Exact Solution for the Flow of Oldroyd-B Fluid Between Coaxial Cylinders

Vatsala Mathur, Kavita Khandelwal

Department of Mathematics
Malaviya National Institute of Technology, Jaipur-302017

Abstract
This paper deals with unsteady unidirectional transient flow of Oldroyd-B fluid between two infinitely long coaxial circular cylinders. At time $t=0^+$, the motion is produced by a constant pressure gradient & the inner cylinder start moving along its axis of symmetry with the constant velocity. The velocity field of the flow of fluid with fractional derivative is obtained by using Hankel and Laplace transforms. The obtained result is presented in terms of the generalized G-functions. Key difference this paper brings from previous work is that in this case inner cylinder is moving with constant velocity. The influence of different values of parameters, constants and fractional coefficients on the velocity field is also analyzed using graphical illustration.

Keywords - Coaxial cylinders; Fractional calculus; Hankel transform; Laplace transform; Oldroyd-B fluids; Velocity field.

1. Introduction

Fluids are generally classified based on their rheological properties. The simplest classification is Newtonian fluid. These fluids are represented using Navier-Stokes theory. The fluids which do not obey Newton’s law of viscosity ($τ = μ \frac{dv}{dy}$), are described as non-Newtonian fluids. Examples of such fluids are blood, saliva, semen, lava, gums, slurries, emulsions, synovial fluid, butter, cheese, jam, ketchup, soup, mayonnaise etc. These fluids have complex molecular structure with non-linear viscoelastic behavior. To study non-Newtonian fluids, many models have been used. Out of these, differential type [1] and rate type [2] have received most of the attention.

Tong [3] used constitutive relation for the flow of non-Newtonian fluid with fractional derivative in an annular pipe as follows

$$\tau = \mu (1 + \lambda) \frac{dv}{dr}, v(r, t),$$

(1)

where $τ$ is tangential tension, $μ$ is the viscosity, $v$ is the velocity, $λ$ and $λ_r$ are relaxation and retardation times respectively.

The starting point of the fractional derivative model of non-Newtonian fluid is usually a classical differential equation being modified by replacing the time derivative of an integer order by the so-called Riemann–Liouville fractional calculus operator [4]. Using fractional approach, the constitutive relation of the generalized Oldroyd-B fluid can be written as

$$(1 + \lambda D_{t}^{α}) \tau = \mu (1 + \lambda) D_{t}^{β} v(r, t),$$

(2)
where $D^\alpha_t$ and $D^\beta_t$ are fractional operators and are defined as \[5\]

\begin{equation}
D^\alpha_t f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \quad 0 \leq \alpha < 1;
\end{equation}

\begin{equation}
\frac{d}{dt} f(t), \quad \alpha = 1,
\end{equation}

where $\Gamma(.)$ is the Gamma function. When $\alpha = \beta = 1$, eq. (2) simplified as eq. (1).

For non-Newtonian fluids, the first exact solution corresponding to motions of Oldroyd-B fluids in cylindrical domains seem to be those of Waters & King [6]. Fetecau [7] worked on unsteady unidirectional transient flows of an Oldroyd-B fluid in unbounded domains which geometrically are axisymmetric pipe-like. He used the theorem of Steklov to obtain exact solutions for flows satisfying no-slip boundary conditions. The unsteady rotational flow of a generalized second grade fluid through a circular cylinder has been considered by Kamran [8]. Exact solutions for the velocity field and the shear stress corresponding to the unsteady flows of a generalized Oldroyd-B fluid in an infinite circular cylinder subject to a longitudinal time dependent shear stress have been obtained by Rubbab [9]. M. Kamran [10] concluded unsteady linearly accelerating flow of a fractional second grade fluid through a circular cylinder. Various other studies have been done recently on non-Newtonian fluids [11-17]

The aim of this paper is to study the flow of Oldroyd-B fluid with fractional derivative between two coaxial cylinders. The solution is obtained by using Hankel and Laplace transform. At time $t = 0^+$ a constant pressure gradient applied and the inner cylinder moves with constant velocity & the outer cylinder held fixed. The obtained result is presented in terms of the generalized G functions.

\section{2. Governing equations}

We consider the unsteady flow of an incompressible Oldroyd-B fluid in coaxial cylinders. Further, following assumptions are considered during this mathematical study. The fluid velocity at the direction of the pipe radius is assumed to be zero. The flows are assumed to be axisymmetric. The axial velocity is assumed to be only relevant to the cylinder radius. The equation of axial flow motion is

\begin{equation}
\rho \frac{\partial v}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{\mu}{\rho} \frac{\partial v}{\partial r} \right) + \frac{1}{r} \frac{\partial p}{\partial \zeta}, \quad 0 \leq r < \infty.
\end{equation}

where $\rho$ is the constant density of the fluid.

Putting eq. (2) in eq. (4), we get

\begin{equation}
\left( 1 + \lambda D^\alpha_t \right) \frac{\partial v}{\partial t} = A + \lambda \lambda \frac{t^\alpha}{\Gamma(1-\alpha)} + b \left( A + \lambda D^\alpha_t \right) \left( \frac{\partial v}{\partial r} + \frac{1}{r} \frac{\partial p}{\partial \zeta} \right),
\end{equation}

where $\nu = \frac{\mu}{\rho}$ is the kinematical viscosity and $-A \lambda = \frac{\partial p}{\partial \zeta}$ is the constant pressure gradient that acts on the liquid in the $z$-direction.

\section{3. Flow through the annular region}

Let us consider an incompressible Oldroyd-B fluid in infinite coaxial circular cylinders. At time $t = 0$, fluid is assumed to be stationary. At time $t > 0$, a constant pressure gradient applied and the inner cylinder moves with constant velocity and the outer cylinder held fixed. Consider that the radius of inner and outer cylinders are $R_1$ and $R_2 (> R_1)$ respectively.

The initial and boundary conditions are

\begin{equation}
v(r,0) = 0, \quad \frac{\partial v(r,0)}{\partial t} = 0, \quad R_1 \leq r \leq R_2,
\end{equation}

\begin{equation}v(R_1,t) = f, \quad v(R_2,t) = 0, \quad t > 0,
\end{equation}

where $f$ is constant.

Making the change of unknown function

\begin{equation}v(r,t) = V(r) + u(r,t),
\end{equation}

where

\begin{equation}V(r) = \frac{A}{4\nu} \left( R_2^2 - r^2 \right) + \frac{A}{4\nu} \ln(r/R_1) \ln(r/R_2).
\end{equation}

Putting eq. (8) in eq. (5), we obtain
(1 + λD^α) \overset{\partial u(r,t)}{\partial t} = u(1 + λD^α) \left( \overset{\partial}{r} + \frac{1}{r} \right) u(r,t) + \frac{\lambda A}{\Gamma(1 - \beta)} t^{-\beta}.
\Rightarrow -\lambda A \frac{t^{-\beta}}{\Gamma(1 - \beta)}.
(10)

Putting eq. (8) in eqs. (6) & (7), we obtain
\begin{align*}
u(r,0) &= -V(r), \quad \overset{\partial}{\nu}(r,0) = 0, \quad (11) \\
u(R_1,t) &= f, \quad \nu(R_2,t) = 0. \quad (12)
\end{align*}

The Hankel Transform method with respect to r is used and is defined as follows
\begin{equation}
\overline{\nu} = \int_{R_1}^{R_2} ru(r,s)\phi_i(s_n, r)dr.
(13)
\end{equation}

The inverse Hankel Transform is
\begin{equation}
u(r,s) = \frac{\pi^2}{2} \sum_{n=1}^{\infty} \frac{s^2 J_0^2(R_2 s_n) \overline{u}(s_n, s) \phi_i(s_n, r)}{J_0^2(R_1 s_n) - J_0^2(R_2 s_n)},
(14)
\end{equation}
where
\begin{equation}
\phi_i(s_n, r) = Y_0(R_1 s_n) J_0(s_n r) - J_0(R_1 s_n) Y_0(s_n r),
\end{equation}
\begin{equation}
s_n \text{ is the positive root of } \phi_i(s_n, R_2) = 0.
\end{equation}

Applying the Hankel transform in eq. (10), we obtain
\begin{align*}
(1 + \lambda D^\alpha) \overset{\partial \overline{u}(s,t)}{\partial t} &= -\omega s^2 (1 + \lambda D^\alpha) \overline{\nu}(s,t) - \frac{2\nu}{\pi} f \\
-2\nu \lambda f + \frac{\lambda A}{\Gamma(1 - \beta)} t^{-\beta} \\
+ \frac{2\lambda A g(s_n) t^{-\beta}}{\nu s_n^2 \Gamma(1 - \beta)}.
(15)
\end{align*}

where
\begin{equation}
g(s_n) = \frac{R_1 \nu s_n}{2} \{ Y_0(R_1 s_n) J_0(R_2 s_n) - J_0(R_1 s_n) Y_0(R_2 s_n) \} - 1.
\end{equation}

Applying the Hankel transform of eq. (11), we obtain
\begin{align*}
\overline{\nu}(s_n, 0) &= -\frac{2A}{\nu s_n^2 \Gamma(1 - \beta)} \left[ J_0(R_1 s_n) \right] - 1, \quad \overset{\partial}{\nu}(s_n, 0) = 0.
(16)
\end{align*}

Applying Laplace transform of eq. (15) and using eq. (16), we obtain
\begin{align*}
\mathbb{L}\{\nu(s, t)\} &= \left[ -\frac{2A}{\nu s_n^2 \Gamma(1 - \beta)} \left[ J_0(R_1 s_n) \right] - 1 \right] (1 + 2s^\alpha + \nu^2 s^\beta) \\
- 2\nu \lambda f + \frac{\lambda A}{\Gamma(1 - \beta)} s^{-\beta} (1 + 2s^\alpha + \nu^2 s^\beta) \\
+ \frac{2\lambda A g(s_n)}{\nu s_n^2 \Gamma(1 - \beta)} s^{-\beta} (1 + 2s^\alpha + \nu^2 s^\beta).
(17)
\end{align*}

Applying Inverse-Laplace transform of eq. (17) and taking into account the following result [18]
\begin{align*}
G_{a,b,c}(d, t) &= L^{-1}\left\{ \frac{q^b}{(q^a - d)^c} \right\} \\
&= \sum_{j=0}^{\infty} \frac{d^j \Gamma(c+j)}{\Gamma(c) \Gamma(c+j+a-b)} t^{(c+j)a-b-1}; \\
\text{Re}(ac - b) > 0, \quad \left| \frac{d}{q^a} \right| < 1.
(18)
\end{align*}

We obtain
\begin{align*}
\overline{\nu}(s_n, t) &= \left[ -\frac{2A}{\nu s_n^2 \Gamma(1 - \beta)} \left[ J_0(R_1 s_n) \right] - 1 \right] \left[ \frac{1}{\lambda} \right] \\
- \frac{1}{\nu} \sum_{k=1}^{m} \sum_{m=0}^{N} \left( \nu s_n \right)^{-\alpha m} \sum_{k=1}^{m} \left( \nu s_n \right)^{-\alpha k} \Gamma_{\nu, a-b, 2\nu}(\lambda^{-\alpha} s_n t) \\
- \frac{2}{\nu} \lambda f \sum_{k=1}^{m} \sum_{m=0}^{N} \left( \nu s_n \right)^{-\alpha k} \Gamma_{\nu, a-b, 2\nu}(\lambda^{-\alpha} s_n t) \\
+ \frac{2\lambda A g(s_n)}{\nu s_n^2 \Gamma(1 - \beta)} \sum_{k=1}^{m} \sum_{m=0}^{N} \left( \nu s_n \right)^{-\alpha m} \Gamma_{\nu, a-b, 2\nu}(\lambda^{-\alpha} s_n t).
(19)
\end{align*}

The expression of the velocity field can be written as
\[
v(r,t) = V(r) + \pi \sum_{n=0}^{\infty} \left[ \frac{J_n(R_j r)}{J_0(R_j r)} \phi(s_j) \right]
\]
\[
\times \left[ 1 - \sum_{n=0}^{\infty} (-1)^n \frac{J_n(R_j r)}{J_0(R_j r)} \sum_{m=1}^{\infty} \frac{g_m}{k} \frac{G_{a,b,n-2,m-1}(-\lambda, t)}{k} \right]
\]
\[
- \lambda \int \sum_{n=0}^{\infty} (-1)^n \frac{J_n(R_j r)}{J_0(R_j r)} \sum_{m=1}^{\infty} \frac{g_m}{k} \frac{G_{a,b,n-2,m-1}(-\lambda, t)}{k}
\]
\[
+ \lambda \sum_{n=0}^{\infty} (-1)^n \frac{J_n(R_j r)}{J_0(R_j r)} \sum_{m=1}^{\infty} \frac{g_m}{k} \frac{G_{a,b,n-2,m-1}(-\lambda, t)}{k} \]
\[
- \lambda \sum_{n=0}^{\infty} (-1)^n \frac{J_n(R_j r)}{J_0(R_j r)} \sum_{m=1}^{\infty} \frac{g_m}{k} \frac{G_{a,b,n-2,m-1}(-\lambda, t)}{k} 
\]
\[
(20) \]

4. Results

As shown in below diagrams, the velocity \( v(r,t) \) given by eq. (20) has been drawn against \( r \) for different values of the time \( t \), \( f \) and some other relevant parameters.

Figure 1 is showing the time dependency on the fluid motion. It can also be clearly seen that the velocity increases, when time increases.

\[ v(r,t) = 0.3, R_1=0.3, R_2=0.5, f=3, t=5s, \nu=0.04, \lambda=7, \alpha=0.3, \beta=0.3, A=4 \]

Figure 2 is showing the dependency of the kinematic viscosity \( \nu \) on the fluid motion. It can also be clearly seen that the velocity decreases, when kinematic viscosity \( \nu \) increases.

Figure 3 is showing the dependency of the retardation time \( \lambda \) on the fluid motion. It can also be clearly seen that the velocity decreases, when \( \lambda \) increases.

Figure 4 is showing the dependency of the relaxation time \( \lambda_r \) on the fluid motion. It can also be clearly seen that the velocity increases, when \( \lambda_r \) increases.
Fig. 4: Profiles of the velocity $v(r,t)$ given by eq. (20) for $R_1=0.3$, $R_2=0.5$, $f=-3$, $t=5s$, $\nu=0.04$, $\lambda=8$, $\alpha=0.3$, $\beta=0.9$, $A=4$ and different values of $\lambda_r$.

Figure 5 is showing the dependency of the fractional parameter $\alpha$ on the fluid motion. It can also be clearly seen that the velocity increases, when $\alpha$ increases.

Fig. 5: Profiles of the velocity $v(r,t)$ given by eq. (20) for $R_1=0.3$, $R_2=0.5$, $f=-3$, $t=6s$, $\nu=0.04$, $\lambda=8$, $\lambda_r=5.5$, $\alpha=1$, $A=4$ and different values of $\beta$.

Figure 6 is showing the dependency of $f$ on the fluid motion. It can also be clearly seen that the velocity decreases, when $f$ increases.

Fig. 6: Profiles of the velocity $v(r,t)$ given by eq. (20) for $R_1=0.3$, $R_2=0.5$, $f=-3$, $t=6s$, $\nu=0.04$, $\lambda=8$, $\lambda_r=2.8$, $\alpha=0.8$, $\beta=0.5$, $A=4$ and different values of $f$.

Figure 7 is showing the dependency of $A$ on the fluid motion. It can also be clearly seen that the velocity increases, when $A$ increases.

Fig. 7: Profiles of the velocity $v(r,t)$ given by eq. (20) for $R_1=0.3$, $R_2=0.5$, $t=5s$, $\nu=0.045$, $\lambda=14$, $\lambda_r=2.8$, $\alpha=0.8$, $\beta=0.5$, $A=4$ and different values of $\lambda_r$.
In all of above, the root $s_n$ has been approximated by 
\[(2n - 1)\pi \over 2(R_2 - R_1)\] 

5. Conclusions

The main objective of this paper is to provide exact solutions for the velocity field for Oldroyd-B fluid between two coaxial circular cylinders where inner cylinder is moving with constant velocity & the outer cylinder is fixed. This solution is obtained by using Hankel transform and Laplace transform methods and the result is presented in terms of generalized G functions. This solution satisfies the governing equation and all imposed initial and boundary conditions. The velocity field is also analyzed using graphical illustration for various parameters, constants and fractional coefficients.

References