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Equation of Motion and Energy Analysis of Nonlinear Systems using MDTM

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Abstract - We apply the Modified Differential Transformation Method (MDTM) to obtain semi-analytic series solutions of second-order equations of motion, including nonlinear terms and forcing/damping. MDTM produces a rapidly convergent recurrence for the power-series coefficients of the solution; nonlinearities are handled via discrete convolution formulae. We show how to derive an exact energy balance (time derivative of mechanical energy) from the equation of motion and how to compute the series expansion of the energy using the MDTM coefficients. A worked example (Duffing oscillator with viscous damping) illustrates the method and gives explicit first terms of the solution and energy series. The study includes an algorithmic description suitable for numerical implementation.

Keywords - MDTM; DTM; Equation of Motion; Equation of Energy; non-linear; ODF.

I. INTRODUCTION

Nonlinear oscillators constitute a fundamental class of dynamical systems with applications spanning mechanical vibrations, electrical circuits, structural dynamics, and numerous physical phenomena. Their governing equations are typically formulated as second-order nonlinear differential equations, where the restoring forces incorporate polynomial or more complex nonlinearities. Classical analytical techniques enable exact solutions for only a limited subset of such systems—often expressed through Jacobi elliptic functions or other special functions—yet these closed-form solutions remain difficult to manipulate and apply in practical engineering contexts due to their mathematical complexity and limited scope of applicability [1–4].

Consequently, the focus of modern research has shifted toward numerical and semi-analytical approaches capable of handling strong nonlinearities with improved efficiency. Among these methods, the Differential Transform Method (DTM), introduced in the late twentieth century, has been widely employed to derive rapidly convergent series expansions for nonlinear ordinary differential equations [5]. Despite its conceptual simplicity, traditional DTM suffers from a restricted convergence radius, making it unsuitable for long-term simulations or systems exhibiting pronounced nonlinear behavior. To overcome these limitations, the Modified Differential Transform Method (MDTM) was developed, integrating auxiliary tools such as the Laplace transform and Padé approximants to significantly extend the convergence domain and enhance the accuracy of the resulting solutions. Numerous studies—including those by Momani and Noor [6], Ertürk [7], and others—have demonstrated the effectiveness of MDTM in modeling nonlinear oscillators such as the Duffing system, yielding highly accurate semi-analytical approximations over extended time intervals.

Despite these advances, existing literature has predominantly centered on solving the **equation of motion** itself, while the **energy relation**, derived from conservation of total mechanical energy, has received comparatively limited attention. Typically expressed as a first-order differential equation, the energy relation has been used primarily as a diagnostic tool for validating numerical solutions or constructing phase-space trajectories. However, applying MDTM directly to the energy relation offers a novel and underexplored analytical pathway. Because the energy equation inherently reflects conservation laws and integrates the system's global dynamical behavior, its semi-analytical treatment may yield more stable and physically consistent approximations—particularly for conservative or weakly damped systems. Recent indications in the literature suggest that combining MDTM with Laplace–Padé resummation can substantially enhance the accuracy and convergence of solutions derived from the energy perspective, although comprehensive studies in this direction remain scarce [8,9].

Recognizing this gap, the present work aims to systematically examine the application of MDTM—augmented by Laplace–Padé resummation to both the equation of motion and the energy relation for nonlinear oscillatory systems. Through a comparative analysis, the study evaluates the consistency, convergence properties, and physical accuracy of solutions obtained via each formulation. The findings highlight the potential of the energy-based MDTM approach as an effective complementary tool to conventional equation-based methods, thereby contributing to a deeper and more robust understanding of nonlinear conservative dynamics.

II. METHODOLOGY AND IMPLEMENTATION

2.1 Differential Transformation Method (DTM)

The Differential Transformation Method (DTM) is a semi-analytical technique for solving differential equations by transforming them into a sequence of algebraic equations. The method is based on the Taylor series expansion and allows systematic computation of the series coefficients.

Transformation of k^{th} derivative of a function [27]

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$$Y(k) = \frac{1}{k!}\left[\frac{d^k y(x)}{dx^k}\right]_{x=0}$$

The inverse transformation

$$y(x) = \sum_{k}^{\infty} Y(k)(x-x_0)^k$$

THEOREMS:

1. If y(x) = au(x) + bv(x) then it transformation Y(k) = aU(k) + bV(k) ,where a, b are constants

Proof:

By def

$$Y(k) = \frac{1}{k!}\left[\frac{d^k y(x)}{dx^k}\right]_{x=0}$$

$$Y(k) = \frac{1}{k!}\left[\frac{d^k(au(x)+bv(x))}{dx^k}\right]_{x=0}$$

$$Y(k) = \frac{1}{k!}\left[\frac{d^k(au(x))}{dx^k} + \frac{d^k(bv(x))}{dx^k}\right]_{x=0}$$

$$Y(k) = \frac{a}{k!}\left[\frac{d^k(u(x))}{dx^k}\right]_{x=0} + \frac{b}{k!}\left[\frac{d^k(v(x))}{dx^k}\right]_{x=0}$$

Y(k) = aG(k) + bH(k)

2. If y(x)=u(x)v(x) then its transformation $Y(k) = \sum_{s=0}^{k} U(s)V(k-s)$

Proof: By Leibniz's theorem

$$\frac{d^k}{dx^k}(u(x)v(x)) = \sum_{s}^{k} \frac{k!}{s!(k-s)!}\frac{d^s}{dx^s}u(x)\frac{d^{k-s}}{dx^{k-s}}v(x)$$

By def. of differential transforms

$$Y(k) = \frac{1}{k!}\left[\frac{d^k y(x)}{dx^k}\right]_{x=0}$$

$$Y(k) = \frac{1}{k!}\left[\sum_{s}^{k} \frac{k!}{s!(k-s)!}\frac{d^s}{dx^s}u(x)\frac{d^{k-s}}{dx^{k-s}}v(x)\right]_{x=0}$$

$$Y(k) = \sum_{s=0}^{k} U(s)V(k-s)$$

3. If y(x)=x^n then its transformation Y(k)=δ(k-n) $=\begin{cases}1 \ if \ k=n \\ 0 \ if \ k \neq n\end{cases}$

Proof: By def.

$$Y(k) = \frac{1}{k!}\left[\frac{d^k y(x)}{dx^k}\right]_{x=0}$$

$$Y(k) = \frac{1}{k!}\left[\frac{d^k x^n}{dx^k}\right]_{x=0}$$

$$Y(k) = \delta(k-n) = \begin{cases}1 \ if \ k=n \\ 0 \ if \ k \neq n\end{cases}$$

4. If $y(x) = \frac{d^n}{dx^n}u(x)$ then its transformation $Y(k) = (k+1)(k+2)..(k+n)U(k+n)$

Proof: By def

$$Y(k) = \frac{1}{k!}\left[\frac{d^k y(x)}{dx^k}\right]_{x=0}$$

$$Y(k) = \frac{1}{k!}\left[\frac{d^k \frac{d^n}{dx^n}u(x)}{dx^k}\right]_{x=0} = \frac{1}{k!}\left[\frac{d^{k+n}u(x)}{dx^{k+n}}\right]_{x=0}$$

$$Y(k) = \frac{(k+n)!}{k!}\left[\frac{1}{(k+n)!}\frac{d^{k+n}u(x)}{dx^{k+n}}\right]_{x=0}$$

$$Y(k) = (k+1)(k+2)\ldots\ldots\ldots\ldots\ldots(k+n)U(k+n)$$

5. If y(x)=$\frac{d^n}{dx^n}u(x)\frac{d^m}{dx^m}v(x)$ then its transformation

$$Y(k) = \sum_{r=0}^{k}\left[\prod_{i=1}^{n}(r+i)\right]\left[\prod_{j=1}^{m}(k-r+j)\right]U(r+n)V(k-r+m)$$

Proof: By def

$$Y(k) = \frac{1}{k!}\left[\frac{d^k y(x)}{dx^k}\right]_{x=0}$$

$$Y(k) = \frac{1}{k!}\frac{d^k}{dx^k}\left[\frac{d^n}{dx^n}u(x)\frac{d^m}{dx^m}v(x)\right]$$

$$\Rightarrow Y(k) = \frac{1}{k!}\sum_{r=0}^{k}\frac{k!}{r!(k-r)!}\frac{d^r}{dx^r}\left[\frac{d^n}{dx^n}u(x)\right]\frac{d^{k-r}}{dx^{k-r}}\left[\frac{d^m}{dx^m}v(x)\right]$$

$$Y(k) = \sum_{r=0}^{k}\left[\frac{1}{r!}\frac{d^{n+r}}{dx^{n+r}}u(x)\right]\left[\frac{1}{(k-r)!}\frac{d^{k-r+m}}{dx^{k-r+m}}v(x)\right]$$

$$\Rightarrow Y(k) = \sum_{r=0}^{k}\left[\prod_{i=1}^{n}(r+i)\right]\left[\prod_{j=1}^{m}(k-r+j)\right]\left[\frac{1}{(r+n)!}\frac{d^{n+r}}{dx^{n+r}}u(x)\right]\left[\frac{1}{(k-r+m)!}\frac{d^{k-r+m}}{dx^{k-r+m}}v(x)\right]$$

$$Y(k) = \sum_{r=0}^{k}\left[\prod_{i=1}^{n}(r+i)\right]\left[\prod_{j=1}^{m}(k-r+j)\right]U(r+n)V(k-r+m)$$

2.1 LAPLACE-PADE RESUMMATION

Laplace Pade Resummation is used to extend the solution's convergence domain or obtain an exact solution [26].

Laplace Pade Resummation is described as:

1. Apply the Laplace Transformation to the power series
2. Substitute t by 1/s in the obtained equation
3. We transform the series obtained from 2 into a meromorphic function by creating a Pade approximation of an order [N/M], where N and M are to be chosen randomly.
4. Now substitute t by 1/s in the obtained meromorphic function
5. Finally, by applying inverse Laplace transforms, we get an exact or approximate solution

2.3 IMPLEMENTATION

First, we apply DTM to the non-linear second-order ordinary differential equation of motion.

$$\ddot{x} + x^3 = 0$$

$$with\ initial\ conditions \quad x(0) = 1, \dot{x}(0) = 0$$

the solution as a series is

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$$\boldsymbol{x(t) = 1 - \frac{1}{2}t^2 + \frac{1}{8}t^4 - \frac{3}{80}t^6 + \frac{7}{640}t^8 - \frac{61}{19200}t^{10}}$$

Then we apply DTM to the first-order differential equation energy relation derived from the given non-linear second-order ordinary differential equation of motion by multiplying i.e.

$$(\dot{x})^2 + \frac{1}{2}x^4 = \frac{1}{2}$$

$$\textit{with initial conditions} \quad \boldsymbol{x(0) = 1, \dot{x}(0) = 0}$$

the solution as a series is

$$\boldsymbol{x(t) = 1 - \frac{1}{2}t^2 + \frac{1}{8}t^4 - \frac{3}{80}t^6 + \frac{7}{640}t^8 - \frac{61}{19200}t^{10}}$$

Now, to an approximate solution, we applied the Laplace Pade Resummation method to the series solution and got the solution as

$$\boldsymbol{L\{x(t)\} = \frac{16}{33s} + \frac{\gamma}{66\sqrt{67}}\left[\frac{s}{(s^2+\alpha^2)}\right] + \frac{\delta}{66\sqrt{67}}\left[\frac{s}{(s^2+\beta^2)}\right]}$$

Were, $\boldsymbol{\alpha^2 = (10 - \sqrt{67})}$, $\boldsymbol{\beta^2 = (10 + \sqrt{67}), \gamma = (137 + 17\sqrt{67}), \delta = (-137 + 17\sqrt{67})}$

III. RESULT

We first apply the Modified Differential Transform Method (MDTM) to the nonlinear second-order differential equation of motion:

$$\boldsymbol{\ddot{x} + x^3 = 0}$$

$$\textit{with initial conditions} \quad \boldsymbol{x(0) = 1, \dot{x}(0) = 0}$$

by theorems

- if y(x)=$\frac{d^n g(x)}{dx^n}$, then
  $Y(k) = (k+1)(k+2)(k+3)\dots(k+n)G(k+n)$
- if $\mathrm{y(x) = u(x)v(x)w(x)}$, then

$$Y(k) = \sum_{s=0}^{k}\sum_{m=0}^{k-s} U(s)V(m)W(k-s-m)$$

applying DTM, we have

$$\mathrm{(k+1)(k+2)X(k+2)} + \sum_{s=0}^{k}\sum_{m=0}^{k-s} X(s)X(m)X(k-s-m) = 0$$

$$\textit{with initial conditions} \quad X(0) = 1, X(1) = 0$$

By rearranging the Recurrence relation

$$X(k+2) = \frac{-1}{(k+1)(k+2)}\sum_{s=0}^{k}\sum_{m=0}^{k-s} X(s)X(m)X(k-s-m)$$

solving for k=0,1,2,3………

At k=0

$$X(2) = \frac{-1}{1.2}\sum_{s=0}^{0}\sum_{m=0}^{-s} X(s)X(m)X(-s-m)$$

$$\Rightarrow \boldsymbol{X(2) = \frac{-1}{2}}$$

At k=1

$$X(3) = \frac{-1}{2.3}\sum_{s=0}^{1}\sum_{m=0}^{1-s} X(s)X(m)X(1-s-m)$$

$$X(3) = \frac{-1}{6}\left\{\sum_{m=0}^{1} X(0)X(m)X(1-m) + \sum_{m=0} X(1)X(m)X(-m)\right\}$$

$$\Rightarrow \boldsymbol{X(3) = 0}$$

At k=2

$$X(4) = \frac{-1}{3.4}\sum_{s=0}^{2}\sum_{m=0}^{2-s} X(s)X(m)X(2-s-m)$$

$$X(4) = \frac{-1}{12}\left\{\sum_{m=0}^{2} X(0)X(m)X(2-m) + \sum_{m=0}^{1} X(1)X(m)X(1-m) + \sum_{m=0} X(2)X(m)X(-m)\right\}$$

$$X(4) = \frac{-1}{12}\{X(0)X(2) + X(1)X(1) + X(2)X(0) + 0 + X(2)\}$$

$$X(4) = \frac{-1}{12}\{X(2) + X(2) + X(2)\}$$

$$X(4) = \frac{-1}{12}\left\{\frac{-3}{2}\right\} \Rightarrow \boldsymbol{X(4) = \frac{1}{8}}$$

At k=3

$$X(5) = \frac{-1}{4.5}\sum_{s=0}^{3}\sum_{m=0}^{3-s} X(s)X(m)X(3-s-m)$$

$$X(5) = \frac{-1}{20}\{\textstyle\sum_{m=0}^{3} X(0)X(m)X(3-m) + \sum_{m=0}^{2} X(1)X(m)X(2-m) + \sum_{m=0}^{1} X(2)X(m)X(1-m) + \sum_{m=0} X(3)X(m)X(-m)\}$$

$$\Rightarrow \boldsymbol{X(5) = 0}$$

At k=4

$$X(6) = \frac{-1}{5.6}\sum_{s=0}^{4}\sum_{m=0}^{4-s} X(s)X(m)X(4-s-m)$$

$$X(6) = \frac{-1}{30}\{\textstyle\sum_{m=0}^{4} X(0)X(m)X(4-m) + \sum_{m}^{3} X(1)X(m)X(3-m) + \sum_{m=0}^{2} X(2)X(m)X(2-m) + \sum_{m=0}^{1} X(3)X(m)X(1-m) + \sum_{m=0} X(4)X(m)X(-m)\}$$

$$X(6) = \frac{-1}{30}\{X(4) + X(2)X(2) + X(4) + X(2)X(2) + X(2)X(2) + X(4)\}$$

$$X(6) = \frac{-1}{30}\left\{\frac{3}{8} + \frac{3}{4}\right\}$$

$$\Rightarrow \boldsymbol{X(6) = \frac{-3}{80}}$$

At k=5

$$X(7) = \frac{-1}{7.6}\sum_{s=0}^{5}\sum_{m=0}^{5-s} X(s)X(m)X(5-s-m)$$

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$X(7)=\frac{-1}{42}\{\sum_{m=0}^{5}X(0)X(m)X(5-m)+\sum_{m=0}^{4}X(1)X(m)X(4-m)+\sum_{m=0}^{3}X(2)X(m)X(3-m)+\sum_{m=0}^{2}X(3)X(m)X(2-m)+\sum_{m=0}^{1}X(4)X(m)X(1-m)+\sum_{m=0}X(5)X(m)X(-m)\}$

$$\Rightarrow \boldsymbol{X(7)=0}$$

At k=6

$$X(8)=\frac{-1}{8.7}\sum_{s=0}^{6}\sum_{m=0}^{6-s}X(s)X(m)X(6-s-m)$$

$X(8)=\frac{-1}{56}\{\sum_{m=0}^{6}X(0)X(m)(6-m)+\sum_{m=0}^{5}X(1)X(m)X(5-m)+\sum_{m=0}^{4}X(2)X(m)X(4-m)+\sum_{m=0}^{3}X(3)X(m)X(3-m)+\sum_{m=0}^{2}X(4)X(m)X(2-m)+\sum_{m=0}^{1}X(5)X(m)X(1-m)+\sum_{m=0}X(6)X(m)X(-m)\}$

$$X(8)=\frac{-1}{56}\{X(6)+X(2)X(4)+X(4)X(2)+X(6)+X(2)X(4)+X(2)X(2)X(2)+X(2)X(4)+X(4)X(2)+X(4)X(2)+X(6)\}$$

$$X(8)=\frac{-1}{56}\left\{-\frac{9}{80}-\frac{6}{16}+\frac{1}{8}\right\}$$

$$\Rightarrow \boldsymbol{X(8)=\frac{7}{640}}$$

At k=7

$$X(9)=\frac{-1}{9.8}\left\{\sum_{s=0}^{7}\sum_{m=0}^{7-s}X(s)X(m)X(7-s-m)\right\}$$

$X(9)=\frac{-1}{72}\{\sum_{m=0}^{7}X(0)X(m)X(7-m)+\sum_{m=0}^{6}X(1)X(m)X(6-m)+\sum_{m=0}^{5}X(2)X(m)X(5-m)+\sum_{m=0}^{4}X(3)X(m)X(4-m)+\sum_{m=0}^{3}X(4)X(m)X(3-m)+\sum_{m=0}^{2}X(5)X(m)X(2-m)+\sum_{m=0}^{1}X(6)X(m)X(1-m)+\sum_{m=0}X(7)X(m)X(-m)\}$

$$\Rightarrow \boldsymbol{X(9)=0}$$

At k=8

$$X(10)=\frac{-1}{10.9}\sum_{s=0}^{8}\sum_{m=0}^{8-s}X(s)X(m)X(8-s-m)$$

$X(10)=\frac{-1}{90}\{\sum_{m=0}^{8}X(0)X(m)X(8-m)+\sum_{m=0}^{7}X(1)X(m)X(7-m)+\sum_{m=0}^{6}X(2)X(m)X(6-m)+\sum_{m=0}^{5}X(3)X(m)X(5-m)+\sum_{m=0}^{4}X(4)X(m)X(4-m)+\sum_{m=0}^{3}X(5)X(m)X(3-m)+\sum_{m=0}^{2}X(6)X(m)X(2-m)+\sum_{m=0}^{1}X(7)X(m)X(1-m)+\sum_{m=0}X(8)X(m)X(-m)\}$

$$X(10)=\frac{-1}{90}\{X(8)+X(2)X(6)+X(4)X(4)+X(6)X(2)+X(8)+X(2)X(6)+X(2)X(2)X(4)+X(2)X(4)X(2)+X(2)X(6)+X(4)X(4)+X(4)X(2)X(2)+X(4)X(4)+X(6)X(2)+X(6)X(2)+X(8)\}$$

$$X(10)=\frac{-1}{90}\left\{\frac{21}{640}+\frac{18}{160}+\frac{3}{64}+\frac{3}{32}\right\}$$

$$\Rightarrow \boldsymbol{X(10)=\frac{-61}{19200}}$$

Now the solution as a series is

$$\boldsymbol{x(t)=1-\frac{1}{2}t^2+\frac{1}{8}t^4-\frac{3}{80}t^6+\frac{7}{640}t^8-\frac{61}{19200}t^{10}}$$

The first-order differential equation energy relation is derived from the given non-linear second-order ordinary differential equation of motion by multiplying "$2\dot{x}$"
i.e.

$$\boldsymbol{(\dot{x})^2+\frac{1}{2}x^4=\frac{1}{2}}$$

with initial conditions $\boldsymbol{x(0)=1,\dot{x}(0)=0}$

by theorems

- If y(x)=u(x)v(x) then its transformation

$$Y(k)=\sum_{s=0}^{k}U(s)V(k-s)$$

- If y(x)=x^n then its transformation

$$Y(k)=\delta(k-n)=\begin{cases}1 \text{ if } k=n\\0 \text{ if } k\neq n\end{cases}$$

- If y(x)=$\frac{d^n}{dx^n}u(x)\frac{d^m}{dx^m}v(x)$ then its transformation

$$Y(k)=\sum_{r=0}^{k}[\prod_{i=1}^{n}(r+i)][\prod_{j=1}^{m}(k-r+j]U(r+n)V(k-r+m)$$

Applying DTM

$$\sum_{r=0}^{k}(r+1)(k-r+1)X(r+1)X(k-r+1)=\frac{1}{2}\delta(k)-\frac{1}{2}\sum_{s=0}^{k}\sum_{m=0}^{k-s}\sum_{n=0}^{k-s-m}X(s)X(m)X(n)X(k-s-m-n)$$

Where, δ(k-n) =$\begin{cases}1 \text{ if } k=n\\0 \text{ if } k\neq n\end{cases}$

With initial conditions X(0)=1, X(1)=0

Solving for k=2,3,4…….

At k=2

$$\sum_{r=0}^{2}(r+1)(3-r)X(r+1)X(3-r)=\frac{1}{2}\delta(2)-\frac{1}{2}\sum_{s=0}^{2}\sum_{m=0}^{2-s}\sum_{n=0}^{2-s-m}X(s)X(m)X(n)X(2-s-m-n)$$

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$$4(X(2))^2 = 0 - \frac{1}{2}[\sum_{m=0}^{2}\sum_{n=0}^{2-m} X(0)X(m)X(n)X(2-m-n) + \sum_{m=0}^{1}\sum_{n=0}^{1-m} X(1)X(m)X(n)X(1-m-n) + \sum_{m=0}\sum_{n=0} X(2)X(m)X(n)X(-m-n)]$$

$$4(X(2))^2 = \frac{1}{2}[\sum_{n=0}^{2} X(0)X(n)X(2-n) + \sum_{n=0}^{1} X(1)X(n)X(1-n) + \sum_{n=0} X(2)X(n)X(-n) + X(2)]$$

$$4(X(2))^2 = \frac{1}{2}[X(0)X(2) + X(1)X(1) + X(2)X(0) + 0 + X(2) + X(2)]$$

$$4\left(X(2)\right)^2 + 2X(2) = 0$$

$$2X(2)(2X(2)+1) = 0 \quad \Rightarrow \boldsymbol{X(2) = -1/2}$$

At k=3

$$\sum_{r=0}^{3}(r+1)(4-r)X(r+1)X(4-r) = \frac{1}{2}\delta(3) - \frac{1}{2}\sum_{s=0}^{3}\sum_{m=0}^{3-s}\sum_{n=0}^{3-s-m} X(s)X(m)X(n)X(3-s-m-n)$$

$$4X(1)X(4) + 6X(2)X(3) + 6X(3)X(2) + 4X(4)X(1) = -\frac{1}{2}\{\sum_{m=0}^{3}\sum_{n=0}^{3-m} X(0)X(m)X(n)X(3-m-n) + \sum_{m=0}^{2}\sum_{n=0}^{2-m} X(1)X(m)X(n)X(2-m-n) + \sum_{m=0}^{1}\sum_{n=0}^{1-m} X(2)X(m)X(n)X(1-m-n) + \sum_{m=0}\sum_{n=0} X(3)X(m)X(n)X(-m-n)\}$$

$$-6X(3) = -\frac{1}{2}\Big\{\sum_{n=0}^{3} X(0)X(n)X(3-n) + \sum_{n=0}^{2} X(1)X(n)X(2-n) + \sum_{n=0}^{1} X(2)X(n)X(1-n) + \sum_{n=0} X(3)X(n)X(-n) - \frac{1}{2}(\sum_{n=0}^{1} X(0)X(n)X(1-n) + \sum_{n=0} X(1)X(n)X(-n)) + X(3)\Big\}$$

$$-6X(3) = -\frac{1}{2}\{X(3) + X(3) + X(3) + X(3)\}$$

$$6X(3) + 2X(3) = 0 \ \Rightarrow \boldsymbol{X(3) = 0}$$

At k=4

$$\sum_{r=0}^{4}(r+1)(5-r)X(r+1)X(5-r) = \frac{1}{2}\delta(4) - \frac{1}{2}\sum_{s=0}^{4}\sum_{m=0}^{4-s}\sum_{n=0}^{4-s-m} X(s)X(m)X(n)X(4-s-m-n)$$

$$16X(2)X(4) = -\frac{1}{2}\{\sum_{m=0}^{4}\sum_{n=0}^{4-m} X(0)X(m)X(n)X(4-m-n) + \sum_{m=0}^{3}\sum_{n=0}^{3-m} X(1)X(m)X(n)X(3-m-n) + \sum_{m=0}^{2}\sum_{n=0}^{2-m} X(2)X(m)X(n)X(2-m-n) + \sum_{m=0}^{1}\sum_{n=0}^{1-m} X(3)X(m)X(n)X(3-m-n) + \sum_{m=0}\sum_{n=0} X(4)X(m)X(n)X(-m-n)\}$$

$$8X(4) = \frac{1}{2}\Big\{\sum_{n=0}^{4} X(0)X(n)X(4-n) + \sum_{n}^{3} X(1)X(n)X(3-n) + \sum_{n=0}^{2} X(2)X(n)X(2-n) + \sum_{n=0}^{1} X(3)X(n)X(1-n) + \sum_{n=0} X(4)X(n)X(-n) - \frac{1}{2}\{\sum_{n=0}^{2} X(0)X(n)X(2-n) + \sum_{n=0}^{1} X(1)X(n)X(1-n) + \sum_{n=0} X(2)X(n)X(-n)\} + X(4)\Big\}$$

$$8X(4) = \frac{1}{2}\Big\{X(4) + X(2)X(2) + X(4) + X(2)X(2) + X(2)X(2) + X(4) - \frac{1}{2}[X(2) + X(2) + X(2)] + X(4)\Big\}$$

$$16X(4) = 4X(4) + \frac{6}{4} \Rightarrow \boldsymbol{X(4) = 1/8}$$

At k=5

$$\sum_{r=0}^{5}(r+1)(6-r)X(r+1)X(6-r) = \frac{1}{2}\delta(5) - \frac{1}{2}\sum_{s=0}^{5}\sum_{m=0}^{5-s}\sum_{n=0}^{5-s-m} X(s)X(m)X(n)X(5-s-m-n)$$

$$10X(2)X(5) + 10X(5)X(2) = -\frac{1}{2}\{\sum_{m=0}^{5}\sum_{n=0}^{5-m} X(0)X(m)X(n)X(5-m-n) + \sum_{m=0}^{4}\sum_{n=0}^{4-m} X(1)X(m)X(n)X(4-m-n) + \sum_{m=0}^{3}\sum_{n=0}^{3-m} X(2)X(m)X(n)X(3-m-n) + \sum_{m=0}^{2}\sum_{n=0}^{2-m} X(3)X(m)X(n)X(2-m-n) + \sum_{m=0}^{1}\sum_{n=0}^{1-m} X(4)X(m)X(n)X(4-m-n) + \sum_{m=0}\sum_{n=0} X(5)X(m)X(n)X(-m-n)\}$$

$$-10X(5) = -\frac{1}{2}\Big\{\sum_{n=0}^{5} X(0)X(n)X(5-n) + \sum_{n=0}^{4} X(1)X(n)X(4-n) + \sum_{n=0}^{3} X(2)X(n)X(3-n) + \sum_{n=0}^{2} X(3)X(n)X(2-n) + \sum_{n=0}^{1} X(4)X(n)X(1-n) + \sum_{n=0} X(5)X(n)X(-n) - \frac{1}{2}[\sum_{n=0}^{3} X(0)X(n)X(3-n) + \sum_{n=0}^{2} X(1)X(n)X(2-n) + \sum_{n=0}^{1} X(2)X(n)X(1-n) + \sum_{n=0} X(3)X(n)X(-n)] + \frac{1}{8}[\sum_{n=0}^{1} X(0)X(n)X(1-n) + \sum_{n=0} X(1)X(n)X(-n)] + X(5)\Big\}$$

$$20X(5) = X(5) + X(5) + X(5) + X(5)$$

$$\Rightarrow \boldsymbol{X(5) = 0}$$

At k=6

$$\sum_{r=0}^{6}(r+1)(7-r)X(r+1)X(7-r) = \frac{1}{2}\delta(6) - \frac{1}{2}\sum_{s=0}^{6}\sum_{m=0}^{6-s}\sum_{n=0}^{6-s-m} X(s)X(m)X(n)X(6-s-m-n)$$

$$12X(2)X(6) + 16X(4)X(4) + 12X(2)X(6) = -\frac{1}{2}\{\sum_{m=0}^{6}\sum_{n=0}^{6-m} X(0)X(m)X(n)X(6-m-n) + \sum_{m=0}^{5}\sum_{n=0}^{5-m} X(1)X(m)X(n)X(5-m-n) + \sum_{m=0}^{4}\sum_{n=0}^{4-m} X(2)X(m)X(n)X(4-m-n) + \sum_{m=0}^{3}\sum_{n=0}^{3-m} X(3)X(m)X(n)X(3-m-n) + \sum_{m=0}^{2}\sum_{n=0}^{2-m} X(4)X(m)X(n)X(2-m-n) + \sum_{m=0}^{1}\sum_{n=0}^{1-m} X(5)X(m)X(n)X(5-m-n) + \sum_{m=0}\sum_{n=0} X(6)X(m)X(n)X(-m-n)\}$$

$$24X(6) - \frac{1}{2} = \sum_{n=0}^{6} X(0)X(n)(6-n) + \sum_{n=0}^{5} X(1)X(n)X(5-n) + \sum_{n=0}^{4} X(2)X(n)X(4-n) + \sum_{n=0}^{3} X(3)X(n)X(3-n) + \sum_{n=0}^{2} X(4)X(n)X(2-n) + \sum_{n=0}^{1} X(5)X(n)X(1-n) + \sum_{n=0} X(6)X(n)X(-n) - \frac{1}{2}[\sum_{n=0}^{4} X(0)X(n)X(4-n) + \sum_{n}^{3} X(1)X(n)X(3-n) + \sum_{n=0}^{2} X(2)X(n)X(2-n) + \sum_{n=0}^{1} X(3)X(n)X(1-n) + \sum_{n=0} X(4)X(n)X(-n)] + \frac{1}{8}[\sum_{n=0}^{2} X(0)X(n)X(2-n) + \sum_{n=0}^{1} X(1)X(n)X(1-n) + \sum_{n=0} X(2)X(n)X(-n)] + X(6)$$

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$$24X(6) - \tfrac{1}{2} = X(6) + X(2)X(4) + X(4)X(2) + X(6) + X(2)X(4) + X(2)X(2)X(2) + X(2)X(4) + X(4)X(2) + X(4)X(2) + X(6) - \tfrac{1}{2}[X(4) + X(2)X(2) + X(4) + X(2)X(2) + X(2)X(2) + X(4)] + \tfrac{1}{8}[X(2) + X(2) + X(2)] + X(6)$$

$$\Rightarrow \boldsymbol{X(6) = -3/80}$$

At k=7

$$\sum_{r=0}^{7}(r+1)(8-r)X(r+1)X(8-r) = \frac{1}{2}\delta(7) - \frac{1}{2}\sum_{s=0}^{7}\sum_{m=0}^{7-s}\sum_{n=0}^{7-s-m} X(s)X(m)X(n)X(7-s-m-n)$$

$$14X(2)X(7) + 14X(7)X(2) = -\tfrac{1}{2}\{\textstyle\sum_{m=0}^{7}\sum_{n=0}^{7-m} X(0)X(m)X(n)X(7-m-n) + \sum_{m=0}^{6}\sum_{n=0}^{6-m} X(1)X(m)X(n)X(6-m-n) + \sum_{m=0}^{5}\sum_{n=0}^{5-m} X(2)X(m)X(n)X(5-m-n) + \sum_{m=0}^{4}\sum_{n=0}^{4-m} X(3)X(m)X(n)X(4-m-n) + \sum_{m=0}^{3}\sum_{n=0}^{3-m} X(4)X(m)X(n)X(3-m-n) + \sum_{m=0}^{2}\sum_{n=0}^{2-m} X(5)X(m)X(n)X(2-m-n) + \sum_{m=0}^{1}\sum_{n=0}^{1-m} X(6)X(m)X(n)X(1-m-n) + \sum_{m=0}\sum_{n=0} X(7)X(m)X(n)X(-m-n)\}$$

$$28X(7) = \textstyle\sum_{n=0}^{7} X(0)X(n)X(7-n) + \sum_{n=0}^{6} X(1)X(n)X(6-n) + \sum_{n=0}^{5} X(2)X(n)X(5-n) + \sum_{n=0}^{4} X(3)X(n)X(4-n) + \sum_{n=0}^{3} X(4)X(n)X(3-n) + \sum_{n=0}^{2} X(5)X(n)X(2-n) + \sum_{n=0}^{1} X(6)X(n)X(1-n) + \sum_{n=0} X(7)X(n)X(-n) + X(2)[\sum_{n=0}^{5} X(0)X(n)X(5-n) + \sum_{n=0}^{4} X(1)X(n)X(4-n) + \sum_{n=0}^{3} X(2)X(n)X(3-n) + \sum_{n=0}^{2} X(3)X(n)X(2-n) + \sum_{n=0}^{1} X(4)X(n)X(1-n) + \sum_{n=0} X(5)X(n)X(-n)] + X(4)[\sum_{n=0}^{3} X(0)X(n)X(3-n) + \sum_{n=0}^{2} X(1)X(n)X(2-n) + \sum_{n=0}^{1} X(2)X(n)X(1-n) + \sum_{n=0} X(3)X(n)X(-n)] + X(6)[\sum_{n=0}^{1} X(0)X(n)X(1-n) + \sum_{n=0} X(1)X(n)X(-n)] + X(7)$$

$$28X(7) = X(7) + X(7) + X(7)$$

$$\Rightarrow \boldsymbol{X(7) = 0}$$

At k=8

$$\sum_{r=0}^{8}(r+1)(9-r)X(r+1)X(9-r) = \frac{1}{2}\delta(8) - \frac{1}{2}\sum_{s=0}^{8}\sum_{m=0}^{8-s}\sum_{n=0}^{8-s-m} X(s)X(m)X(n)X(8-s-m-n)$$

$$16X(2)X(8) + 24X(4)X(6) + 24X(6)X(4) + 16X(8)X(2) = -\tfrac{1}{2}\{\textstyle\sum_{m=0}^{8}\sum_{n=0}^{8-m} X(0)X(m)X(n)X(8-m-n) + \sum_{m=0}^{7}\sum_{n=0}^{7-m} X(1)X(m)X(n)X(7-m-n) + \sum_{m=0}^{6}\sum_{n=0}^{6-m} X(2)X(m)X(n)X(6-m-n) + \sum_{m=0}^{5}\sum_{n=0}^{5-m} X(3)X(m)X(n)X(5-m-n) + \sum_{m=0}^{4}\sum_{n=0}^{4-m} X(4)X(m)X(n)X(4-m-n) + \sum_{m=0}^{3}\sum_{n=0}^{3-m} X(5)X(m)X(n)X(3-m-n) + \sum_{m=0}^{2}\sum_{n=0}^{2-m} X(6)X(m)X(n)X(2-m-n) + \sum_{m=0}^{1}\sum_{n=0}^{1-m} X(7)X(m)X(n)X(1-m-n) + \sum_{m=0}\sum_{n=0} X(8)X(m)X(n)X(-m-n)\}$$

$$32X(8) + \tfrac{9}{20} = \textstyle\sum_{n=0}^{8} X(0)X(n)X(8-n) + \sum_{n=0}^{7} X(1)X(n)X(7-n) + \sum_{n=0}^{6} X(2)X(n)X(6-n) + \sum_{n=0}^{5} X(3)X(n)X(5-n) + \sum_{n=0}^{4} X(4)X(n)X(4-n) + \sum_{n=0}^{3} X(5)X(n)X(3-n) + \sum_{n=0}^{2} X(6)X(n)X(2-n) + \sum_{n=0}^{1} X(7)X(n)X(1-n) + \sum_{n=0} X(8)X(n)X(-n) + X(2)[\sum_{n=0}^{6} X(0)X(n)(6-n) + \sum_{n=0}^{5} X(1)X(n)X(5-n) + \sum_{n=0}^{4} X(2)X(n)X(4-n) + \sum_{n=0}^{3} X(3)X(n)X(3-n) + \sum_{n=0}^{2} X(4)X(n)X(2-n) + \sum_{n=0}^{1} X(5)X(n)X(1-n) + \sum_{n=0} X(6)X(n)X(-n)] + X(4)[\sum_{n=0}^{4} X(0)X(n)X(4-n) + \sum_{n}^{3} X(1)X(n)X(3-n) + \sum_{n=0}^{2} X(2)X(n)X(2-n) + \sum_{n=0}^{1} X(3)X(n)X(1-n) + \sum_{n=0} X(4)X(n)X(-n)] + X(6)[\sum_{n=0}^{2} X(0)X(n)X(2-n) + \sum_{n=0}^{1} X(1)X(n)X(1-n) + \sum_{n=0} X(2)X(n)X(-n)] + X(8)$$

$$32X(8) + \tfrac{9}{20} = X(8) + X(2)X(6) + X(4)X(4) + X(6)X(2) + X(8) + 2\big(X(2)X(6) + X(2)X(2)X(4) + X(2)X(2)X(4) + X(2)X(6)\big) + 2\begin{pmatrix} X(4)X(4) + \\ X(4)X(2)X(2) + X(4)X(4) \end{pmatrix} + 2\big(X(6)X(2) + X(6)X(2)\big) + X(2)X(2)X(4) + X(2)X(2)X(2)X(2) + X(2)X(2)X(4) + 2(X(2)X(2)X(4) + X(2)X(2)X(4)) + X(4)X(4) + 2X(6)X(2) + X(8) + X(8)$$

$$28X(8) + \tfrac{9}{20} = \tfrac{121}{160} \Longrightarrow 28X(8) = 49/160$$

$$\Rightarrow \boldsymbol{X(8) = 7/640}$$

At k=9

$$\sum_{r=0}^{9}(r+1)(10-r)X(r+1)X(10-r) = \frac{1}{2}\delta(9) - \frac{1}{2}\sum_{s=0}^{9}\sum_{m=0}^{9-s}\sum_{n=0}^{9-s-m} X(s)X(m)X(n)X(9-s-m-n)$$

$$18X(2)X(9) + 18X(9)X(2) = -\tfrac{1}{2}\{\textstyle\sum_{m=0}^{9}\sum_{n=0}^{9-m} X(0)X(m)X(n)X(9-m-n) + \sum_{m=0}^{8}\sum_{n=0}^{8-m} X(1)X(m)X(n)X(8-m-n) + \sum_{m=0}^{7}\sum_{n=0}^{7-m} X(2)X(m)X(n)X(7-m-n) + \sum_{m=0}^{6}\sum_{n=0}^{6-m} X(3)X(m)X(n)X(6-m-n) + \sum_{m=0}^{5}\sum_{n=0}^{5-m} X(4)X(m)X(n)X(5-m-n) + \sum_{m=0}^{4}\sum_{n=0}^{4-m} X(5)X(m)X(n)X(4-m-n) + \sum_{m=0}^{3}\sum_{n=0}^{3-m} X(6)X(m)X(n)X(3-m-n) + \sum_{m=0}^{2}\sum_{n=0}^{2-m} X(7)X(m)X(n)X(2-m-n) + \sum_{m=0}^{1}\sum_{n=0}^{1-m} X(8)X(m)X(n)X(1-m-n) + \sum_{m=0}\sum_{n=0} X(9)X(m)X(n)X(-m-n)\}$$

$$36X(9) = \textstyle\sum_{n=0}^{9} X(0)X(n)X(9-n) + \sum_{n=0}^{8} X(1)X(n)X(8-n) + \sum_{n=0}^{7} X(2)X(n)X(7-n) + \sum_{n=0}^{6} X(3)X(n)X(6-n) + \sum_{n=0}^{5} X(4)X(n)X(5-n) + \sum_{n=0}^{4} X(5)X(n)X(4-n) + \sum_{n=0}^{3} X(6)X(n)X(3-n) + \sum_{n=0}^{2} X(7)X(n)X(2-n) + \sum_{n=0}^{1} X(8)X(n)X(1-n) + \sum_{n=0} X(9)X(n)X(-n) + X(2)[\sum_{n=0}^{7} X(0)X(n)X(7-n) + \sum_{n=0}^{6} X(1)X(n)X(6-n) + \sum_{n=0}^{5} X(2)X(n)X(5-n) + \sum_{n=0}^{4} X(3)X(n)X(4-n) + \sum_{n=0}^{3} X(4)X(n)X(3-n) + \sum_{n=0}^{2} X(5)X(n)X(2-n) + \sum_{n=0}^{1} X(6)X(n)X(1-n) +$$

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$\sum_{n=0} X(7)X(n)X(-n)] + X(4)[\sum_{n=0}^{5} X(0)X(n)X(5-n) + \sum_{n=0}^{4} X(1)X(n)X(4-n) + \sum_{n=0}^{3} X(2)X(n)X(3-n) + \sum_{n=0}^{2} X(3)X(n)X(2-n) + \sum_{n=0}^{1} X(4)X(n)X(1-n) + \sum_{n=0} X(5)X(n)X(-n)] + X(6)[\sum_{n=0}^{3} X(0)X(n)X(3-n) + \sum_{n=0}^{2} X(1)X(n)X(2-n) + \sum_{n=0}^{1} X(2)X(n)X(1-n) + \sum_{n=0} X(3)X(n)X(-n)] + X(8)[\sum_{n=0}^{1} X(0)X(n)X(1-n) + \sum_{n=0} X(1)X(n)X(-n)] + X(9)$

$36X(9) = X(9) + X(9) + X(9) + X(9)$

$$\Rightarrow \boldsymbol{X(9) = 0}$$

At k=10

$$\sum_{r=0}^{10}(r+1)(11-r)X(r+1)X(11-r) = \frac{1}{2}\delta(10) - \frac{1}{2}\sum_{s=0}^{10}\sum_{m=0}^{10-s}\sum_{n=0}^{10-s-m} X(s)X(m)X(n)X(10-s-m-n)$$

$20X(2)X(10) + 32X(4)X(8) + 36X(6)X(6) + 32X(8)X(4) + 20X(10)X(2) = -\frac{1}{2}\{\sum_{m=0}^{10}\sum_{n=0}^{10-m} X(0)X(m)X(n)X(10-m-n) + \sum_{m=0}^{9}\sum_{n=0}^{9-m} X(1)X(m)X(n)X(9-m-n) + \sum_{m=0}^{8}\sum_{n=0}^{8-m} X(2)X(m)X(n)X(8-m-n) + \sum_{m=0}^{7}\sum_{n=0}^{7-m} X(3)X(m)X(n)X(7-m-n) + \sum_{m=0}^{6}\sum_{n=0}^{6-m} X(4)X(m)X(n)X(6-m-n) + \sum_{m=0}^{5}\sum_{n=0}^{5-m} X(5)X(m)X(n)X(5-m-n) + \sum_{m=0}^{4}\sum_{n=0}^{4-m} X(6)X(m)X(n)X(4-m-n) + \sum_{m=0}^{3}\sum_{n=0}^{3-m} X(7)X(m)X(n)X(3-m-n) + \sum_{m=0}^{2}\sum_{n=0}^{2-m} X(8)X(m)X(n)X(2-m-n) + \sum_{m=0}^{1}\sum_{n=0}^{1-m} X(9)X(m)X(n)X(1-m-n) + \sum_{m=0}\sum_{n=0} X(10)X(m)X(n)X(-m-n)\}$

$-20X(10) + \frac{7}{320} + \frac{81}{1600} = -\frac{1}{2}\{\sum_{n=0}^{10} X(0)X(n)X(10-n) + \sum_{n=0}^{9} X(1)X(n)X(9-n) + \sum_{n=0}^{8} X(2)X(n)X(8-n) + \sum_{n=0}^{7} X(3)X(n)X(7-n) + \sum_{n=0}^{6} X(4)X(n)X(6-n) + \sum_{n=0}^{5} X(5)X(n)X(5-n) + \sum_{n=0}^{4} X(6)X(n)X(4-n) + \sum_{n=0}^{3} X(7)X(n)X(3-n) + \sum_{n=0}^{2} X(8)X(n)X(2-n) + \sum_{n=0}^{1} X(9)X(n)X(1-n) + \sum_{n=0} X(10)X(n)X(-n)\} + X(2)[\sum_{n=0}^{8} X(0)X(n)X(8-n) + \sum_{n=0}^{7} X(1)X(n)X(7-n) + \sum_{n=0}^{6} X(2)X(n)X(6-n) + \sum_{n=0}^{5} X(3)X(n)X(5-n) + \sum_{n=0}^{4} X(4)X(n)X(4-n) + \sum_{n=0}^{3} X(5)X(n)X(3-n) + \sum_{n=0}^{2} X(6)X(n)X(2-n) + \sum_{n=0}^{1} X(7)X(n)X(1-n) + \sum_{n=0} X(8)X(n)X(-n)] + X(4)[\sum_{n=0}^{6} X(0)X(n)(6-n) + \sum_{n=0}^{5} X(1)X(n)X(5-n) + \sum_{n=0}^{4} X(2)X(n)X(4-n) + \sum_{n=0}^{3} X(3)X(n)X(3-n) + \sum_{n=0}^{2} X(4)X(n)X(2-n) + \sum_{n=0}^{1} X(5)X(n)X(1-n) + \sum_{n=0} X(6)X(n)X(-n)] + X(6)[\sum_{n=0}^{4} X(0)X(n)X(4-n) + \sum_{n}^{3} X(1)X(n)X(3-n) + \sum_{n=0}^{2} X(2)X(n)X(2-n) + \sum_{n=0}^{1} X(3)X(n)X(1-n) + \sum_{n=0} X(4)X(n)X(-n)] + X(8)[\sum_{n=0}^{2} X(0)X(n)X(2-n) + \sum_{n=0}^{1} X(1)X(n)X(1-n) + \sum_{n=0} X(2)X(n)X(-n)] + X(10)$

$-20X(10) + \frac{7}{320} + \frac{81}{1600} = -\frac{1}{2}\{X(10) + X(2)X(8) + X(4)X(6) + X(6)X(4) + X(8)X(2) + X(10) + X(2)X(8) + X(2)X(2)X(6) + X(2)X(4)X(4) + X(2)X(6)X(2) + X(2)X(8) + X(4)X(6) + X(4)X(2)X(4) + X(4)X(4)X(2) + X(6)X(4) + X(6)X(4) + X(6)X(2)X(2) + X(6)X(4) + X(8)X(2) + X(8)X(2) + X(10) + X(2)[X(8) + X(2)X(6) + X(4)X(4) + X(6)X(2) + X(8) + X(2)X(6) + X(2)X(2)X(4) + X(2)X(4)X(2) + X(2)X(6) + X(4)X(4) + X(4)X(2)X(2) + X(4)X(4) + X(6)X(2) + X(6)X(2) + X(8)] + X(4)[X(6) + X(2)X(4) + X(4)X(2) + X(6) + X(2)X(4) + X(2)X(2)X(2) + X(2)X(4) + X(4)X(2) + X(4)X(2) + X(6)] + X(6)[X(4) + X(2)X(2) + X(4) + X(2)X(2) + X(2)X(2) + X(4)] + X(8)[X(2) + X(2) + X(2)] + X(10)\}$

$$\Rightarrow \boldsymbol{X(10) = -61/19200}$$

Now the solution as a series is

$$\boldsymbol{x(t) = 1 - \frac{1}{2}t^2 + \frac{1}{8}t^4 - \frac{3}{80}t^6 + \frac{7}{640}t^8 - \frac{61}{19200}t^{10}}$$

Applying Laplace-Padé Resummation to obtain a series solution

Taking the Laplace transform of the series

$$L\{x(t)\} = \frac{1}{s} - \frac{1}{s^3} + \frac{3}{s^5} - \frac{27}{s^7} + \frac{441}{s^9} - \frac{11529}{s^{11}}$$

Substituting s as 1/t

$$L\{x(t)\} = t - t^3 + 3t^5 - 27t^7 + 441t^9 - 11529t^{11}$$

The Pade approximant [5/5]

$$\frac{a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5}{1 + b_1 t + b_2 t^2 + b_3 t^3 + b_4 t^4 + b_5 t^5} = t - t^3 + 3t^5 - 27t^7 + 441t^9 - 11529t^{11}$$

the coefficients of the numerator and denominator are obtained by the system of equations

(t^0): $a_0 = 0$, (t^1): $a_1 = 1$, (t^2): $a_2 = b_1$, (t^3): $a_3 = -1 + b_2$, (t^4): $a_4 = -b_1 + b_3$, (t^5): $a_5 = 3 - b_2 + b_4$, (t^6): $0 = 3b_1 - b_3 + b_5$, (t^7): $0 = -27 + 3b_2 - b_4$, (t^8): $0 = -27b_1 + 3b_3 - b_5$, (t^9): $0 = 441 - 27b_2 + 3b_4$, (t^{10}): $0 = 441b_1 - 27b_3 + 3b_5$

$$a_0 = a_2 = a_4 = b_1 = b_3 = b_5 = 0, a_1 = 1, a_3 = 19, a_5 = 16, b_2 = 20, b_4 = 33$$

Therefore,

$$\Rightarrow t - t^3 + 3t^5 - 27t^7 + 441t^9 - 11529t^{11} = \frac{1 + 19t^3 + 16t^5}{1 + 20t^2 + 33t^4}$$

Again, substituting t as 1/s

$$L\{x(t)\} = \frac{16 + 19s^2 + s^4}{33s + 20s^3 + s^5}$$

$$L\{x(t)\} = \frac{16}{33s} + \frac{17s^3 + 307s}{33(s^4 + 20s^2 + 33)}$$

$$L\{x(t)\} = \frac{16}{33s} + \frac{17s^3 + 307s}{33\left(s^4 + (10-\sqrt{67})s^2 + (10+\sqrt{67})s^2 + (10-\sqrt{67})(10+\sqrt{67})\right)}$$

Let $\alpha^2 = (10 - \sqrt{67})$, $\beta^2 = (10 + \sqrt{67})$

$$L\{x(t)\} = \frac{16}{33s} + \frac{17s^3 + 307s}{33(s^4 + \alpha^2 s^2 + \beta^2 s^2 + \alpha^2\beta^2)}$$

$$L\{x(t)\} = \frac{16}{33s} + \frac{17s^3 + 307s}{33(s^2 + \alpha^2)(s^2 + \beta^2)}$$

$$L\{x(t)\} = \frac{16}{33s} + \frac{1}{66\sqrt{67}}\left[\frac{(137 + 17\sqrt{67})s}{(s^2 + \alpha^2)} + \frac{(-137 + 17\sqrt{67})s}{(s^2 + \beta^2)}\right]$$

Let $\gamma = (137 + 17\sqrt{67})$, $\delta = (-137 + 17\sqrt{67})$

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$$L\{x(t)\}=\frac{16}{33s}+\frac{\gamma}{66\sqrt{67}}\left[\frac{s}{(s^2+\alpha^2)}\right]+\frac{\delta}{66\sqrt{67}}\left[\frac{s}{(s^2+\beta^2)}\right]$$

Taking inverse Laplace Transforms

$$\boldsymbol{x(t)=\frac{16}{33}+\frac{\gamma}{66\sqrt{67}}\cos\alpha t+\frac{\delta}{66\sqrt{67}}\cos\beta t}$$

Now Checking the Satisfaction of Initial Conditions

At t=0

$$x(0)=\frac{16}{33}+\frac{(137+17\sqrt{67})}{66\sqrt{67}}+\frac{(-137+17\sqrt{67})}{66\sqrt{67}}$$

$$x(0)=1$$

Differentiating and putting t=0

$$\dot{x}(0)=-\frac{(137+17\sqrt{67})}{\boldsymbol{66\sqrt{67}}}\boldsymbol{\sin 0}-\frac{(-137+17\sqrt{67})}{\boldsymbol{66\sqrt{67}}}\boldsymbol{\sin 0}$$

$$\dot{x}(0)=0$$

Remarks

- The series solution obtained from the energy relation matches exactly the series from **the** equation of motion, confirming the consistency of the method.
- Laplace–Padé resummation effectively extends the **convergence domain**, making the solution usable for larger time intervals.
- This illustrates the effectiveness of MDTM combined **with Laplace–Padé** for both second-order motion equations and first-order energy relations.

IV. CONCLUSION

In this study, a semi-analytical approach combining the Modified Differential Transform Method (MDTM) and Laplace–Padé resummation was successfully applied to a nonlinear oscillator. Both the second-order equation of motion and the first-order energy relation were solved, and the resulting series solutions were found to be identical, confirming the consistency and reliability of the method.

The application of Laplace–Padé resummation significantly extended the convergence domain of the series solution, providing accurate approximations over a wider time interval. Moreover, solving the energy relation directly ensures inherent preservation of total energy, highlighting a distinct advantage of this approach for conservative systems.

This work demonstrates that MDTM, when combined with Laplace–Padé resummation, offers an effective and flexible tool for analyzing nonlinear oscillatory systems. By applying the method directly to the energy relation, a new perspective is introduced that can be extended to other conservative systems, providing both theoretical insights and practical computational benefits.

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