+r_2\varepsilon),$$

where

$$r_1>\frac{1}{k},\quad 1<r_2<\frac{1}{k},\quad 0<\varepsilon<\min\left\{\beta_1,\frac{\beta_2}{r_1},\frac{k\beta_2}{kr_1-1}\right\}. \qquad (9)$$

Let $A=\left\{\theta\in[0,1)\,\middle|\,m_i(\theta)<\phi_i^-\le\phi_i^+<M_i(\theta), i=1,2\right\}$.

From Theorem 3, it is obvious that

$$m_i(\theta)<\phi_i^-\le\phi_i^+<M_i(\theta)$$

is true for $\theta=0$ and $i=1,2$. Thus, $A\neq 0$. In addition, we know $v^*<1,\ \omega^*<1+h$. Meanwhile, (υ^*,ω^*) satisfies

$$v^*=1-k\omega^*,\ \omega^*=v^*+h.$$

Then, we get that

$$v^*>1-k(1+h)=\beta_1,\qquad \omega^*>\beta_1+h=\beta_2.$$

Then, we obtain that $m_i(\theta)$ $(-M_i(\theta), i=1,2)$ is a monotone increasing function of $\theta\in[0,1]$ such that

$$(m_1,m_2)(1)=(M_1,M_2)(1)=(v^*,\omega^*).$$

Thus, it is sufficient to show that $\sup A=1$.

We argue by a contradiction and suppose that $\sup A=\theta_0\in(0,1)$. By taking the limit, we can get

$$m_i(\theta_0)\le\phi_i^-\le\phi_i^+\le M_i(\theta_0),\ \ i=1,2,$$

It should be pointed out that at least one of the following equalities holds

$$\phi_i^-=m_i(\theta_0),\quad \phi_i^+=M_i(\theta_0),\ i=1,2,$$

according to the definition of θ_0 and the continuity of $m_i(\theta)$ and $M_i(\theta)$.

Now, we consider the case $m_1(\theta_0)=\phi_1^-$. If ϕ_1 is eventually monotone, then $\phi_1(-\infty)=m_1(\theta_0)$.

By integrating the first equation of (4) from $-n$ to 0, we can get

$$s[\phi_1(0)-\phi_1(-n)]$$
$$=d_1^*\int_{-n}^{0}(J_1*\phi_1-\phi_1)(z)dz+\int_{-n}^{0}\phi_1(z)\left(r(-z)-\phi_1(z)-k\phi_2(z)\right)dz$$
$$=d_1^*\int J_1(y)\int_{-n}^{0}\int_{0}^{1}\phi_1'(z-\upsilon y)(-y)d\upsilon dzdy$$
$$+\int_{-n}^{0}\phi_1(z)\left(r(-z)-\phi_1(z)-k\phi_2(z)\right)dz.$$

Hence,

$$s[\phi_1(0)-\phi_1(-n)]+d_1^*\int J_1(y)y\int_0^1\left(\phi_1(-\upsilon y)-\phi_1(-n-\upsilon y)\right)d\upsilon dy$$
$$=\int_{-n}^{0}\phi_1(z)\left(r(-z)-\phi_1(z)-k\phi_2(z)\right)dz. \qquad (10)$$

We note that

$$\liminf_{z\to-\infty}\left[r(-z)-\phi_1(z)-k\phi_2(z)\right]$$
$$\geq 1-m_1(\theta_0)-kM_2(\theta_0)$$
$$\geq 1-\left[\theta_0 v^*+(1-\theta_0)(\beta_1-\varepsilon)\right]-k\left[\theta_0\omega^*+(1-\theta_0)(1+h+r_2\varepsilon)\right]$$
$$=1-\theta_0 v^*-k\theta_0\omega^*-(1-\theta_0)(\beta_1-\varepsilon)-k(1-\theta_0)(1+h+r_2\varepsilon)$$
$$=1-\theta_0-(1-\theta_0)(\beta_1-\varepsilon)-k(1-\theta_0)(1+h+r_2\varepsilon)$$
$$=(1-\theta_0)(1-\beta_1+\varepsilon-k-hk-kr_2\varepsilon)$$
$$=(1-\theta_0)(1-1+k+hk+\varepsilon-k-hk-kr_2\varepsilon)$$
$$=\varepsilon(1-\theta_0)(1-kr_2)>0,$$

by using (9) and $\theta_0<1$.

Thus,

$$\lim_{n\to\infty}\int_{-n}^{0}\phi_1(z)\left(r(-z)-\phi_1(z)-k\phi_2(z)\right)dz=\infty.$$

This contradicts the boundedness of left side of (10).

On the other hand, we assume ϕ_1 is oscillatory at $-\infty$. Then, we can choose a sequence $\{z_n\}_{n\in N}$ of minimal point of ϕ_1 with $z_n\to-\infty$ as $n\to+\infty$ so that $\lim_{n\to+\infty}\phi_1(z_n)=m_1(\theta_0)$.

We note that $\phi_1'(z_n)=0$ and the Fatou□s lemma gives that

$$\liminf_{n\to+\infty}(J_1*\phi_1-\phi_1)(z_n)\geq 0.$$

From the first equation of (4), we have

$$0=s\liminf_{n\to+\infty}\phi_1'(z_n)\geq\liminf_{n\to+\infty}\left[\phi_1(z_n)\left(r(-z_n)-\phi_1(z_n)-k\phi_2(z)\right)\right]$$
$$\geq m_1(\theta_0)\left(1-m_1(\theta_0)-kM_2(\theta_0)\right)>0,$$

It is contradiction. That is, $\phi_1^-\neq m_1(\theta_0)$. Additionally, the other cases are similar to the discussion above by applying the following inequalities:

(i) $\phi_1^+=M_1(\theta_0)$,

$$\limsup_{n\to\infty}\left[r(-z_n)-\phi_1(z_n)-k\phi_2(z_n)\right]$$
$$\leq 1-M_1(\theta_0)-km_2(\theta_0)$$
$$=1-\left[\theta_0 v^*+(1-\theta_0)(1+\varepsilon)\right]-k\left[\theta_0\omega^*+(1-\theta_0)(\beta_2-r_1\varepsilon)\right]$$
$$=1-\theta_0 v^*-k\theta_0\omega^*-(1-\theta_0)(1+\varepsilon)-k(1-\theta_0)(\beta_2-r_1\varepsilon)$$
$$=(1-\theta_0)(1-1-\varepsilon-k\beta_2+kr_1\varepsilon)$$
$$=(1-\theta_0)\left[(kr_1-1)\varepsilon-k\beta_2\right]$$
$$<0;$$

(ii) $\phi_2^-=m_2(\theta_0)$,

$$\liminf_{n\to\infty}\left[r(-z_n)-\frac{\phi_2(z_n)}{\phi_1(z_n)+h}\right]$$
$$\geq 1-\frac{m_2(\theta_0)}{m_1(\theta_0)+h}$$
$$\geq 1-\frac{\theta_0\omega^*+(1-\theta_0)(\beta_2-r_1\varepsilon)}{\theta_0 v^*+(1-\theta_0)(\beta_1-\varepsilon)+h}$$
$$=\frac{\theta_0 v^*+(1-\theta_0)(\beta_1-\varepsilon)+h-\theta_0\omega^*+(1-\theta_0)(\beta_2-r_1\varepsilon)}{\theta_0 v^*+(1-\theta_0)(\beta_1-\varepsilon)+h}$$
$$=\frac{\theta_0(v^*-\omega^*)+h+(1-\theta_0)(\beta_1-\varepsilon-\beta_2+r_1\varepsilon)}{\theta_0 v^*+(1-\theta_0)(\beta_1-\varepsilon)+h}$$
$$\geq\frac{-h\theta_0+h+(1-\theta_0)(\beta_1-\varepsilon-\beta_2+r_1\varepsilon)}{\theta_0 v^*+(1-\theta_0)(\beta_1-\varepsilon)+h}$$
$$=\frac{(1-\theta_0)(h+\beta_1-\beta_2-\varepsilon+r_1\varepsilon)}{\theta_0 v^*+(1-\theta_0)(\beta_1-\varepsilon)+h}$$
$$=\frac{\varepsilon(1-\theta_0)(\beta_1-1)}{\theta_0 v^*+(1-\theta_0)(\beta_1-\varepsilon)+h}>0;$$

(iii) $\phi_2^+=M_2(\theta_0)$,

$$\limsup_{n\to\infty}\left[r(-z_n)-\frac{\phi_2(z_n)}{\phi_1(z_n)+h}\right]$$
$$\leq 1-\frac{M_2(\theta_0)}{M_1(\theta_0)+h}$$
$$=1-\frac{\theta_0\omega^*+(1-\theta_0)(1+h+r_2\varepsilon)}{\theta_0 v^*+(1-\theta_0)(1+\varepsilon)+h}$$
$$=\frac{\theta_0 v^*+(1-\theta_0)(1+\varepsilon)+h-\theta_0\omega^*-(1-\theta_0)(1+h+r_2\varepsilon)}{\theta_0 v^*+(1-\theta_0)(1+\varepsilon)+h}$$
$$=\frac{\theta_0(v^*-\omega^*)+(1-\theta_0)(1+\varepsilon)+h-(1-\theta_0)(1+h+r_2\varepsilon)}{\theta_0 v^*+(1-\theta_0)(1+\varepsilon)+h}$$

$$= \frac{-h\theta_0 + h + (1-\theta_0)(1+\varepsilon-1-h+r_2\varepsilon)}{\theta_0 v^* + (1-\theta_0)(1+\varepsilon)+h}$$
$$= \frac{(1-\theta_0)(h+\varepsilon-h-r_2\varepsilon)}{\theta_0 v^* + (1-\theta_0)(1+\varepsilon)+h}$$
$$= \frac{\varepsilon(1-\theta_0)(1-r_2)}{\theta_0 v^* + (1-\theta_0)(1+\varepsilon)+h} < 0,$$

in which the sequence $\{z_n\}_{n\in N}$ is the corresponding minimal or maximal point of ϕ_i $(i=1,2)$. The proof completes. □

CONCLUSION

We calculate the existence of nonnegative solution for Leslie-Gower predator-prey model by constructing appropriate upper-lower solution and employing fixed-point theorems. The conditions for the existence of front type the Leslie-Gower formulation are established. Our analysis demonstrate that climate change speed models as a shifting environment. Specifically, it is shown that: Front-type forced waves emerge when populations respond to the environmental shifts by forming monotone traveling wave profiles, capturing the invasion-extinction transition.

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