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Evaluation of Unified Integrals Involving Products of Generalized M-Series and Incomplete H-Functions

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Abstract:
This work presents a unified approach to evaluate a class of definite integrals involving the product of an M-series and an incomplete H-function. These integrals are evaluated in terms of the incomplete H-function, yielding generalized and unified expressions. Several special cases are derived by specifying the parameters of the M-series and the incomplete H-function, which include the Fox H-function, incomplete Fox-Wright functions and incomplete generalized hypergeometric functions. The unified results presented here are broad in scope and hold significant applicability in various fields such as science, engineering, and finance.

Keywords: **M-series, Incomplete H-function, Improper integral**.

1. INTRODUCTION AND PRELIMINARIES

This section provides a brief overview of essential definitions and notations have been investigated in several prior studies $[1-8]$ related to incomplete function, M-series, and Unified integral, which are used throughout this work.

Incomplete Gamma Function (IGF)

The standard incomplete gamma function $\gamma(r,x)$ and $\Gamma(r,x)$ expressed by

$$\gamma(r,x) = \int_0^x t^{r-1} e^{-t} dt \; ; \qquad (\Re(r) > 0; x \geq 0) \tag{1}$$

$$\Gamma(r,x) = \int_x^\infty t^{r-1} e^{-t} dt \; ; \qquad (x \geq 0; \Re(r) > 0 \text{ when } x = 0) \tag{2}$$

such that their sum yields the complete gamma function:

$$\gamma(r,x) + \Gamma(r,x) = \Gamma(r) \; ; \qquad (\Re(r) > 0) \tag{3}$$

Incomplete Generalized Hypergeometric function (IGHF)

The incomplete generalized hypergeometric function ${}_p\gamma_q$ and ${}_p\Gamma_q$ introduced by Srivastava et al. $[9]$ through Mellin-Barnes integral representation involving the *IGF* $\gamma(r,x)$ and $\Gamma(r,x)$ as given below

$$ {}_p\Gamma_q \left[\begin{matrix} (a_1, x); (a_j, a_p)_{(2,p)}; \\ (b_1, b_q)_{(1,q)}; \end{matrix} \; z \right] = \frac{\prod_{j=1}^{q} \Gamma(b_j)}{\prod_{j=1}^{p} \Gamma(a_j)} \sum_{l=0}^{\infty} \frac{\Gamma(a_1 + l) \prod_{j=2}^{p} (a_j + l)}{\prod_{j=1}^{q} \Gamma(b_j + l)} \frac{z^l}{\lfloor l} $$

$$ = \frac{1}{2\pi i} \frac{\prod_{j=1}^{q} \Gamma(b_j)}{\prod_{j=1}^{p} \Gamma(a_j)} \int_{\Im} \frac{\Gamma(a_1 + \eta, x) \prod_{j=2}^{p} \Gamma(a_1 + \eta)}{\prod_{j=1}^{q} \Gamma(b_j + \eta)} \Gamma(-\eta)(-z)^{\eta} \, d\eta \; ; $$

$$ (|\arg(-\eta)| < \pi) \tag{4}$$

and

$$ {}_p\gamma_q \left[\begin{matrix} (a_1, x); (a_j, a_p)_{(2,p)}; \\ (b_1, b_q)_{(1,q)}; \end{matrix} \; z \right] = \frac{\prod_{j=1}^{q} \Gamma(b_j)}{\prod_{j=1}^{p} \Gamma(a_j)} \sum_{l=0}^{\infty} \frac{\gamma(a_1 + l) \prod_{j=2}^{p} (a_j + l)}{\prod_{j=1}^{q} \Gamma(b_j + l)} \frac{z^l}{\lfloor l} $$

$$ = \frac{1}{2\pi i} \frac{\prod_{j=1}^{q} \Gamma(b_j)}{\prod_{j=1}^{p} \Gamma(a_j)} \int_{\Im} \frac{\gamma(a_1 + \eta, x) \prod_{j=2}^{p} \Gamma(a_1 + \eta)}{\prod_{j=1}^{q} \Gamma(b_j + \eta)} \Gamma(-\eta)(-z)^{\eta} \, d\eta \; ; $$

$$ (|\arg(-\eta)| < \pi) \tag{5}$$

here $\Im$ is a Mellin-Barnes type contour extending from $\xi - i\infty$ to $\xi + i\infty$ with $(\xi \in \Re)$,and indented, when necessary to separate the sets of poles of the integral in each case.

Incomplete H-function

Srivastava et al. $[10]$ (equation (2.1) -(2.4)) define incomplete H-function as follows:

$$\Gamma_{p,q}^{m,n}(z)=\Gamma_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_1,A_1,x);(a_j,A_j)_{(2,p)}\\(b_j,B_j)_{(1,q)}\end{matrix}\right.\right]$$

$$=\Gamma_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_1,A_1,x);(a_2,A_2),...,(a_p,A_p)\\(b_1,B_1),(b_2,B_2),...,(b_q,B_q)\end{matrix}\right.\right]=\frac{1}{2\pi i}\int_{\Im}\phi(\eta,x)z^{-\eta}d\eta, \tag{6}$$

where

$$\phi(\eta,x)=\frac{\Gamma(1-a_1-A_1\eta,x)\prod_{j=1}^{m}\Gamma(b_j+B_j\eta)\prod_{j=2}^{n}\Gamma(1-a_j-A_j\eta)}{\prod_{j=m+1}^{q}(1-b_j-B_j\eta)\prod_{j=n+1}^{p}\Gamma(a_j+A_j\eta)}$$

and

$$\gamma_{p,q}^{m,n}(z)=\gamma_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_1,A_1,x);(a_j,A_j)_{(2,p)}\\(b_j,B_j)_{(1,q)}\end{matrix}\right.\right]$$

$$=\gamma_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_1,A_1,x);(a_2,A_2),...,(a_p,A_p)\\(b_1,B_1),(b_2,B_2),...,(b_q,B_q)\end{matrix}\right.\right]=\frac{1}{2\pi i}\int_{\Im}\phi(\eta,x)z^{-\eta}d\eta, \tag{7}$$

where

$$\phi(\eta,x)=\frac{\gamma(1-a_1-A_1\eta,x)\prod_{j=1}^{m}\Gamma(b_j+B_j\eta)\prod_{j=2}^{n}\Gamma(1-a_j-A_j\eta)}{\prod_{j=m+1}^{q}(1-b_j-B_j\eta)\prod_{j=n+1}^{p}\Gamma(a_j+A_j\eta)}$$

The incomplete H-function as defined in equation (6) and (7) respectively, exist for all $x\geq 0$ under the same set of conditions and contour specifications as presented in the work of Kilabs et al. [11], Mathai and Saxena [12], and Mathai et al. [13].

The previously mentioned functions admit numerous special cases, a few of which are enumerated below:

(i) By putting m=1, n=p and replacing q by q+1 with relevant parameters, the functions (6) & (7) converges to incomplete Fox-Wright functions ${}_p\psi_q^{(\Gamma)}$ and ${}_p\psi_q^{(\gamma)}$ (see for details, [10] [P. 132, Equation (6.3) and (6.4)]):

$$\Gamma_{p,q+1}^{1,p}\left[-z\left|\begin{matrix}(1-a_1,A_1,x);(1-a_1,A_j)_{(2,p)}\\(0,1);(1-b_j,B_j)_{(1,q)}\end{matrix}\right.\right]={}_p\psi_q^{(\Gamma)}\left[\begin{matrix}(a_1,A_1,x);(a_1,A_1)_{(2,p)};\\(b_j,B_j)_{(1,q)};\end{matrix}\;z\right] \tag{8}$$

and

$$\gamma_{p,q+1}^{1,p}\left[-z\left|\begin{matrix}(1-a_1,A_1,x);(1-a_1,A_j)_{(2,p)}\\(0,1);(1-b_j,B_j)_{(1,q)}\end{matrix}\right.\right]={}_p\psi_q^{(\gamma)}\left[\begin{matrix}(a_1,A_1,x);(a_1,A_1)_{(2,p)};\\(b_j,B_j)_{(1,q)};\end{matrix}\;z\right] \tag{9}$$

(ii) Additionally putting $x=0$ in equation (8), incomplete Fox-Wright function converges to Fox-Wright function (see for details, [14] ([P. 39, Equation (2.6.11)]):

$${}_p\psi_q^{(\Gamma)}\left[\begin{matrix}(a_1,A_1,0);(a_j,A_j)_{(2,p)};\\(b_j,B_j)_{(1,q)};\end{matrix}\;z\right]={}_p\psi_q\left[\begin{matrix}(a_1,A_1);(a_j,A_j)_{(2,p)};\\(b_j,B_j)_{(1,q)};\end{matrix}\;z\right] \tag{10}$$

(iii) If we take $A_j=B_j=1(j=1,...,p,\ j=1,...q)$ in equation (8) & (9), then the incomplete H-function converges to IGHF ${}_p\gamma_q$ and ${}_p\Gamma_q$ (see details in Srivastava et al. [9]):

$${}_p\psi_q^{(\Gamma)}\left[\begin{matrix}(a_1,1,x);(a_j,1)_{(2,p)};\\(b_j,1)_{(1,q)};\end{matrix}\;z\right]={}_p\Gamma_q\left[\begin{matrix}(a_1,x);a_2,...,a_p;\\b_1,...,b_p;\end{matrix}\;z\right] \tag{11}$$

and

$${}_p\psi_q^{(\gamma)}\left[\begin{matrix}(a_1,1,x);(a_j,1)_{(2,p)};\\(b_j,1)_{(1,q)};\end{matrix}\;z\right]={}_p\gamma_q\left[\begin{matrix}(a_1,x);a_2,...,a_p;\\b_1,...,b_p;\end{matrix}\;z\right] \tag{12}$$

M-series

Sharma et al. [15] proposed and examined the generalized M-series in the following form:

$${}_rM_s^{\alpha,\beta}\left[\begin{matrix}e_1,...,e_r;\\f_1,...,f_r;\end{matrix}\;z\right]={}_rM_s^{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{(e_1)_k\cdots(e_r)_k}{(f_1)_k\cdots(f_r)_k}\frac{z^k}{\Gamma(\alpha k+\beta)};\quad(\alpha,\beta,z\in\mathbb{C},\Re(\alpha)>0) \tag{13}$$

From the table of integration, series, and products by I. S. Gradshteyn, M. I. Ryzhik the following Integral formula ([17], p. 377 Equation (3.257)) is given as:

$$\int_0^\infty\left[\left(uy+\frac{v}{y}\right)^2+w\right]^{-\delta-1}dy=\frac{\sqrt{\pi}\,\Gamma\left(\delta+\frac{1}{2}\right)}{2uw^{\delta+\frac{1}{2}}\Gamma(\delta+1)};\quad\left(u>0,v<0,w>0,\delta>-\frac{1}{2}\right) \tag{14}$$

2.Main Results

This section presents certain integrals involving the product of M-series and Incomplete H-function.

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THEOREM 1: Suppose that $\delta \in \mathbb{C}$; with $R(\delta)>0$ and $y>0,\ u>0,\ v<0,\ w>0,\ \delta>\frac{-1}{2},\ \frac{-1}{2}<u-v-w<\frac{1}{2}$ then the following integration hold:

$$\int_0^\infty \left[\left(uy+\frac{v}{y}\right)^2+w\right]^{-\delta-1} {}_rM_s^{\alpha,\beta}\left[\begin{matrix} e_1,\dots,e_r; \\ f_1,\dots,f_r; \end{matrix} \frac{z_1}{\left(uv+\frac{v}{y}\right)^2+w}\right]\times$$
$$\Gamma_{p,q}^{m,n}\left[\frac{z_2}{\left(uv+\frac{v}{y}\right)^2+w}\left|\begin{matrix}(a_1,A_1,x);(a_2,A_2),\dots,(a_p,A_p) \\ (b_1,B_1),(b_2,B_2),\dots,(b_q,B_q)\end{matrix}\right.\right]dy$$
$$=\frac{\sqrt{\pi}}{2uw^{\delta+\frac{1}{2}}}\sum_{k=0}^{\infty}\frac{(e_1)_k\cdots(e_r)_k}{(f_1)_k\cdots(f_s)_k}\frac{1}{\Gamma(\alpha k+\beta)}\left(\frac{z_1}{w}\right)^k\times$$
$$\Gamma_{p+1,q+1}^{m,n+1}\left[\frac{z_2}{w}\left|\begin{matrix}(a_1,A_1,x);(a_2,A_2),\dots,(a_p,A_p);\left(-\delta-k+\frac{1}{2},1\right) \\ (b_1,B_1),\dots,(a_q,B_q);(-\delta-k,1)\end{matrix}\right.\right] \quad (15)$$

assume that the conditions for the incomplete H-function $\Gamma_{p,q}^{m,n}$ in equation [6] are fulfilled.

Proof: L.H.S. of equation (15) Let

$$L=\int_0^\infty \left[\left(uy+\frac{v}{y}\right)^2+w\right]^{-\delta-1} {}_rM_s^{\alpha,\beta}\left[\begin{matrix} e_1,\dots,e_r; \\ f_1,\dots,f_r; \end{matrix} \frac{z_1}{\left(uv+\frac{v}{y}\right)^2+w}\right]\times$$
$$\Gamma_{p,q}^{m,n}\left[\frac{z_2}{\left(uv+\frac{v}{y}\right)^2+w}\left|\begin{matrix}(a_1,A_1,x);(a_2,A_2),\dots,(a_p,A_p) \\ (b_1,B_1),(b_2,B_2),\dots,(b_q,B_q)\end{matrix}\right.\right]dy$$

By the using of equation (6) and (13) in L we get

$$L=\frac{1}{2\pi i}\int_0^\infty\left[\left(uy+\frac{v}{y}\right)^2+w\right]^{-\delta-1}\times$$
$$\sum_{k=0}^{\infty}\frac{(e_1)_k\cdots(e_r)_k}{(f_1)_k\cdots(f_s)_k}\frac{1}{\Gamma(\alpha k+\beta)}\left[z_1\left\{\left(uy+\frac{v}{y}\right)^2+w\right\}^{-1}\right]^k\times$$
$$\int_{\Im}\left[z_2\left\{\left(uy+\frac{v}{y}\right)^2+w\right\}^{-1}\right]^{-\eta}\phi(\eta,x)\,d\eta\,dy$$
$$=\sum_{k=0}^{\infty}\frac{(e_1)_k\cdots(e_r)_k}{(f_1)_k\cdots(f_s)_k}\frac{z_1^k}{\Gamma(\alpha k+\beta)}\times$$
$$\frac{1}{2\pi i}\int_0^\infty\left[\left(uy+\frac{v}{y}\right)^2+w\right]^{-\delta-1-k+\eta}\int_{\Im}\phi(\eta,x)z_2^{-\eta}\,d\eta\,dy$$

Now by changing the order of integration, we get

$$=\sum_{k=0}^{\infty}\frac{(e_1)_k\cdots(e_r)_k}{(f_1)_k\cdots(f_s)_k}\frac{z_1^k}{\Gamma(\alpha k+\beta)}\times$$
$$\frac{1}{2\pi i}\int_{\Im}\phi(\eta,x)z_2^{-\eta}\int_0^\infty\left[\left(uy+\frac{v}{y}\right)^2+w\right]^{-\delta-1-k+\eta}dy\,d\eta$$

By using the result of (14), we get

$$=\sum_{k=0}^{\infty}\frac{(e_1)_k\cdots(e_r)_k}{(f_1)_k\cdots(f_s)_k}\frac{z_1^k}{\Gamma(\alpha k+\beta)}\times$$
$$\frac{1}{2\pi i}\int_{\Im}\frac{\sqrt{\pi}\,\Gamma\left(\delta+k-\eta+\frac{1}{2}\right)}{2uw^{\left(\delta+k-\eta+\frac{1}{2}\right)}\Gamma(\delta+k-\eta+1)}\phi(\eta,x)z_2^{-\eta}\,d\eta$$
$$=\frac{\sqrt{\pi}}{2uw^{\left(\delta+k-\eta+\frac{1}{2}\right)}}\sum_{k=0}^{\infty}\frac{(e_1)_k\cdots(e_r)_k}{(f_1)_k\cdots(f_s)_k}\frac{z_1^k}{\Gamma(\alpha k+\beta)}\times$$
$$\frac{1}{2\pi i}\int_{\Im}\frac{\Gamma\left(\delta+k-\eta+\frac{1}{2}\right)}{\Gamma(\delta+k-\eta+1)}\phi(\eta,x)z_2^{-\eta}\,d\eta$$
$$=\frac{\sqrt{\pi}}{2uw^{\left(\delta+\frac{1}{2}\right)}}\sum_{k=0}^{\infty}\frac{(e_1)_k\cdots(e_r)_k}{(f_1)_k\cdots(f_s)_k}\frac{\left(\frac{z_1}{w}\right)^k}{\Gamma(\alpha k+\beta)}\times$$
$$\frac{1}{2\pi i}\int_{\Im}\frac{\Gamma\left(\delta+k-\eta+\frac{1}{2}\right)}{\Gamma(\delta+k-\eta+1)}\phi(\eta,x)\left(\frac{z_2}{w}\right)^{-\eta}d\eta$$

Through the interpretation of equation (6), the desired result is derived

$$\Rightarrow\frac{\sqrt{\pi}}{2uw^{\delta+\frac{1}{2}}}\sum_{k=0}^{\infty}\frac{(e_1)_k\cdots(e_r)_k}{(f_1)_k\cdots(f_s)_k}\frac{1}{\Gamma(\alpha k+\beta)}\left(\frac{z_1}{w}\right)^k\times$$
$$\Gamma_{p+1,q+1}^{m,n+1}\left[\frac{z_2}{w}\left|\begin{matrix}(a_1,A_1,x);(a_2,A_2),\dots,(a_p,A_p);\left(-\delta-k+\frac{1}{2},1\right) \\ (b_1,B_1),\dots,(a_q,B_q);(-\delta-k,1)\end{matrix}\right.\right]$$

Hence Theorem 1 is proved.

THEOREM 2: Suppose that $\delta \in \mathbb{C}$; with $R(\delta)>0$ and $y>0,\ u>0,\ v<0,\ w>0,\ \delta>\frac{-1}{2},\ \frac{-1}{2}<u-v-w<\frac{1}{2}$ then the following integration hold:

$$\int_0^\infty \left[\left(uy+\frac{v}{y}\right)^2+w\right]^{-\delta-1} {}_rM_s^{\alpha,\beta}\left[\begin{matrix} e_1,\dots,e_r; \\ f_1,\dots,f_r; \end{matrix} \frac{z_1}{\left(uv+\frac{v}{y}\right)^2+w}\right]\times$$
$$\gamma_{p,q}^{m,n}\left[\frac{z_2}{\left(uv+\frac{v}{y}\right)^2+w}\left|\begin{matrix}(a_1,A_1,x);(a_2,A_2),\dots,(a_p,A_p) \\ (b_1,B_1),(b_2,B_2),\dots,(b_q,B_q)\end{matrix}\right.\right]dy$$
$$=\frac{\sqrt{\pi}}{2uw^{\delta+\frac{1}{2}}}\sum_{k=0}^{\infty}\frac{(e_1)_k\cdots(e_r)_k}{(f_1)_k\cdots(f_s)_k}\frac{1}{\Gamma(\alpha k+\beta)}\left(\frac{z_1}{w}\right)^k\times$$
$$\gamma_{p+1,q+1}^{m,n+1}\left[\frac{z_2}{w}\left|\begin{matrix}(a_1,A_1,x);(a_2,A_2),\dots,(a_p,A_p);\left(-\delta-k+\frac{1}{2},1\right) \\ (b_1,B_1),\dots,(a_q,B_q);(-\delta-k,1)\end{matrix}\right.\right] \quad (16)$$

assume that the conditions for the incomplete H-function $\gamma_{p,q}^{m,n}$ in equation [7] are fulfilled.

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Proof: L.H.S. of equation (16)

Let

$$L=\int_0^\infty\left[\left(uy+\frac{v}{y}\right)^2+w\right]^{-\delta-1}{}_rM_s^{\alpha,\beta}\left[\begin{matrix}e_1,\dots,e_r;\\ f_1,\dots,f_r;\end{matrix}\frac{z_1}{\left(uv+\frac{v}{y}\right)^2+w}\right]\times$$

$$\gamma_{p,q}^{m,n}\left[\frac{z_2}{\left(uv+\frac{v}{y}\right)^2+w}\left|\begin{matrix}(a_1,A_1,x);(a_2,A_2),\dots,(a_p,A_p)\\(b_1,B_1),(b_2,B_2),\dots,(b_q,B_q)\end{matrix}\right.\right]dy$$

By the using of equation (7) and (13) in L we get

$$L=\frac{1}{2\pi i}\int_0^\infty\left[\left(uy+\frac{v}{y}\right)^2+w\right]^{-\delta-1}\times$$

$$\sum_{k=0}^\infty\frac{(e_1)_k\cdots(e_r)_k}{(f_1)_k\cdots(f_s)_k}\frac{1}{\Gamma(\alpha k+\beta)}\left[z_1\left\{\left(uy+\frac{v}{y}\right)^2+w\right\}^{-1}\right]^k\times$$

$$\int_\Im\left[z_2\left\{\left(uy+\frac{v}{y}\right)^2+w\right\}^{-1}\right]^{-\eta}\phi(\eta,x)\,d\eta\,dy$$

$$=\sum_{k=0}^\infty\frac{(e_1)_k\cdots(e_r)_k}{(f_1)_k\cdots(f_s)_k}\frac{z_1^k}{\Gamma(\alpha k+\beta)}\times$$

$$\frac{1}{2\pi i}\int_0^\infty\left[\left(uy+\frac{v}{y}\right)^2+w\right]^{-\delta-1-k+\eta}\int_\Im\phi(\eta,x)z_2^{-\eta}\,d\eta\,dy$$

Now by changing the order of integration, we get

$$=\sum_{k=0}^\infty\frac{(e_1)_k\cdots(e_r)_k}{(f_1)_k\cdots(f_s)_k}\frac{z_1^k}{\Gamma(\alpha k+\beta)}\times$$

$$\frac{1}{2\pi i}\int_\Im\phi(\eta,x)z_2^{-\eta}\int_0^\infty\left[\left(uy+\frac{v}{y}\right)^2+w\right]^{-\delta-1-k+\eta}dy\,d\eta$$

By using the result of (14), we get

$$=\sum_{k=0}^\infty\frac{(e_1)_k\cdots(e_r)_k}{(f_1)_k\cdots(f_s)_k}\frac{z_1^k}{\Gamma(\alpha k+\beta)}\times$$

$$\frac{1}{2\pi i}\int_\Im\frac{\sqrt{\pi}\,\Gamma\left(\delta+k-\eta+\frac{1}{2}\right)}{2uw^{\left(\delta+k-\eta+\frac{1}{2}\right)}\Gamma(\delta+k-\eta+1)}\phi(\eta,x)z_2^{-\eta}\,d\eta$$

$$=\frac{\sqrt{\pi}}{2uw^{\left(\delta+k-\eta+\frac{1}{2}\right)}}\sum_{k=0}^\infty\frac{(e_1)_k\cdots(e_r)_k}{(f_1)_k\cdots(f_s)_k}\frac{z_1^k}{\Gamma(\alpha k+\beta)}\times$$

$$\frac{1}{2\pi i}\int_\Im\frac{\Gamma\left(\delta+k-\eta+\frac{1}{2}\right)}{\Gamma(\delta+k-\eta+1)}\phi(\eta,x)z_2^{-\eta}\,d\eta$$

$$=\frac{\sqrt{\pi}}{2uw^{\left(\delta+\frac{1}{2}\right)}}\sum_{k=0}^\infty\frac{(e_1)_k\cdots(e_r)_k}{(f_1)_k\cdots(f_s)_k}\frac{\left(\frac{z_1}{w}\right)^k}{\Gamma(\alpha k+\beta)}\times$$

$$\frac{1}{2\pi i}\int_\Im\frac{\Gamma\left(\delta+k-\eta+\frac{1}{2}\right)}{\Gamma(\delta+k-\eta+1)}\phi(\eta,x)\left(\frac{z_2}{w}\right)^{-\eta}d\eta$$

Through the interpretation of equation (7), the desired result is derived

$$\Rightarrow\frac{\sqrt{\pi}}{2uw^{\delta+\frac{1}{2}}}\sum_{k=0}^\infty\frac{(e_1)_k\cdots(e_r)_k}{(f_1)_k\cdots(f_s)_k}\frac{1}{\Gamma(\alpha k+\beta)}\left(\frac{z_1}{w}\right)^k\times$$

$$\gamma_{p+1,q+1}^{m,n+1}\left[\frac{z_2}{w}\left|\begin{matrix}(a_1,A_1,x);(a_2,A_2),\dots,(a_p,A_p);\left(-\delta-k+\frac{1}{2},1\right)\\(b_1,B_1),\dots,(a_q,B_q);(-\delta-k,1)\end{matrix}\right.\right]$$

Hence the Theorem 2 is proved.

2. Special Cases

We illustrate in this part some significant special cases corresponding to the principal result of Theorem 1 and Theorem 2.

Corollary 1.

For $\delta\in\mathbb{C}$ with $\Re(\delta)>0$ and y>0, u>0, v<0, w>0, $\delta>-\frac{1}{2}$

Furthermore, setting $m=1, p=n$ and replacing q with $q+1$, in equation (6) and (7), and assume that the incomplete H-function reduces to the incomplete Wright function as given in equation (8) and (9), we obtain the result here from those in Theorems 1 and 2. The corresponding integral formula as given:

$$\int_0^\infty\left[\left(uy+\frac{v}{y}\right)^2+w\right]^{-\delta-1}{}_rM_s^{\alpha,\beta}\left[\begin{matrix}e_1,\dots,e_r;\\ f_1,\dots,f_r;\end{matrix}\frac{z_1}{\left(uv+\frac{v}{y}\right)^2+w}\right]\times$$

$${}_p\Psi_q^{(\Gamma)}\left[\frac{z_2}{\left(uv+\frac{v}{y}\right)^2+w}\left|\begin{matrix}(a_1,A_1,x);(a_2,A_2),\dots,(a_p,A_p)\\(b_1,B_1),(b_2,B_2),\dots,(b_q,B_q)\end{matrix}\right.\right]dy$$

$$=\frac{\sqrt{\pi}}{2uw^{\delta+\frac{1}{2}}}\sum_{k=0}^\infty\frac{(e_1)_k\cdots(e_r)_k}{(f_1)_k\cdots(f_s)_k}\frac{1}{\Gamma(\alpha k+\beta)}\left(\frac{z_1}{w}\right)^k\times$$

$${}_{p+1}\Psi_{q+1}^{(\Gamma)}\left[\frac{z_2}{w}\left|\begin{matrix}(a_1,A_1,x);(a_2,A_2),\dots,(a_p,A_p);\left(\delta+k+\frac{1}{2},1\right)\\(b_1,B_1),\dots,(a_q,B_q);(\delta+k+1,1)\end{matrix}\right.\right] \quad (17)$$

and

$$\int_0^\infty \left[\left(uy+\frac{v}{y}\right)^2+w\right]^{-\delta-1} {}_rM_s^{\alpha,\beta}\left[\begin{matrix} e_1,\ldots,e_r; \\ f_1,\ldots,f_r;\end{matrix}\ \frac{z_1}{\left(uv+\frac{v}{y}\right)^2+w}\right]\times$$
$${}_p\Psi_q^{(\gamma)}\left[\frac{z_2}{\left(uv+\frac{v}{y}\right)^2+w}\middle|\begin{matrix}(a_1,A_1,x);(a_2,A_2),\ldots,(a_p,A_p)\\ (b_1,B_1),(b_2,B_2),\ldots,(b_q,B_q)\end{matrix}\right]dy$$
$$=\frac{\sqrt{\pi}}{2uw^{\delta+\frac{1}{2}}}\sum_{k=0}^{\infty}\frac{(e_1)_k\cdots(e_r)_k}{(f_1)_k\cdots(f_s)_k}\frac{1}{\Gamma(\alpha k+\beta)}\left(\frac{z_1}{w}\right)^k\times$$
$${}_{p+1}\Psi_{q+1}^{(\gamma)}\left[\frac{z_2}{w}\middle|\begin{matrix}(a_1,A_1,x);(a_2,A_2),\ldots,(a_p,A_p);\left(\delta+k+\frac{1}{2},1\right)\\ (b_1,B_1),\ldots,(a_q,B_q);(\delta+k+1,1)\end{matrix}\right] \quad (18)$$

under the assumption that each element of equation (17) and (18) are exists.

Corollary 2:

For $\delta\in\mathbb{C}$ with $\Re(\delta)>0$ and y>0, u>0, v<0, w>0, $\delta>-\frac{1}{2}$.

Furthermore, setting $A_j=B_j=1,\ (\forall\, j=1,\ldots,p;\ \forall\, j=1,\ldots,q)$ in equation (8) and (9), and assume that the incomplete H-function reduces to the incomplete generalized hypergeometric function as given in equation (11) and (12), we obtain the result here from those in Theorems 1 and 2. The corresponding integral formula as given:

$$\int_0^\infty \left[\left(uy+\frac{v}{y}\right)^2+w\right]^{-\delta-1} {}_rM_s^{\alpha,\beta}\left[\begin{matrix} e_1,\ldots,e_r; \\ f_1,\ldots,f_r;\end{matrix}\ \frac{z_1}{\left(uv+\frac{v}{y}\right)^2+w}\right]\times$$
$${}_p\Gamma_q\left[\begin{matrix}(a_1,x);a_2,\ldots,a_p;\\ b_1,\ldots,b_p;\end{matrix}\ \frac{z_2}{\left(uv+\frac{v}{y}\right)^2+w}\right]dy$$
$$=\frac{\sqrt{\pi}}{2uw^{\delta+\frac{1}{2}}}\sum_{k=0}^{\infty}\frac{(e_1)_k\cdots(e_r)_k}{(f_1)_k\cdots(f_s)_k}\frac{1}{\Gamma(\alpha k+\beta)}\left(\frac{z_1}{w}\right)^k\times$$
$${}_{p+1}\Gamma_{q+1}\left[\frac{z_2}{w}\middle|\begin{matrix}(a_1,x);a_2,\ldots,a_p;\left(-\delta-k+\frac{1}{2}\right)\\ b_1,\ldots,b_p;(-\delta-k)\end{matrix}\right] \quad (19)$$

and

$$\int_0^\infty \left[\left(uy+\frac{v}{y}\right)^2+w\right]^{-\delta-1} {}_rM_s^{\alpha,\beta}\left[\begin{matrix} e_1,\ldots,e_r; \\ f_1,\ldots,f_r;\end{matrix}\ \frac{z_1}{\left(uv+\frac{v}{y}\right)^2+w}\right]\times$$
$${}_p\gamma_q\left[\begin{matrix}(a_1,x);a_2,\ldots,a_p;\\ b_1,\ldots,b_p;\end{matrix}\ \frac{z_2}{\left(uv+\frac{v}{y}\right)^2+w}\right]dy$$
$$=\frac{\sqrt{\pi}}{2uw^{\delta+\frac{1}{2}}}\sum_{k=0}^{\infty}\frac{(e_1)_k\cdots(e_r)_k}{(f_1)_k\cdots(f_s)_k}\frac{1}{\Gamma(\alpha k+\beta)}\left(\frac{z_1}{w}\right)^k\times$$
$${}_{p+1}\gamma_{q+1}\left[\frac{z_2}{w}\middle|\begin{matrix}(a_1,x);a_2,\ldots,a_p;\left(-\delta-k+\frac{1}{2}\right)\\ b_1,\ldots,b_p;(-\delta-k)\end{matrix}\right] \quad (20)$$

under the assumption that each element of equation (19) and (20) are exists.

Corollary 3:

For $\delta\in\mathbb{C}$ with $\Re(\delta)>0$ and y>0, u>0, v<0, w>0, $\delta>-\frac{1}{2}$

Furthermore, setting $x=0$, in equation (8), and assume that the incomplete H-function reduces to the incomplete Fox-Wright generalized hypergeometric function as given in equation (10), also M-series reduces into unity, we obtain the result here from those in Theorems 1. The corresponding integral formula as given:

$$\int_0^\infty \left[\left(uy+\frac{v}{y}\right)^2+w\right]^{-\delta-1}\times$$
$${}_p\Psi_q\left[\begin{matrix}(a_1,A_1),\ldots,(a_p,A_p);\\ (b_j,B_j)_{(1,q)};\end{matrix}\ z\left\{\left(uy+\frac{v}{y}\right)^2+w\right\}^{-1}\right]dy$$
$$=\frac{\sqrt{\pi}}{2uw^{\delta+\frac{1}{2}}}\,{}_{p+1}\Psi_{q+1}\left[\frac{z}{w}\middle|\begin{matrix}(a_1,A_1),\ldots,(a_p,A_p);\left(\delta+\frac{1}{2},1\right)\\ (b_1,B_1),\ldots,(a_q,B_q);(\delta+1,1)\end{matrix}\right] \quad (21)$$

under the assumption that each element of equation (21) are exists.

3. CONCLUSION

In the present study, we have established several significant integrals involving the product of the M-series and incomplete H-functions, represented in terms of incomplete H-functions themselves. Additionally, we have presented certain special cases by assigning certain values to the parameters of the M-series and incomplete H-functions-such as Fox's H-function, incomplete Fox–Wright functions, Fox–Wright functions, and incomplete generalized hypergeometric functions. Some previously known results are also included as special cases. The integrals derived in this analysis are of a general form and may serve as a foundation for developing numerous results relevant to practical and applied contexts

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