Exact Solutions of Some Nonlinear Partial Differential Equations Using Homotopy Perturbation Method Linked with Laplace Transforms and the Pade's Approximants

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Abstract: **In this paper, obtaining exact solution of non linear partial differential equations by using Homotopy Perturbation method via Laplace transform and Pades Approximants. Numerical examples including Cauchy problem and non linear gas dynamics equation are given to show that the effectiveness of HPM-Lap-Pade's technique.**
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1. INTRODUCTION

In the recent years, much attention has been devoted to the newly developed methods to construct the analytic solution of first order non-linear partial differential equations which arise in gas dynamics, water waves, elastic dynamics, chemical reactions, transport of pollutants, flood waves in rivers, traffic flow and a wide range of biological and ecological systems. The non linear partial differential equations are generally difficult to solve and difficult to find their exact solution.

In this paper, the closed form of exact solution of non-linear partial differential equation is obtained by applying Laplace transform with pade's approximants on the second order truncated solution series obtained by Homotopy perturbation method.

The numerical solution of first order nonlinear partial differential equation has been given considerable attention in recent years by introducing various methods and techniques. For example D.J.Evans et.al [1] used Decomposition method to investigate the solution of Gas Dynamics equation. This method provides the solution in a rapidly convergent series where the series may lead to the exact solution if it exists. Y.Keskin et.al [6] introduced Reduced Differential Transform method to find the approximate analytic solution to non linear partial differential equations.

2. BASIC IDEA OF HOMOTOPY PERTURBATION METHOD

The Homotopy perturbation method is powerful and efficient technique to find the solution of linear and nonlinear differential equations. This method was first introduced by He ([2], [3]).The method which is coupling of traditional perturbation and homotopy in topology deforms continuously to a simple problem.

The Homotopy perturbation method has been successfully applied to solve nonlinear Gas Dynamics equation [5], quadratic Riccati differential equation of fractional order [9], nonlinear heat conduction problems [7], Bratu type problems [8].

The basic idea of this method is the following. We consider the non linear differential equation

$$A(u) - f(r) = 0 \qquad r \in \Omega \qquad (1)$$

With boundary conditions

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \qquad r \in \Gamma \qquad (2)$$

Where A is the general differential operator, B is the boundary operator and $f(r)$ is a known analytic function. Γ is the boundary of the domain Ω

The operator A can be generally divided in to two parts L and N, where L is linear, whereas N is non linear. Therefore, Eq. (1) can be rewritten as follows:

$$L(u) + N(u) - f(r) = 0 \qquad r \in \Omega \qquad (3)$$

Using homotopy technique, we construct the homotopy as:

$v(r,p): \Omega\, X\, [0,1] \rightarrow \Re$ which satisfies:

$$H(v,p) = (1-p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0, \quad r \in \Omega \qquad (4)$$

Where $p \in [0,1]$ is an embedding parameter and u_0 is the first approximation that satisfy the boundary conditions. We can write this homotopy equation as follows:

$$H(v,p) = L(v) - L(u\_0\,) + pL(u\_0\,) + p(N(v) - f(r)\,) = 0, \quad r \in \Omega \qquad (5)$$

Obviously from Eqns. (4) and (5), we have

$$H(v,o) = L(u) - L(u_0) = 0 \qquad (6)$$

$$H(v,1) = L(u) + N(u) - f(r) = 0 \qquad (7)$$

The embedded parameter p monotonically changes from zero to unity as the trivial problem $L(v) - L(u_0)$ is continuously deformed to the original problem $A(u) - f$. Due to the fact that $p \in [0,1]$can be considered as a small parameter, hence we consider the solution of Eqn.(4) as a power series in p as the following:

$$v = v_0 + pv_1 + p^2 v_2 + \cdots \quad (8)$$

Setting $p = 1$ results in the approximate solution for Eqn.(1)

$$u(x) = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \cdots \quad (9)$$

The series is convergent for most of the cases and rate of convergence depend on $A(v)$,(see [2])

3. NUMERICAL EXAMPLES

Example 3.1 We consider the Cauchy problem for porous medium equation with source term, which is the simple model for a nonlinear heat propagation in reactive medium.

$$\frac{\partial u}{\partial t} = a\frac{\partial}{\partial x}\left(u^{-2}\frac{\partial u}{\partial x}\right) + bu \quad (10)$$

With the initial condition $u(x,0) = x^{-1}$ (11)

To solve the system (10) by HPM, we construct the homotopy as

$$\frac{\partial v}{\partial t} - \frac{\partial u_0}{\partial t} = p\left[a\frac{\partial}{\partial x}\left(v^{-2}\frac{\partial v}{\partial x}\right) + bv - \frac{\partial u_0}{\partial t}\right] \quad (12)$$

Substituting the Eqn.(8) in to the Eqn.(14) and equating the coefficient of p, we get the following:

$$p^0 : \frac{\partial u_0}{\partial t} - \frac{\partial u_0}{\partial t} = 0 \qquad u_0\,(x,0) = x^{-1} \quad (13)$$

$$p^1 : \frac{\partial u_1}{\partial t} = a\frac{\partial}{\partial x}\left(u_0^{-2}\frac{\partial u_0}{\partial x}\right) + bu_0 - x^{-1} \qquad u_1\,(x,0) = 0 \quad (14)$$

$$p^2 : \frac{\partial u_2}{\partial t} = a\frac{\partial}{\partial x}\left(-2u_1u_0^{-3}\,\frac{\partial u_0}{\partial x} + u_0^{-2}\frac{\partial u_1}{\partial x}\right) + bu_1 \qquad u_2(x,0) = 0 \quad (15)$$

If we solve the above equations $u_0, u_1$and u_2, we get the following results:

$$u_0 = x^{-1} \quad (16)$$

$$u_1 = btx^{-1} \quad (17)$$

$$u_2 = b\frac{t^2}{2}x^{-1} \quad (18)$$

Substituting the Eqns.(16), (17) and (18) in to the Eqn.(9) and truncate the series solution by second order perturbation ie. The approximate solution is:

$$u(x,t) = \sum_{n=0}^{2} u_n\,(x,t) = u_0 + u_1 + u_2 = x^{-1}\left(1 + bt + \frac{bt^2}{2}\right) \quad (19)$$

Taking the Laplace transform of the Eqn. (19)

$$L[u(x,t)] = x^{-1}\left(\frac{1}{s} + \frac{b}{s^2} + \frac{b}{s^3}\right) \quad (20)$$

For the sake of simplicity, let $s = \frac{1}{t}$ then

$$L[u(x,t)] = x^{-1}(t + bt^2 + bt^3) \quad (21)$$

Its $\left[\frac{L}{M}\right]$ pades approximant with $L \geq 1$ and $M \geq 1$yields

$$\left[\frac{L}{M}\right] = x^{-1}\left(\frac{t}{1-bt}\right) \quad (22)$$

Replace $t = \frac{1}{s}$, we obtain $\left[\frac{L}{M}\right]$ in terms of s as follows: $\left[\frac{L}{M}\right] = x^{-1}\left(\frac{1}{s-b}\right)$

By using the inverse Laplace transform to $\left[\frac{L}{M}\right]$, we obtain the exact solution:

$$u(x,t) = x^{-1}e^{bt} \quad (23)$$

Example 3.2 we consider the non linear homogeneous gas dynamics equation

$$\frac{\partial u}{\partial t} + \frac{1}{2}(u^2)_x - u(1-u) = 0, \qquad 0 \leq x \leq 1, t > 0 \quad (24)$$

With initial condition

$$u(x,0) = e^{-x} \quad (25)$$

We construct a homotopy as follows:

$$\frac{\partial v}{\partial t} - \frac{\partial u_0}{\partial t} = p\left[v - v^2 - \frac{1}{2}(v^2)_x - \frac{\partial u_0}{\partial t}\right] \quad (26)$$

Substituting the Eqn.(8) in to the Eqn.(26) and equating the powers of p in both sides, we have

$$p^0 : \frac{\partial u_0}{\partial t} - \frac{\partial u_0}{\partial t} = 0 \qquad u_0\,(x,0) = e^{-x} \quad (27)$$

$$p^1: \frac{\partial u_1}{\partial t} = u_0 - u_0^2 - \frac{1}{2}(u_0^2)_x \qquad u_1(x,0) = 0 \tag{28}$$

$$p^2: \frac{\partial u_2}{\partial t} = u_1 - 2u_0 u_1 - (u_0 u_1)_x \qquad u_2(x,0) = 0 \tag{29}$$

If we solve above equations for u_0, u_1 and u_2, we get the following results:

$$u_0 = e^{-x} \tag{30}$$

$$u_1 = te^{-x} \tag{31}$$

$$u_2 = \frac{t^2}{2} e^{-x} \tag{32}$$

Substituting the Eqns.(30), (31) and (32) in to the Eqn.(9) and truncate the series solution by second order perturbation

$$u(x,t) = \sum_{n=0}^{2} u_n(x,t) = u_0 + u_1 + u_2 = e^{-x}\left(1 + t + \frac{t^2}{2}\right) \tag{33}$$

Taking the Laplace transform of the Eqn. (33)

$$L[u(x,t)] = e^{-x}\left(\frac{1}{s} + \frac{1}{s^2} + \frac{1}{s^3}\right) \tag{34}$$

For the sake of simplicity, let $s = \frac{1}{t}$ then

$$L[u(x,t)] = e^{-x}(t + t^2 + t^3) \tag{35}$$

Its $\left[\frac{L}{M}\right]$ pades approximant with $L \geq 1$ and $M \geq 1$yields

$$\left[\frac{L}{M}\right] = x^{-1}\left(\frac{t}{1-t}\right) \tag{36}$$

Replace $t = \frac{1}{s}$, we obtain $\left[\frac{L}{M}\right]$ in terms of s as follows:

$$\left[\frac{L}{M}\right] = x^{-1}\left(\frac{1}{s-1}\right)$$

By using the inverse Laplace transform to $\left[\frac{L}{M}\right]$, we obtain the exact solution:

$$u(x,t) = e^{-x+t} \tag{37}$$

4. CONCLUSION

HPM-Laplace-Pades approximants enables us to get the exact solution of the presented first order non-linear partial differential equations. One can also apply this technique to other nonlinear problems.

REFERENCE

[1] D.J.Evans, H.Bulut, A new approach to the gas dynamics equation: an application of the decomposition method, International Journal of Computer Mathematics, 79(7), 2002, 817-822.

[2] J.H.He, Homotopy perturbation method, Comput.Methods.Appl.Mech.Engg, 178(1999), 257-262.

[3] J.H.He, Homotopy perturbation method: A new non-linear analytical technique, Appl.Math.Comput. 135(2003), 73-79.

[4] J.H.He, Homotopy perturbation method with two expanding parameters, Indian J.Physics, (88), 2014, 193 – 196.

[5] H.Jafari, C.Chun, S. Seifi, M.Saeidy, Analytical solution of non-linear Gas Dynamics Equation by Homotopy analysis method, Applications and applied mathematics: An International Journal, 4(1), 2009, 149 -154.

[6] Y.Keskin, G.Oturanc: Reduced Differential transform Method for partial differential equations, International Journal of Non Linear Sciences and Numerical Simulation, 10(6),2009, 741 – 749.

[7] T.A.Nofel, Application of the Homotopy perturbation method to non linear heat conduction problems and Fractional Vander Pol Damped Nonlinear Oscillator, Journal of Applied Mathematics, 5(2014), 852-861.

[8] Olusola Ezekiel Abolarin, New improved variational Homotopy perturbation method for Bratu-type problems, American journal of computational Mathematics,3 (2013), 110-113.

[9] [9]. Zaid Odibat, Saher Momani, Modified Homotopy perturbation method: Application to quadratic Riccati ifferential equation with fractional order, Chaos, solitons & fractals, 36(2008), 167-174.