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Existence of Solution for Variable-order FDEs with IBCs

Osama Ahmad[1*] and Adnan Ali [1]

vilniaus universitetas

Abstract

This paper presents a comprehensive analytical approach to establish the existence of solutions for variable-order fractional differential equations (FDEs) with integral boundary conditions (IBCs) in the Caputo derivative sense. The investigation delves into the scrutiny of existence and uniqueness of solutions, guided by the specified assumptions (**A1**)–(**A4**). The theoretical foundation of this investigation is firmly rooted in the Banach contraction principle (BCP), the Schauder fixed point theorem (SFPT), and the preconditions of the Arzelà-Ascoli theorem. To substantiate the theoretical findings, the paper incorporates numerical examples that not only serve as empirical validation but also affirm the reliability and robustness of the obtained solutions. Through these examples, the efficacy of the developed methods is vividly demonstrated, further enhancing the credibility of the analytical approach. The utilization of established assumptions and numerical validations ensures a consistent and rigorous investigation into the variable-order FDEs considered in this paper.

Keywords: Fractional differential equations, Integral boundary conditions, Banach contraction principle, Schauder fixed point theorem, Arzelà-Ascoli theorem.

1 Introduction

The investigation into the emerging discipline of fractional calculus (FC) is framed within the historical development of calculus and its evolutionary path. The study of FC finds its roots in the pioneering works of L'Hospital and Leibniz, who made significant contributions to the understanding of fractional derivatives [1, 2]. Traditional differential equations of integer-order (IO) have served as fundamental tools for modeling dynamic systems across various disciplines, including physics, biology, engineering, and economics [3]. These equations trace their origins to the seminal contributions of luminaries such as Newton and Leibniz.

Fractional differential equations have been widely used for modeling a wide range of physical processes and phenomena. They find applications in areas such as structural probability theory, cell growth, quantum mechanics, linear and nonlinear system dynamics, astrophysics, and electrodynamics [4–6]. To analyze fractional derivatives, several fractional differential operators have been developed, including Caputo, Febrizio, Atangana-Baleonu-Caputo, and Riemann-Liouville [7–9]. FO differential equations are investigated from numerous perspectives, encompassing existence theory and numerical approximation [16–20]. Researchers have utilized fixed-point theory, particularly the BCP and SFPT, to explore the existence and stability analysis of solutions for the differential equations [21, 22]. These fixed-point theorems play a crucial role in examining the existence of solutions [23–25].

Recent research has placed notable emphasis on visualizing solutions of integral fractional boundary value problems (FBVPs) to gain insights into their behavior in physical settings. This visualization aspect is crucial for understanding complex phenomena like heat conduction and fluid flow. The integration of qualitative analysis, leveraging tools such as fractional Green's functions and topological degree theory, with visualization techniques offers a comprehensive approach to studying integral FBVPs and their applications in real-world scenarios [10–12]. Of particular interest are IBCs, which significantly influence phenomena such as heat conduction, fluid flow, and viscoelasticity. IBCs impose restrictions on physical processes over the entire interval of consideration, providing a more holistic perspective than localized boundary conditions [13–15].

Building on the aforementioned results, the authors investigated a class of pantograph implicit FO differential equations with anti-periodic boundary conditions [26]. Specifically, the authors focused on studying the equation:

$$\begin{cases} {}^{C}_{0}\mathcal{D}^{\omega}_{\eta}\psi(\eta) = f(\psi, \psi(\eta), \psi(\lambda\eta), {}^{C}_{0}\mathcal{D}^{\omega}_{\eta}\psi(\eta)), \eta \in [0,T], \quad 2 < \omega \leq 3, \\ \psi(0) = -\psi(T), {}^{C}_{0}\mathcal{D}^{\alpha}_{\eta}\psi(0) = -{}^{C}_{0}\mathcal{D}^{\alpha}_{\eta}\psi(T), {}^{C}_{0}\mathcal{D}^{\beta}_{\eta}\psi(0) = -{}^{C}_{0}\mathcal{D}^{\beta}_{\eta}\psi(T) \end{cases} \tag{1}$$

where, $0 < \lambda < 1$, $0 < \alpha < 1$, $1 < \beta < 2$, and $f : [0,T] \times \mathbb{R}^3 \to \mathbb{R}$ is a continuous function. The author investigates the qualitative theory of problem (1) using fixed-point theory and focusing on Hyers-Ulam types of stability. The study explores the significance of variable-order operators in FC, an area that has garnered considerable attention [1–6]. Previous studies have examined boundary value problems (BVPs) related to various forms of FDEs. For instance, in [27], the author established a solution for a nonlinear FDE with order $\omega \in (2,3]$ and the author in [28] obtain a solution for implicit fractional-order differential equations. The study presented in [29] addressed implicit FO differential equations and discussed the existence of solutions for two-point BVPs involving singular FDEs of variable order. Furthermore, [30] conducted a qualitative analysis of problem (2) within the context of variable-order derivatives.

$$\begin{cases} {}^{C}\mathcal{D}_{0+}^{\omega(\eta)}\psi(\eta) = f_1(\eta, \psi(\eta), I_{0+}^{\omega(\eta)}\psi(\eta)), \quad \eta \in J \\ \psi(0) = 0, \psi(T) = 0, \end{cases} \tag{2}$$

where, $J = [0, T]$, $0 < T < \infty$, $\omega(\eta) : J \to (1, 2]$ represents the variable order of the fractional derivatives, $f_1 : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

The primary contribution in [30] lies in examining the existence of a solution to the variable-order BVP. This is achieved through the transformation of the variable-order BVP into a comparable standard Caputo BVP, which maintains a fractional constant order. The investigation centers on the existence of solutions to the adjusted BVP by utilizing Darbo's fixed-point theorem and adopting the definition of Ulam-Hyers type stability. In [31], a study on a BVP for Hadamard fractional differential equations of variable order, focusing on existence criteria and stability, is conducted. The existence-uniqueness solution for a Caputo type variable order fractional differential equation, presenting new Ulam–Hyers stability results [32]. Furthermore, a nonlinear variable order fractional differential system incorporating a p-Laplacian operator is formulated in [33]. The investigation concentrates on establishing the existence of solutions and analyzing Hyers–Ulam stability, with an application in a waterborne disease model.

Motivated by the above studies,the primary aim of this paper is to explore the impact of the variable-order function $\omega(\eta)$ on the solutions of FDEs and the importance of solution existence and uniqueness. To address this objective, the problem is formulated as follows:

$$\begin{cases} {}^{c}\mathcal{D}_{0}^{\omega(\eta)}\psi(\eta) = f(\eta), \\ \psi(0) = \psi_0 + \int_0^1 \frac{(1-\zeta)^{\delta-1}}{\Gamma(\delta)} g(\zeta, \psi(\zeta))\mathbf{d}\zeta, \ \ 0 < \delta < 1 \end{cases} \tag{3}$$

where

$$f(\eta) = f(\eta, \psi(\eta), {}^{c}\mathcal{D}_{0}^{\omega(\eta)}\psi(\eta)),$$

and

$$g(\zeta) = g(\zeta, \psi(\zeta)).$$

In equation (3), the functions $f(\eta)$, $g(\zeta)$ are known functions, while the variable-order fractional derivative operator, denoted as ${}^{c}\mathcal{D}_{0}^{\omega(\eta)}$, plays a significant role. The order of the derivative, represented by the function $\omega(\eta)$, is not constant but varies. We have described this function as a mapping: $\omega : [0, 1) \to (0, 1)$, such that $0 < \omega(\eta) < 1$ for all $\eta \in [0, 1)$.

The paper further endeavors to address a lacuna in existing literature by examining innovative mathematical methodologies from fixed point theory to ascertain the presence of solutions to the BVP outlined in equation (3). The investigation will draw inspiration from the research conducted by [26], which lays the groundwork for exploring the existence and uniqueness of solutions for variable-order FDEs characterized by

IBCs. Mathematical tools including the BCP, SFPT, and Arzelà-Ascoli theorem will be utilized throughout the study.

2 Preliminaries

Some useful definitions and lemmas will be introduced in this section.

Definition 1. *([34]) The variable-order Riemann–Liouville integral of function $\psi(\eta)$ is defined as:*

$$ {}^{RL}I_{0,\eta}^{\omega(\eta)}\psi(\eta) = \frac{1}{\Gamma(\omega(\eta))}\int_0^\eta (\eta - s)^{\omega(\eta)-1}\psi(s)\,ds, \quad \eta > 0, \omega(\eta) > 0. \tag{4}$$

Definition 2. *([34]) The variable-order Riemann–Liouville derivative of function $\psi(\eta)$ is defined as:*

$$ {}^{RL}\mathcal{D}_{0,\eta}^{\omega(\eta)}\psi(\eta) = \frac{1}{\Gamma(n-\omega(\eta))}\frac{d^n}{d\eta^n}\int_0^\eta (\eta - s)^{n-\omega(\eta)-1}\psi(s)\,ds, \tag{5}$$

where $n \in \mathbb{N}$ such that $n > \omega(\eta)$, and $\eta, \omega(\eta) > 0$.
Definition 3. *([34]) The variable-order Caputo derivative of $\psi(\eta)$ is given as:*

$$ {}^{C}\mathcal{D}_{0,\eta}^{\omega(\eta)}\psi(\eta) = \frac{1}{\Gamma(n-\omega(\eta))}\int_0^\eta (\eta - s)^{n-1-\omega(\eta)}\psi'(s)\,ds, \quad \eta > 0, \omega(\eta) > 0. \tag{6}$$

Definition 4. *([34]) The definitions of variable-order derivatives (5) and (6) are not often equivalent; however, they can be linked by the following relationship [35]:*

$$ {}^{RL}\mathcal{D}_{0,\eta}^{\omega(\eta)}\psi(\eta) = \sum_{k=0}^{n-1}\frac{\psi^{(k)}(0)\eta^{k-\omega(\eta)}}{\Gamma(1+k-\omega(\eta))} + {}^{C}\mathcal{D}_{0,\eta}^{\omega(\eta)}\psi(\eta), \tag{7}$$

$$ {}^{C}\mathcal{D}_{0,\eta}^{\omega(\eta)}\psi(\eta) = \frac{1}{\Gamma(n-\omega(\eta))}\int_0^\eta (\eta - s)^{n-1-\omega(\eta)}\psi'(s)\,ds, \quad \eta > 0, \omega(\eta) > 0. \tag{8}$$

Lemma 1. *([19]) Let $\psi \in B(0,1)$, then the solution of fractional differential equation*

$$ {}^{c}\mathcal{D}^{\omega}\psi(\eta) = 0,$$

with order $n-1 \le \omega < n$, is given by

$$\psi(\eta) = c_0 + c_1\eta + \cdots + c_{n-1}\eta^{n-1}, \;\; i = 0, 1, \cdots, n-1.$$

Lemma 2. *([19]) Suppose that $\psi \in C(0,1)$, with derivative of fractional order ω, then*

$$\mathcal{I}^{\omega}\,{}^{c}\mathcal{D}^{\omega}\psi(\eta) = \psi(\eta) + c_0 + c_1\eta + \cdots + c_{n-1}\eta^{n-1}, i = 0, 1, \cdots, n-1.$$

2.1 Nonlinear analysis

Example 1. *Consider the Banach space* $\mathrm{C}'(P,Q)$ *defined as follows:*

$$\mathrm{C}'(P,Q) = \{\psi \in \mathrm{C}(P,Q) : {}^{c}\mathcal{D}^{\omega-1}\psi \in \mathrm{C}(P,Q)\},$$

where P *and* Q *are suitable intervals, and* ψ *represents a continuous function from* P *to* Q*. The norm in this space is defined as:*

$$\|\psi\|_{\mathrm{C}'} = \max\{\|\psi\|_\infty, \|{}^{c}\mathcal{D}^{\omega-1}\psi\|_\infty\}.$$

Consider two specific functions ψ_1 *and* ψ_2 *defined within this Banach space as follows:*

$$\psi_1(x) = x^2, \quad \textit{for } x \in P,$$
$$\psi_2(x) = \sin(\pi x), \quad \textit{for } x \in P.$$

It can be verified that both ψ_1 *and* ψ_2 *belong to* $\mathrm{C}'(P,Q)$*, as they are continuous functions and their fractional derivatives exist and are continuous as well.*

Next, let's compute the norms of these functions:

$$\|\psi_1\|_{\mathrm{C}''} = \max\{\|\psi_1\|_\infty, \|{}^{c}\mathcal{D}^{\omega-1}\psi_1\|_\infty\} = \max\{1,2\} = 2,$$

$$\|\psi_2\|_{\mathrm{C}''} = \max\{\|\psi_2\|_\infty, \|{}^{c}\mathcal{D}^{\omega-1}\psi_2\|_\infty\} = \max\{1,\pi\} = \pi.$$

Certainly, ψ_1 *and* ψ_2 *are elements of* $\mathrm{C}'(P,Q)$*, with their norms being finite. Additionally, these norms serve as measures of the maximum absolute value of the functions and their fractional derivatives within this Banach space.*

3 Existence and uniqueness

Lemma 3. *Let* $f : \mathcal{J} \times \mathcal{R}^2 \to \mathcal{R}$ *is continuous, then the BVP(3) i-e*

$$\begin{aligned} &{}^{c}\mathcal{D}_0^{\omega(\eta)}\psi(\eta) = f(\eta),\ 0<\omega(\eta)<1,\ \eta\in[0,1),\\ &\psi(0) = \psi_0 + \int_0^1 \frac{(1-\zeta)^{\delta-1}}{\Gamma(\delta)} g(\zeta)\mathbf{d}\zeta,\ 0<\delta<1, \end{aligned} \tag{9}$$

where

$$f(\eta) = f(\eta, \psi(\eta), {}^{c}\mathcal{D}_0^{\omega(\eta)}\psi(\eta)),$$

and

$$g(\zeta) = g(\zeta, \psi(\zeta)),$$

has a solution

$$\psi(\eta) = \psi_0 + \int_0^1 \frac{(1-\zeta)^{\delta-1}}{\Gamma(\delta)} g(\zeta)\mathbf{d}\zeta + \frac{1}{\Gamma(\omega(\eta))}\int_0^\eta (\eta-\zeta)^{\omega(\zeta)-1} f(\zeta)\mathbf{d}\zeta. \tag{10}$$

Proof. With the help of the Lemma 1,

$$\psi(\eta) = c_0 + \frac{1}{\Gamma(\omega(\eta))}\int_0^\eta (\eta-\zeta)^{\omega(\zeta)-1} f(\zeta)\mathbf{d}\zeta, \tag{11}$$

by using the IBCs

$$\psi(0) = c_0 + \frac{1}{\Gamma(\omega(\eta))}\int_0^0 (\eta-\zeta)^{\omega(\zeta)-1} f(\zeta)\mathbf{d}\zeta, \tag{12}$$

$$\psi(0) = \psi_0 + \int_0^1 \frac{(1-\zeta)^{\delta-1}}{\Gamma(\delta)} g(\zeta)\mathbf{d}\zeta, \ \ 0<\delta<1. \tag{13}$$

$$\psi(0) = c_0$$

□

Corollary 1. *On the basis of Lemma 3, solution of (3) is given as:*

$$\psi(\eta) = \psi_0 + \int_0^1 \frac{(1-\zeta)^{\delta-1}}{\Gamma(\delta)} g(\zeta)\mathbf{d}\zeta + \frac{1}{\Gamma(\omega(\eta))}\int_0^\eta (\eta-\zeta)^{\omega(\zeta)-1} f(\zeta)\mathbf{d}\zeta, \tag{14}$$

Now for the main result, we considered the following assumptions:

*(**A1**).* $f : \mathcal{J} * \mathcal{R}^2 \to \mathcal{R}$ *is continuous and* $\exists$ *constant* $Ł_m > 0$ *and* $0 < Ł_n < 1$ *with* $| f(\eta,u,v) - f(\eta,\bar{u},\bar{v}) | \leq \ Ł_m \mid u-\bar{u} \mid + Ł_n \mid v - \bar{v} \mid \quad \forall\, \eta \in \mathcal{J} \ \ \& \ \ u,\bar{u},v,\bar{v} \in \mathcal{R}.$

*(**A2**).* $g : \mathcal{J} * \mathcal{R} \to \mathcal{R}$ *is continuous and* $\exists$ *constant* $Ł_g > 0$ *with* $| g(\eta,u) - g(\eta,\bar{u}) | \leq \ Ł_g \mid u - \bar{u} \mid, \ \ \forall\, \eta \in \mathcal{J} \ \ \& \ \ u,\bar{u} \in \mathcal{R}.$

*(**A3**). There exist a non decreasing and continuous function* $\psi : \mathcal{R}^+ \to \mathcal{R}^+$ *and* $h \in C(\mathcal{J},\mathcal{R})$*, such that* $| f(\eta,u,v) | < h(\eta)\psi(| V |)$ *for* $\eta \in \mathcal{J}$ *and* $u,v \in \mathcal{R}_j$ *and* $\exists$ *a constant* $\zeta > 0$ *such that* $\{| Y_0 | + \frac{K}{\Gamma(\delta+1)} + \frac{h^*\psi(\zeta)}{\Gamma(\omega(1)+1)}\} \leq \zeta$ *where* $h^* = Sup\{h(\zeta), \zeta \in \mathcal{J})\}$ *and* $| \psi | \leq \zeta$.

*(**A4**).* $| g(\eta,\psi(\eta)) | \leq \ K \mid \psi(\eta) \mid$.

Consider the set $B = \{\psi \in X \mid {}^c\mathcal{D}_0^{\omega(\eta)}\psi \in X\}$ *and the operator* $\mathcal{T} : \beta \to \beta$ *define as:*

$$\mathcal{T}\psi(\eta) = \psi(\eta),$$

$$\begin{aligned}\mathcal{T}\psi(\eta) &= \psi_0 + \int_0^1 \frac{(1-\zeta)^{\delta-1}}{\Gamma(\delta)} g(\zeta,\psi(\zeta))\mathbf{d}\zeta \\ &\quad + \frac{1}{\Gamma(\omega(\eta))}\int_0^\eta (\eta-\zeta)^{\omega(\zeta)-1} f(\zeta,\psi(\zeta),{}^c\mathcal{D}_0^{\omega(\zeta)}\psi(\zeta))\mathbf{d}\zeta,\end{aligned}$$

$$\begin{aligned}
{}^{c}\mathcal{D}_0^{\omega(\eta)}\mathcal{T}\psi(\eta) &= {}^{c}\mathcal{D}_0^{\omega(\eta)}\{\psi_0 + \int_0^1 \frac{(1-\zeta)^{\delta-1}}{\Gamma(\delta)} g(\zeta,\psi(\zeta))\mathbf{d}\zeta \\
&+ I_0^{\omega(\eta)} f(\zeta,\psi(\zeta),{}^{c}\mathcal{D}_0^{\omega(\eta)}\psi(\zeta))\mathbf{d}\zeta\}, \\
{}^{c}\mathcal{D}_0^{\omega(\eta)}\mathcal{T}\psi(\eta) &= f(\zeta,\psi(\zeta),{}^{c}\mathcal{D}_0^{\omega(\eta)}\psi(\zeta)).
\end{aligned}$$

and

$$\begin{aligned}
{}^{c}\mathcal{D}_0^{\omega(\eta)}\psi(\eta) &= {}^{c}\mathcal{D}_0^{\omega(\eta)}\{\psi_0 + \int_0^1 \frac{(1-\zeta)^{\delta-1}}{\Gamma(\delta)} g(\zeta,\psi(\zeta))\mathbf{d}\zeta \\
&+ \frac{1}{\Gamma(\omega(\eta))}\int_0^\eta (\eta-\zeta)^{\omega(\zeta)-1} f(\zeta,\psi(\zeta),{}^{c}\mathcal{D}_0^{\omega(\zeta)}\psi(\zeta))\mathbf{d}\zeta\}, \qquad (15)\\
&= {}^{c}\mathcal{D}_0^{\omega(\eta)}\{\psi_0 + \int_0^1 \frac{(1-\zeta)^{\delta-1}}{\Gamma(\delta)} g(\zeta,\psi(\zeta))\mathbf{d}\zeta\} \\
&+ {}^{c}\mathcal{D}_0^{\omega(\eta)}\{\frac{1}{\Gamma(\omega(\eta))}\int_0^\eta (\eta-\zeta)^{\omega(\zeta)-1} f(\zeta,\psi(\zeta),{}^{c}\mathcal{D}_0^{\omega(\zeta)}\psi(\zeta))\mathbf{d}\zeta\}, \\
&= 0 + {}^{c}\mathcal{D}_0^{\omega(\eta)} I_0^{\omega(\eta)} f(\zeta,\psi(\zeta),{}^{c}\mathcal{D}_0^{\omega(\eta)}\psi(\zeta)), \\
{}^{c}\mathcal{D}_0^{\omega(\eta)}\psi(\eta) &= f(\zeta,\psi(\zeta),{}^{c}\mathcal{D}_0^{\omega(\eta)}\psi(\zeta)), \qquad (16)
\end{aligned}$$

*Now by using (**A1**) and equation (16), we have*

$$\begin{aligned}
\mid f(\zeta,\psi(\zeta),{}^{c}\mathcal{D}_0^{\omega(\eta)}\psi(\zeta)) - f(\zeta,\bar{\psi}(\zeta),{}^{c}\mathcal{D}_0^{\omega(\eta)}\bar{\psi}(\zeta)) \mid &\le \text{Ł}_m \mid \psi(\zeta)-\bar{\psi}(\zeta) \\
&+ \text{Ł}_n \mid {}^{c}\mathcal{D}_0^{\omega(\eta)}\psi(\zeta) - {}^{c}\mathcal{D}_0^{\omega(\eta)}\bar{\psi}(\zeta) \mid, \\
\mid f(\zeta,\psi(\zeta),{}^{c}\mathcal{D}_0^{\omega(\eta)}\psi(\zeta)) - f(\bar{\zeta},\bar{\psi}(\zeta),{}^{c}\mathcal{D}_0^{\omega(\eta)}\bar{\psi}(\zeta)) \mid &- \text{Ł}_n \mid {}^{c}\mathcal{D}_0^{\omega(\eta)}\psi(\zeta) \\
&- {}^{c}\mathcal{D}_0^{\omega(\eta)}\bar{\psi}(\zeta) \mid \\
&\le \text{Ł}_m \mid \psi(\zeta)-\bar{\psi}(\zeta) \mid, \\
\mid f(\zeta,\psi(\zeta),{}^{c}\mathcal{D}_0^{\omega(\eta)}\psi(\zeta)) - f(\bar{\zeta},\bar{\psi}(\zeta),{}^{c}\mathcal{D}_0^{\omega(\eta)}\bar{\psi}(\zeta)) \mid &- \text{Ł}_n \mid f(\zeta,\psi(\zeta)){}^{c}\mathcal{D}_0^{\omega(\eta)}\psi(\zeta) \\
&- f(\bar{\zeta},\bar{\psi}(\zeta){}^{c}\mathcal{D}_0^{\omega(\eta)}\bar{\psi}(\zeta) \mid \\
&\le \text{Ł}_m \mid \psi(\zeta)-\bar{\psi}(\zeta) \mid, \\
\mid f(\zeta,\psi(\zeta),{}^{c}\mathcal{D}_0^{\omega(\eta)}\psi(\zeta)) - f(\bar{\zeta},\bar{\psi}(\zeta),{}^{c}\mathcal{D}_0^{\omega(\eta)}\bar{\psi}(\zeta)) \mid (1-\text{Ł}_n) &\le \text{Ł}_m \mid \psi(\zeta)-\bar{\psi}(\zeta) \mid, \\
\mid f(\zeta,\psi(\zeta),{}^{c}\mathcal{D}_0^{\omega(\eta)}\psi(\zeta)) - f(\bar{\zeta},\bar{\psi}(\zeta),{}^{c}\mathcal{D}_0^{\omega(\eta)}\bar{\psi}(\zeta)) \mid &\le \text{Ł}_c \mid \psi(\zeta)-\bar{\psi}(\zeta) \mid, \qquad (17)
\end{aligned}$$

where

$$\text{Ł}_c = \frac{\text{Ł}_m}{1-\text{Ł}_n}.$$

Theorem 1. *If the assumptions (**A1**) and (**A2**) hold and $\text{Ł}_b < 1$, then problem (3) has a unique solution.*

Proof. Since f is continuous on $\mathcal{J} = [0,1]$ and ${}^{c}\mathcal{D}_{0}^{\omega(\eta)} \in C(\mathcal{J}, \mathcal{R})$. For this let $\psi(\eta), \bar{\psi}(\eta) \in B$, we have

$$
\begin{aligned}
\|\mathcal{T}\psi(\eta) - \mathcal{T}\bar{\psi}(\eta)\| &= \max_{\eta\in\mathcal{J}} \mid \{\psi_0 + \int_0^1 \frac{(1-\zeta)^{\delta-1}}{\Gamma(\delta)} g(\zeta, \psi(\zeta))\mathbf{d}\zeta \\
&+ \frac{1}{\Gamma(\omega(\eta))} \int_0^\eta (\eta-\zeta)^{\omega(\zeta)-1} f(\zeta, \psi(\zeta), {}^{c}\mathcal{D}_0^{\omega(\zeta)}\psi(\zeta))\mathbf{d}\zeta\} \\
&- \{\psi_0 + \int_0^1 \frac{(1-\zeta)^{\delta-1}}{\Gamma(\delta)} g(\zeta, \bar{\psi}(\zeta))\mathbf{d}\zeta \\
&+ \frac{1}{\Gamma(\omega(\eta))} \int_0^\eta (\eta-\zeta)^{\omega(\zeta)-1} f(\zeta, \bar{\psi}(\zeta), {}^{c}\mathcal{D}_0^{\omega(\eta)}\bar{\psi}(\zeta))\mathbf{d}\zeta\} \mid, \\
&= \int_0^1 \frac{(1-\zeta)^{\delta-1}}{\Gamma(\delta)} g(\zeta, \psi(\zeta))\mathbf{d}\zeta - \int_0^1 \frac{(1-\zeta)^{\delta-1}}{\Gamma(\delta)} g(\zeta, \bar{\psi}(\zeta))\mathbf{d}\zeta \\
&+ \frac{1}{\Gamma(\omega(\eta))} \int_0^1 (\eta-\zeta)^{\omega(\zeta)-1} f(\zeta, \psi(\zeta), {}^{c}\mathcal{D}_0^{\omega(\zeta)}\psi(\zeta))\mathbf{d}\zeta \\
&- \frac{1}{\Gamma(\omega(\eta))} \int_0^1 (\eta-\zeta)^{\omega(\zeta)-1} f(\zeta, \bar{\psi}(\zeta), {}^{c}\mathcal{D}_0^{\omega(\zeta)}\bar{\psi}(\zeta))\mathbf{d}\zeta, \\
\|\mathcal{T}\psi(\eta) - \mathcal{T}\bar{\psi}(\eta)\| &= \max_{\eta\in\mathcal{J}}\{\int_0^1 \frac{(1-\zeta)^{\delta-1}}{\Gamma(\delta)} \mid g(\zeta, \psi(\zeta)) - g(\zeta, \bar{\psi}(\zeta))) \mid \mathbf{d}\zeta \\
&+ \frac{1}{\Gamma\omega(\eta)} \int_0^1 (\eta-\zeta)^{\omega(\zeta)-1} \mid f(\zeta, \psi(\zeta), {}^{c}\mathcal{D}_0^{\omega(\zeta)}\psi(\zeta)) \\
&- f(\zeta, \bar{\psi}(\zeta), {}^{c}\mathcal{D}_0^{\omega(\eta)}\bar{\psi}(\zeta) \mid \mathbf{d}\zeta\},
\end{aligned}
$$

Using assumption $(\mathbf{A1})$, $(\mathbf{A2})$ and inequality (17), we have

$$
\begin{aligned}
\|\mathcal{T}\psi(\eta) - \mathcal{T}\bar{\psi}(\eta)\| &\leq \max_{\eta\in\mathcal{J}}[\int_0^1 \frac{(1-\zeta)^{\delta-1}}{\Gamma(\delta)} \L_g \mid \psi(\zeta) - \bar{\psi}(\zeta) \mid \mathbf{d}\zeta \\
&+ \frac{1}{\Gamma(\omega(\eta))} \int_0^\eta \{(\eta-\zeta)^{\omega(\zeta)-1} \L_c \mid \psi(\zeta) - \bar{\psi}(\zeta) \mid \mathbf{d}\zeta\}], \\
&\leq \{\L_g \mid \psi(\zeta) - \bar{\psi}(\zeta) \mid \int_0^1 \frac{(1-\zeta)^{\delta-1}}{\Gamma(\delta)} \mathbf{d}\zeta \\
&+ \frac{1}{\Gamma(\omega(\eta))} \L_c \mid \psi(\zeta) - \bar{\psi}(\zeta) \mid \int_0^\eta (\eta-\zeta)^{\omega(\zeta)-1}\mathbf{d}\zeta\}, \\
&\leq \{\L_g \mid \psi(\zeta) - \bar{\psi}(\zeta) \mid (\frac{(1-\zeta)^{\delta-1}}{\delta\Gamma(\delta)} \mid_0^1) \\
&+ \frac{1}{\Gamma(\omega(\eta))} \L_c \mid \psi(\zeta) - \bar{\psi}(\zeta) \mid (\frac{(\eta-\zeta)^{\omega(\eta)-1}\omega(\eta)}{\omega(\eta)} \mid_0^\eta)\}, \\
&\leq \L_g \mid \psi(\zeta) - \bar{\psi}(\zeta) \mid (\frac{1}{\delta\Gamma(\delta)})
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{1}{\Gamma(\omega(\eta))}\text{Ł}_c \mid \psi(\zeta)-\bar{\psi}(\zeta) \mid \max_{\eta\in\mathcal{J}}\Big(\frac{(\eta)^{\omega(\eta)}}{\omega(\eta)}\Big),\\
&\le \text{Ł}_g \mid \psi(\zeta)-\bar{\psi}(\zeta) \mid \Big(\frac{1}{\Gamma(\delta+1)}\Big)\\
&+ \frac{1}{\Gamma(\omega(1)+1)}\text{Ł}_c \mid \psi(\zeta)-\bar{\psi}(\zeta) \mid,\\
&\le \frac{\text{Ł}_g}{\Gamma(\delta+1)} \mid \psi(\zeta)-\bar{\psi}(\zeta) \mid\\
&+ \frac{1}{\Gamma(\omega(1)+1)}\text{Ł}_c \mid \psi(\zeta)-\bar{\psi}(\zeta) \mid,\\
&\le \frac{\text{Ł}_g}{\Gamma(\delta+1)} \mid \psi(\zeta)-\bar{\psi}(\zeta) \mid\\
&+ \frac{1}{\Gamma(\omega(1)+1)}\text{Ł}_c \mid \psi(\zeta)-\bar{\psi}(\zeta) \mid,\\
&\le \Big(\frac{\text{Ł}_g}{\Gamma(\delta+1)} + \frac{1}{\Gamma(\omega(1)+1)}\text{Ł}_c\Big) \mid \psi(\zeta)-\bar{\psi}(\zeta) \mid,\\
&\le \Big(\frac{\text{Ł}_g}{\Gamma(\delta+1)} + \frac{1}{\Gamma(\omega(1)+1)}\text{Ł}_c\Big) \mid \psi(\zeta)-\bar{\psi}(\zeta) \mid, \qquad (18)
\end{aligned}
$$

where $\text{Ł}_c = \frac{\text{Ł}_m}{1-\text{Ł}_n}$ and $\text{Ł}_b = \frac{\text{Ł}_g}{\Gamma(\sigma+1)} + \frac{\text{Ł}_c}{\Gamma(\omega(\eta)+1)}$. Which proves that $\mathcal{T}$ is a contraction mapping. So by BCP, problem (3) has a unique solution. □

Theorem 2. *Consider (**A1**)–(**A1**) hold, then BVP (3) has at least one solution.*

Proof. To show that $\mathcal{T}$ satisfies the assumption of SFPT.
Let us consider a closed convex subset $E = \{\psi \in C(\mathcal{J},\mathcal{R})\ , \|\psi\| \le \zeta\}ofC(\mathcal{J},\mathcal{R})$

- **Step 1:**
  To show that $\mathcal{T}$ is continuous.
  Let us consider a sequence $\{\psi_n\}$, such that $\psi_n \to \psi$ in $C(\mathcal{J},\mathcal{R})$ and let $P > 0$ such that for each $\eta \in \mathcal{J}$, we have

$$
\begin{aligned}
\|\mathcal{T}\psi_n(\eta) - \mathcal{T}\psi(\eta) &= \max_{\eta\in\mathcal{J}}\Big\{\int_0^1 \frac{(1-\zeta)^{\delta-1}}{\Gamma(\delta)} \mid g(\zeta,\psi_n(\zeta)) - g(\zeta,\psi(\zeta)) \mid \mathbf{d}\zeta\\
&+ \frac{1}{\Gamma(\omega(\eta))}\int_0^{\eta}(\eta-\zeta)^{\omega(\zeta)-1} \mid f(\zeta,\psi_n(\zeta),{}^{c}\mathcal{D}_0^{\omega(\zeta)}\psi_n(\zeta))\\
&- f(\zeta,\psi(\zeta),{}^{c}\mathcal{D}_0^{\omega(\zeta)}\psi(\zeta) \mid \mathbf{d}\zeta\Big\}. \qquad (19)
\end{aligned}
$$

By assumptions (**A1**), (**A2**) and inequality (17), we have

$$
\begin{aligned}
\|\mathcal{T}\psi_n(\eta) - \mathcal{T}\psi(\eta)\| &\le \max\Big\{\int_0^1 \frac{(1-\zeta)^{\delta-1}}{\Gamma(\delta)}\text{Ł}_g \mid \psi_n(\zeta) - \psi(\zeta) \mid \mathbf{d}\zeta\\
&+ \frac{1}{\Gamma(\omega(\eta))}\int_0^{\eta}(\eta-\zeta)^{\omega(\zeta)-1}(\text{Ł}_c \mid \psi_n(\zeta)-\psi(\zeta) \mid)\mathbf{d}\zeta\Big\},
\end{aligned}
$$

$$\begin{aligned}
\|\mathcal{T}\psi_n(\eta)-\mathcal{T}\psi(\eta)\| &\leq \{\L_g \mid \psi_n(\zeta)-\psi(\zeta)\mid \int_0^1 \frac{(1-\zeta)^{\delta-1}}{\Gamma(\delta)}\mathbf{d}\zeta \\
&+ \frac{1}{\Gamma(\omega(\eta))}\L_c \mid \psi_n(\zeta)-\psi(\zeta)\mid \int_0^1 (\eta-\zeta)^{\omega(\zeta)-1}\mathbf{d}\zeta\}, \\
&\leq \max\{\L_g \mid \psi_n(\zeta)-\psi(\zeta)\mid (\frac{(1-\zeta)^{\delta-1}}{\Gamma(\delta)} \mid_0^1) \\
&+ \frac{1}{\Gamma(\omega(\eta))}\L_c \mid \psi_n(\zeta)-\psi(\zeta)\mid (\frac{(\eta-\zeta)^{\omega(\eta)}}{\omega(\eta)} \mid_0^\eta)\}, \\
&\leq \L_g \mid \psi_n(\zeta)-\psi(\zeta)\mid (\frac{1}{\Gamma(\delta)}) \\
&+ \frac{1}{\Gamma(\omega(\eta)+1)}\L_c \mid \psi_n(\zeta)-\psi(\zeta)\mid, \\
&\leq (\frac{\L_g}{\Gamma(\delta+1)}+\frac{1}{\Gamma(\omega(\eta)+1)}\L_c) \mid \psi_n(\zeta)-\psi(\zeta)\mid, \qquad (20)
\end{aligned}$$

$\|\mathcal{T}\psi_n-\mathcal{T}\psi\| \to 0$, this implies that $\psi_n \to \psi$,therefore $\mathcal{T}$ is a continuous linear operator.

- **Step 2:**
  The image of a bounded set under $\mathcal{T}$ is bounded.
  Let $\psi \in E$, we have to show that $\mathcal{T}(\psi) \in E$, for each $\eta \in \mathcal{J}$,

$$\begin{aligned}
\|\mathcal{T}\psi(\eta)\| &\leq \max_{\eta\in\mathcal{J}}\{\mid \psi_0 + \int_0^1 \frac{(1-\zeta)^{\delta-1}}{\Gamma(\delta)} g(\zeta,\psi(\zeta))\mathbf{d}\zeta \\
&+ \frac{1}{\Gamma(\omega(\eta))}\int_0^\eta (\eta-\zeta)^{\omega(\zeta)-1} f(\zeta,\psi(\zeta),{}^{c}\mathcal{D}_0^{\omega(\zeta)}\psi(\zeta))\mathbf{d}\zeta \mid\}, \qquad (21) \\
&\leq \max_{\eta\in\mathcal{J}}\{\mid \psi_0 \mid + \int_0^1 \frac{(1-\zeta)^{\delta-1}}{\Gamma(\delta)} \mid g(\zeta,\psi(\zeta)) \mid \mathbf{d}\zeta \\
&+ \frac{1}{\Gamma(\omega(\eta))}\int_0^\eta (\eta-\zeta)^{\omega(\zeta)-1} \mid f(\zeta,\psi(\zeta),{}^{c}\mathcal{D}_0^{\omega(\zeta)}\psi(\zeta)) \mid \mathbf{d}\zeta\}, \qquad (22)
\end{aligned}$$

by assumption (**A3**), let $K = \int_0^1 \mid g(\zeta,\psi(\zeta)) \mid \mathbf{d}\zeta$, then we have

$$\begin{aligned}
&\leq \max_{\eta\in\mathcal{J}}\{\mid \psi_0 \mid + K\int_0^1 \frac{(1-\zeta)^{\delta-1}}{\Gamma(\delta)}\mathbf{d}\zeta \\
&+ \frac{1}{\Gamma(\omega(\eta))}\int_0^\eta (\eta-\zeta)^{\omega(\zeta)-1} h(\zeta)\,\psi \mid^{c}\mathcal{D}_0^{\omega(\zeta)}\psi(\zeta)) \mid \mathbf{d}\zeta\}, \\
&\leq \mid \psi_0 \mid + K\frac{1}{\Gamma(\delta+1)} + \frac{1}{\Gamma(\omega(1))} h^*\,\psi\|\psi\| \frac{1}{\omega(1)},
\end{aligned}$$

Where $h^* = Sup\{(h(\zeta), \zeta \in \mathcal{J})\}$.

$$\begin{aligned} \|\mathcal{T}\psi(\eta)\| &\leq \frac{K}{\Gamma(\delta+1)} + \frac{h^*\,\psi(\zeta)}{\Gamma(\omega(1)+1)}, \\ \|\mathcal{T}\psi(\eta)\| &\leq \ \zeta, \end{aligned} \tag{23}$$

which show that

$$\mathcal{T}(E) \subseteq E.$$

- **Step 3:**
  To Show that $\mathcal{T}$ is equi-continuous on $C[\mathcal{J},\mathcal{R}]$.
  Let $\eta_1, \eta_2 \in \mathcal{J}$, $\eta_1 < \eta_2$ and $\psi \in E$, then

$$\begin{aligned} \|\mathcal{T}\psi(\eta_2) - \mathcal{T}\psi(\eta_1)\| \leq & \max_{\eta\in\mathcal{J}} \{\psi_0 + \int_0^1 \frac{(1-\zeta)^{\delta-1}}{\Gamma(\delta)} \mid g(\zeta,\psi(\zeta)) \mid \mathbf{d}\zeta \\ & + \frac{1}{\Gamma(\omega(\eta_2))} \int_0^{\eta_2} (\eta_2-\zeta)^{\omega(\zeta)-1} \mid f(\zeta,\psi(\zeta),{}^c\mathcal{D}_0^{\omega(\zeta)}\psi(\zeta)) \mid \mathbf{d}\zeta \\ & - \psi_0 - \int_0^1 \frac{(1-\zeta)^{\delta-1}}{\Gamma(\delta)} \mid g(\zeta,\psi(\zeta)) \mid \mathbf{d}\zeta \\ & - \frac{1}{\Gamma(\omega(\eta_1))} \int_0^{\eta_1} (\eta_1-\zeta)^{\omega(\zeta)-1} \mid f(\zeta,\psi(\zeta),{}^c\mathcal{D}_0^{\omega(\zeta)}\psi(\zeta)) \mid \mathbf{d}\zeta\,\}, \end{aligned} \tag{24}$$

$$\begin{aligned} & \leq \max_{\eta\in\mathcal{J}} \{ \int_0^1 \frac{(1-\zeta)^{\delta-1}}{\Gamma(\delta)} \mid g(\zeta,\psi(\zeta)) \mid \mathbf{d}\zeta \\ & - \int_0^1 \frac{(1-\zeta)^{\delta-1}}{\Gamma(\delta)} \mid g(\zeta,\psi(\zeta)) \mid \mathbf{d}\zeta \\ & + \frac{1}{\Gamma(\omega(\eta_2))} \int_0^{\eta_2} (\eta_2-\zeta)^{\omega(\zeta)-1} \mid f(\zeta,\psi(\zeta),{}^c\mathcal{D}_0^{\omega(\zeta)}\psi(\zeta)) \mid \mathbf{d}\zeta \\ & - \frac{1}{\Gamma(\omega(\eta_1))} \int_0^{\eta_1} (\eta_1-\zeta)^{\omega(\zeta)-1} \mid f(\zeta,\psi(\zeta),{}^c\mathcal{D}_0^{\omega(\zeta)}\psi(\zeta)) \mid \mathbf{d}\zeta\,\}, \\ & \leq \max_{\eta\in\mathcal{J}} \{ \frac{1}{\Gamma\omega(\eta_2)} \int_0^{\eta_2} (\eta_2-\zeta)^{\omega(\zeta)-1} \mid f(\zeta,\psi(\zeta),{}^c\mathcal{D}_0^{\omega(\zeta)}\psi(\zeta)) \mid \mathbf{d}\zeta \\ & - \frac{1}{\Gamma\omega(\eta_1)} \int_0^{\eta_1} (\eta_1-\zeta)^{\omega(\zeta)-1} \mid f(\zeta,\psi(\zeta),{}^c\mathcal{D}_0^{\omega(\zeta)}\psi(\zeta)) \mid \mathbf{d}\zeta\,\}, \\ & \leq \{ \frac{1}{\Gamma\omega(\eta_2)} \int_0^{\eta_1} (\eta_2-\zeta)^{\omega(\zeta)-1} \mid f(\zeta,\psi(\zeta),{}^c\mathcal{D}_0^{\omega(1)}\psi(\zeta)) \mid \mathbf{d}\zeta \\ & + \int_{\eta_1}^{\eta_2} (\eta_2-\zeta)^{\omega(\zeta)-1} \mid f(\zeta,\psi(\zeta),{}^c\mathcal{D}_0^{\omega(1)}\psi(\zeta)) \mid \mathbf{d}\zeta \\ & - \frac{1}{\Gamma\omega(\eta_1)} \int_0^{\eta_1} (\eta_1-\zeta)^{\omega(\zeta)-1} \mid f(\zeta,\psi(\zeta),{}^c\mathcal{D}_0^{\omega(1)}\psi(\zeta)) \mid \mathbf{d}\zeta\,\}, \\ & \leq \mid f(\zeta,\psi(\zeta),{}^c\mathcal{D}_0^{\omega(1)}\psi(\zeta) \mid \{ \frac{1}{\Gamma\omega(\eta_2)} \int_0^{\eta_1} (\eta_2-\zeta)^{\omega(\zeta)-1} \mathbf{d}\zeta \end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{\Gamma\omega(\eta_1)} \int_0^{\eta_1} (\eta_1-\zeta)^{\omega(\zeta)-1} \mathbf{d}\zeta \\
&+ \frac{1}{\Gamma\omega(\eta_2)} \int_{\eta_1}^{\eta_2} (\eta_2-\zeta)^{\omega(\zeta)-1} \mid f(\zeta,\psi(\zeta),{}^c\mathcal{D}_0^{\omega(1)}\psi(\zeta)) \mid \mathbf{d}\zeta \}, \\
&\le \mid f(\zeta,\psi(\zeta),{}^c\mathcal{D}_0^{\omega(1)}\psi(\zeta) \mid [\frac{1}{\Gamma\omega(\eta_2)+1}\{-(\eta_2-\zeta)^{\omega(\eta_2)}\} \mid_0^{\eta_2} \\
&- \frac{1}{\Gamma\omega(\eta_1)+1}\{-(\eta_1-\zeta)^{\omega(\eta_1)}\} \mid_0^{\eta_1} + \frac{1}{\Gamma\omega(\eta_2)+1}\{-(\eta_2-\zeta)^{\omega(\eta_2)}\} \mid_{\eta_1}^{\eta_2}], \\
&\le \mid f(\zeta,\psi(\zeta),{}^c\mathcal{D}_0^{\omega(1)}\psi(\zeta) \mid \frac{1}{\Gamma(\omega(\eta_2)+1)}(\eta_2)^{\omega(\eta_2)} \\
&- \frac{{\eta_1}^{\omega(\eta_0)}}{\Gamma(\omega(\eta_1)+1)} + \frac{(\eta_2-\eta_1)^{\omega(\eta_1)}}{\Gamma(\omega(\eta_2)+1)}, \\
\|\mathcal{T}\psi(\eta_2)-\mathcal{T}\psi(\eta_1)\| &\le \mid f(\zeta,\psi(\zeta),{}^c\mathcal{D}_0^{\omega(1)}\psi(\zeta) \mid \{\frac{(\eta_2)^{\omega(\eta_2)}}{\Gamma(\omega(\eta_2)+1)}, \\
&- \frac{(\eta_1)^{\omega(\eta_1)}}{\Gamma(\omega(\eta_1)+1)} + \frac{(\eta_2-\eta_1)^{\omega(\eta_1)}}{\Gamma(\omega(\eta_2)+1)}\}, \qquad (25)
\end{aligned}
$$

As $\eta_2 \to \eta_1$, then $\|\mathcal{T}\psi(\eta_2)-\mathcal{T}\psi(\eta_1)\| \to 0$
Step 1-3 implies that $\mathcal{T}$ satisfy all the condition of $Arzela'r - Aseoli, Theorem$.
Hence $\mathcal{T}$ is completely continuous.
So by SFPT $\mathcal{T}$ has a fixed point $\psi \in E$, which is a solution of BVP (3).

□

4 Examples

In this section, the validity of the proposed approach in Section 3 is verified via two numerical examples.
Example 2. *Consider a Caputo fractional differential equation with boundary conditions as*

$$
\begin{aligned}
{}^c\mathcal{D}_0^{\frac{\eta^2}{2}}\psi(\eta) =& \eta^2\Big(\frac{\psi(\eta)}{13+|\psi(\eta)|} + \frac{{}^c\mathcal{D}_0^{\frac{\eta^2}{2}}\psi(\eta)}{17+|{}^c\mathcal{D}_0^{\frac{\eta^2}{2}}\psi(\eta)|}\Big), \eta \in [0,1) \\
\psi(0) =& \psi_0 + \int_0^1 \frac{(1-\eta)^{\frac{1}{2}}}{\Gamma(\frac{1}{2})}\frac{\psi(\eta)}{11+|\psi(\eta)|}.
\end{aligned}
\qquad (26)
$$

Clearly, $\omega(\eta) = \frac{\eta^2}{2}$, $\delta = \frac{1}{2}$ *and* $f(\eta,\psi(\eta),{}^c\mathcal{D}_0^{\frac{\eta^2}{2}}\psi(\eta)) = \eta^2(\frac{\psi(\eta)}{13+|\psi(\eta)|} + \frac{{}^c\mathcal{D}_0^{\frac{\eta^2}{2}}\psi(\eta)}{17+|{}^c\mathcal{D}_0^{\frac{\eta^2}{2}}\psi(\eta)|})$ *and* $g(\eta,\psi(\eta)) = \frac{\psi(\eta)}{11+|\psi(\eta)|}$.

Let $\psi, \bar{\psi} \in [\mathcal{J}, \mathcal{R}]$*, one has*

$$\begin{aligned}|f(\eta,\psi(\eta),{}^c\mathcal{D}_0^{\frac{\eta^2}{2}}\psi(\eta)) - f(\eta,\bar{\psi}(\eta),{}^c\mathcal{D}_0^{\frac{\eta^2}{2}}\bar{\psi}(\eta))| &\le |\eta^2||(\frac{\psi(\eta)}{13+|\psi(\eta)|} + \frac{{}^c\mathcal{D}_0^{\frac{\eta^2}{2}}\psi(\eta)}{17+|{}^c\mathcal{D}_0^{\frac{\eta^2}{2}}\psi(\eta)|} \\ &\quad - \frac{\bar{\psi}(\eta)}{13+|\bar{\psi}(\eta)|} - \frac{{}^c\mathcal{D}_0^{\frac{\eta^2}{2}}\bar{\psi}(\eta)}{17+|{}^c\mathcal{D}_0^{\frac{\eta^2}{2}}\bar{\psi}|})| \qquad (27)\end{aligned}$$

$$\begin{aligned}&\le |\eta^2|((|\frac{\psi(\eta)}{13+|\psi(\eta)|} - \frac{\bar{\psi}(\eta)}{13+|\bar{\psi}(\eta)|}| + |\frac{{}^c\mathcal{D}_0^{\frac{\eta^2}{2}}\psi(\eta)}{17+|{}^c\mathcal{D}_0^{\frac{\eta^2}{2}}\psi(\eta)|} - \frac{{}^c\mathcal{D}_0^{\frac{\eta^2}{2}}\bar{\psi}(\eta)}{17+|{}^c\mathcal{D}_0^{\frac{\eta^2}{2}}\bar{\psi}(\eta)|}|),\\ &\le |\frac{\psi(\eta)}{13} - \frac{\bar{\psi}(\eta)}{13}| + |\frac{{}^c\mathcal{D}_0^{\frac{1}{2}}\psi(\eta)}{17} - \frac{{}^c\mathcal{D}_0^{\frac{1}{2}}\bar{\psi}(\eta)}{17}|,\\ &\le \frac{1}{13}|\psi(\eta) - \bar{\psi}(\eta)| + \frac{1}{17}|{}^c\mathcal{D}_0^{\frac{1}{2}}\psi(\eta) -{}^c\mathcal{D}_0^{\frac{1}{2}}\bar{\psi}(\eta)|,\end{aligned}$$

and

$$\begin{aligned}|g(\eta,\psi(\eta) - g(\eta,\bar{\psi}(\eta)| &\le |\frac{\psi(\eta)}{11+|\psi(\eta)|} - \frac{\bar{\psi}(\eta)}{11+|\bar{\psi}(\eta)|}|,\\ &\le |\frac{\psi(\eta)}{11} - \frac{\bar{\psi}(\eta)}{11}|,\\ &\le \frac{1}{11}|\psi(\eta) - \bar{\psi}(\eta)|, \qquad (28)\end{aligned}$$

Here $L_m = \frac{1}{13}$, $L_n = \frac{1}{17}$ *and* $L_g = \frac{1}{11}$. *Hence,*

$$L_b \approx \frac{2}{11\sqrt{\pi}} + \frac{17}{208\ \Gamma(1+\frac{\eta^2}{2})} < 1,$$

for any value of $\eta \in [0,1)$.
Now, if we choose $\eta = 0.4,\ 0.6,\ and\, 0.8$*, the values of* L_b *are* 0.187741, 0.191059, *and* 0.193936 *respectively. So by Theorem* (1)*, problem* (26) *has unique solution.*

Example 3. *Consider a Caputo fractional differential equation with boundary conditions as*

$$\begin{aligned}{}^c\mathcal{D}_0^{\eta^3}\psi(\eta) &= \eta^2(\frac{\psi(\eta)}{15+|\psi(\eta)|} + \frac{{}^c\mathcal{D}_0^{\eta^3}\psi(\eta)}{17+|{}^c\mathcal{D}_0^{\eta^3}\psi(\eta)|}), \eta \in [0,1)\\ \psi(0) &= \psi_0 + \int_0^1 \frac{(1-\eta)^{\frac{1}{2}}}{\Gamma(\frac{1}{3})}\frac{\psi(\eta)}{13+|\psi(\eta)|}.\end{aligned} \qquad (29)$$

Clearly, $\omega(\eta)=\eta^3$, $\delta=\frac{1}{3}$ *and* $f(\eta,\psi(\eta),{}^c\mathcal{D}_0^{\eta^3}\psi(\eta))=\eta^2(\frac{\psi(\eta)}{15+|\psi(\eta)|}+\frac{{}^c\mathcal{D}_0^{\eta^3}\psi(\eta)}{17+|{}^c\mathcal{D}_0^{\eta^3}\psi(\eta)|})$ *and* $g(\eta,\psi(\eta))=\frac{\psi(\eta)}{11+|\psi(\eta)|}$.
Let $\psi,\bar{\psi}\in[\mathcal{J},\mathcal{R}]$, *then*

$$
\begin{aligned}
|f(\eta,\psi(\eta),{}^c\mathcal{D}_0^{\eta^3}\psi(\eta))-f(\eta,\bar{\psi}(\eta),{}^c\mathcal{D}_0^{\eta^3}\bar{\psi}(\eta))| \leq |\eta^2|\ \ |(\frac{\psi(\eta)}{15+|\psi(\eta)|}&+\frac{{}^c\mathcal{D}_0^{\eta^3}\psi(\eta)}{17+|{}^c\mathcal{D}_0^{\eta^3}\psi(\eta)|}\\
&-\frac{\bar{\psi}(\eta)}{15+|\bar{\psi}(\eta)|}-\frac{{}^c\mathcal{D}_0^{\eta^3}\bar{\psi}(\eta)}{17+|{}^c\mathcal{D}_0^{\eta^3}\bar{\psi}|})|,
\end{aligned}
$$

$$
\begin{aligned}
&\leq|\eta^2|(|\frac{\psi(\eta)}{15+|\psi(\eta)|}-\frac{\bar{\psi}(\eta)}{15+|\bar{\psi}(\eta)|}|+|\frac{{}^c\mathcal{D}_0^{\eta^3}\psi(\eta)}{17+|{}^c\mathcal{D}_0^{\eta^3}\psi(\eta)|}-\frac{{}^c\mathcal{D}_0^{\eta^3}\bar{\psi}(\eta)}{17+|{}^c\mathcal{D}_0^{\eta^3}\bar{\psi}(\eta)|}|),\\
&\leq|\frac{\psi(\eta)}{15}-\frac{\bar{\psi}(\eta)}{15}|+|\frac{{}^c\mathcal{D}_0^1\psi(\eta)}{17}-\frac{{}^c\mathcal{D}_0^1\bar{\psi}(\eta)}{17}|,\\
&\leq\frac{1}{15}|\psi(\eta)-\bar{\psi}(\eta)|+\frac{1}{17}|{}^c\mathcal{D}_0^1\psi(\eta)-{}^c\mathcal{D}_0^1\bar{\psi}(\eta)|,
\end{aligned}
$$

and

$$
\begin{aligned}
|g(\eta,\psi(\eta)-g(\eta,\bar{\psi}(\eta)|&\leq|\frac{\psi(\eta)}{13+|\psi(\eta)|}-\frac{\bar{\psi}(\eta)}{13+|\bar{\psi}(\eta)|}|,\\
&\leq|\frac{\psi(\eta)}{13}-\frac{\bar{\psi}(\eta)}{13}|,\\
&\leq\frac{1}{13}|\psi(\eta)-\bar{\psi}(\eta)|, \qquad (30)
\end{aligned}
$$

Here $\L_m=\frac{1}{15}$, $\L_n=\frac{1}{17}$ *and* $\L_g=\frac{1}{13}$. *Hence*

$$
\L_b\approx\frac{1}{13\,\Gamma(\frac{4}{3})}+\frac{17}{240\ \ \Gamma(1\ +\ \eta^3)}<1,
$$

for any value of $\eta\in[0,1)$.
Now, if we choose $\eta=0.4,\ 0.6,$ *and* 0.8, *the value of* $\L_b$ *are* 0.159401, 0.163633, *and* 0.166029 *respectively, which are less than 1. Hence by Theorem* (1), *problem* (26) *has unique solution.*

5 Conclusion

In this paper, the existence of solutions for variable-order FDEs with IBCs in the Caputo derivative sense is established. Through rigorous analysis guided by specified assumptions (A1–A4) and leveraging foundational theorems such as the BCP and the

SFPT, the reliability and robustness of the proposed analytical approach are demonstrated. Numerical examples further validate the findings and underscore the efficacy of the proposed methods. The work contributes to the advancement of understanding in variable-order FDEs and provides insights for both theoretical developments and practical applications. Future research avenues can explore extensions to more complex systems, investigate alternative numerical methods, and explore applications across diverse scientific and engineering disciplines.

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