Edge-Odd Gracefulness Of The Wheel Graph $W_{1,n}$

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Abstract

A (p,q)- connected graph is an edge-odd graceful graph if there exists an injective map $f : E(G) \rightarrow \{1,3,5,...,2q-1\}$ so that the induced map $f^+ : V(G) \rightarrow \{0,1,2,3,...2k-1\}$ defined by $f^+(x) \equiv \sum_{xy \in E(G)} f(xy) \pmod{2k}$ where $k = max\{p,q\}$ makes all edges distinct and odd. In this paper the edge-odd gracefulness of the wheel graph $W_{1,n}$ is obtained for $n \geq 4$.

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1. Introduction

All the graphs considered in this paper are simple, undirected, finite and connected. The terms not defined here may be found in [1]. A vertex labelling of a graph G is a function $f : V(G) \rightarrow Z^+$. An injective vertex labelling $f : V(G) \rightarrow \{0,1,2,...,|E(G)|\}$ *is* called graceful labelling if the induced function $f^+ : E(G) \rightarrow Z^+$, defined by $f(uv) = | f(u) - f(v)|$, is injective.[2]

A (p,q)- connected graph is said to be an edge-odd graceful graph if there exists an injective function $f : E(G) \rightarrow \{1,3,5,...,2q-1\}$ such that the induced function $f^+ : V(G) \rightarrow \{0,1,2,3,...2k-1\}$, defined by $f^+(x) \equiv \sum_{xy \in E(G)} f(xy) \pmod{2k}$ where $k = \max\{p,q\}$ is injective.[3][4]

The sum of two graphs G_1 and G_2, denoted by $G_1 + G_2$, is defined as a graph G whose vertex set is $V(G_1)$ U $V(G_2)$ and the edge set is $E(G_1)$ U $E(G_2)$ U $\{ xy : x \in V(G_1), y \in V(G_2)\}$. A wheel $W_{1,n}$ is a graph $C_n + K_1$. Further the vertices of C_n in $W_{1,n}$ are called its rim vertices and the vertex of K_1 is called its centre. An edge in a wheel which joins a rim vertex to the centre is called its spoke. Throughout this paper, we denote the centre of a wheel $W_{1,n}$ by v_0 and the rim vertices by $v_1,v_2,v_3,...v_n$, so that the edges v_0v_i , $1\leq i \leq n$ are the spoke edges and v_iv_{i+1} , $1 \leq i \leq n-1$ and v_nv_1 are the rim edges.

Theorem 1.1. For a given integer $n \geq 4$, the wheel graph $W_{1,n}$ is edge-odd graceful.

Proof: Define a function f: $E(W_{1,n})$: $\rightarrow \{1,3,...,4n-1\}$ depending upon n is even or odd as follows.
Case 1: n is odd

$$f(v_iv_j) = \begin{cases} n+2j, & \text{if } i = 0 \\ 4n+1-2i, & \text{if } j = i+1 \text{ and } 1 \leq i \leq (n-1)/2 \\ 2n+1-2i, & \text{if } j = i+1 \text{ and } (n+1)/2 \leq i \leq n-1 \\ 1, & \text{if } i = n, j = 1 \end{cases}$$

By the definition of f as in the above, it follows that

$$f^+(v_i) = \begin{cases} 3n \equiv \text{mod} 4n, & \text{if } \underline{i} = 0 \\ n+(4-2i) \equiv \text{ mod } 4n, & \text{if } 1 \leq i \leq (n-1)/2 \\ 2n+3 \equiv \text{mod } 4n, & \text{if } i = (n+1)/2 \\ n+(4-2i) \equiv \text{mod } 4n, & \text{if } (n+3)/2 \leq i \leq n \end{cases}$$

Since $n \geq 4$ we have $4 - 2i \leq 2$ and even, $2n \geq 8$ and hence it is clear that $f^+(v_i) \neq f^+(v_j)$ whenever $i \neq j$.

Case 2: n is even

$$f(v_iv_j) = \begin{cases} n+2j+1, & \text{if } i = 0 \\ 4n+1-2i, & \text{if } j = i+1 \text{and } 1 \leq i \leq (n-2)/2 \\ 2n+1-2i, & \text{if } j = i+1 \text{ and } n/2 \leq i \leq n-1 \\ 1, & \text{if } i = n \text{ and } j = 1 \end{cases}$$

By the definition of f as in the above, it follows that

$$f^{+}(v_i) = \begin{cases} 2n & \equiv \bmod 4n, \quad \text{if } i = 0 \\ n + (5-2i) & \equiv \bmod 4n, \quad \text{if } 1 \leq i \leq (n-2)/2 \\ 2n + 5 & \equiv \bmod 4n, \quad \text{if } i = n/2 \\ n + (5-2i) & \equiv \bmod 4n, \quad \text{if } (n+2)/2 \leq i \leq n \end{cases}$$

Since $n \geq 4$ we have $5 - 2i \leq 3$ and odd, $2n \geq 8$ and hence it is clear that $f^{+}(v_i) \neq f^{+}(v_j)$ whenever $i \neq j$. Thus f is an edge odd graceful labelling for any $n \geq 4$.

Theorem 1.2. The connected graph $W_{1,3} \cong K_4$ is not edge odd graceful.

Proof: Since $W_{1,3}$ is a 3-regular graph having 4 vertices and 6 edges. By the definition of edge-odd graceful labelling, each edge of $W_{1,3}$ can take only one of the label in {1,3,5,7,9,11}. Again by definition of edge-odd graceful labelling we should have a unique label at each vertex whose value is the sum of the edge labels incident at it under modulo 12 in case of $W_{1,3}$. Since each vertex is of degree 3, the only possible sums with three numbers are given below. Let the sum of the edge labels $\{a,b,c\}$ incident at a vertex be denoted by S.

case 1: $S = 1$
{1,3,9},{1,5,7}and {5,9,11}
case 2: $S = 3$
{1,3,11},{1,5,9},{3,5,7}and {7,9,11}
case 3: $S = 5$
{1,5,11},{1,7,9}and {3,5,9}
case 4: $S = 7$
{1,7,11},{3,5,11}and {3,7,9}
case 5: $S = 9$
{1,3,5},{3,7,11}and {5,7,9}
case 6: $S = 11$
{1,3,7},{3,9,11}and {5,7,11}

Now for a wheel $W_{1,3}$ an edge-odd graceful labelling exists, then we must be able to get 4 such sets from the above sets such that each set is from one of the cases above and the intersection between the sets is exactly at one number and that number is appearing only in those two sets.

Now w.l.o.g. let one of the vertices has the sum $S = 1$, then we have the following cases.

Case 1.1.1: {1,3,9},{3,5,7} and {1,5,11}.Now the last set is now forced to be as {7,9,11}. Then both {7,9,11} and {3,5,7} are from *case* 2, which is a contradiction to the fact that there should be only one set from each case. Otherwise it violates the condition that there should be unique label at each vertex.

Case 1.1.2: {1,3,9},{3,5,7} and {1,7,11}.Now the last set is now forced to be as {5,9,11}. Then both {1,3,9} and {5,9,11} are from *case* 1, which is a contradiction to the fact that there should be only one set from each case.

Case 1.1.3: {1,3,9},{7,9,11} and {1,5,11}.Now the last set is now forced to be as {3,5,7}. Then both {3,5,7} and {7,9,11} are from *case* 2, which is a contradiction to the fact that there should be only one set from each case.

Case 1.1.4: {1,3,9},{7,9,11} and {3,5,11}.Now the last set is now forced to be as {1,5,7}. Then both {1,5,7} and {1,3,9} are from *case* 1, which is a contradiction to the fact that there should be only one set from each case.

Case 1.1.5: {1,3,9},{1,5,11} and {3,5,7}.Now the last set is now forced to be as {7,9,11}. Then both {3,5,7} and {7,9,11} are from *case* 2, which is a contradiction to the fact that there should be only one set from each case.

Case 1.1.6: {1,3,9},{1,5,11} and {7,9,11}.Now the last set is now forced to be as {3,5,7}. Then both {3,5,7} and {7,9,11} are from *case* 2, which is a contradiction to the fact that there should be only one set from each case.

Case 1.1.7: {1,3,9},{1,5,11} and {3,7,11}.Now the last set is now forced to be as {5,9,7}. Then both {5,9,7} and {3,7,11} are from *case* 5, which is a contradiction to the fact that there should be only one set from each case.

Case 1.1.8: {1,3,9},{1,7,11} and {3,5,7}.Now the last set is now forced to be as {5,9,11}. Then both {5,9,11} and {1,3,9} are from *case* 1, which is a contradiction to the fact that there should be only one set from each case.

Case 1.1.9: {1,3,9},{1,7,11} and {5,7,9}.Now the last set is now forced to be as {3,5,11}. Then both {1,7,11} and {3,5,11} are from *case* 4, which is a contradiction to the fact that there should be only one set from each case.

Case 1.1.10: {1,3,9},{3,5,11} and {7,9,11}.Now the last set is now forced to be as {1,5,7}. Then both {1,5,7} and {1,3,9} are from *case* 1, which is a contradiction to the fact that there should be only one set from each case.

Case 1.1.11: {1,3,9},{3,5,11} and {5,7,9}.Now the last set is now forced to be as {1,7,11}. Then both {1,7,11} and {3,5,11} are from *case* 4, which is a contradiction to the fact that there should be only one set from each case.

Case 1.1.12: {1,3,9},{3,7,11} and {1,5,11}.Now the last set is now forced to be as {5,7,9}. Then both {5,7,9} and {3,7,11} are from *case* 5, which is a contradiction to the fact that there should be only one set from each case.

Case 1.2.1: {1,5,7},{1,3,11} and {3,5,9}.Now the last set is now forced to be as {7,9,11}. Then both {7,9,11} and {1,3,11} are from *case* 2, which is a contradiction to the fact that there should be only one set from each case.

Case 1.2.2: {1,5,7},{1,3,11} and {3,7,9}.Now the last set is now forced to be as {5,9,11}. Then both {5,9,11} {1,5,7} and are from *case* 1, which is a

contradiction to the fact that there should be only one set from each case.
Case 1.2.3: {1,5,7},{7,9,11} and {3,5,9}.Now the last set is now forced to be as {1,3,11}. Then both {1,3,11} and {7,9,11} are from *case* 2, which is a contradiction to the fact that there should be only one set from each case.
Case 1.2.4: {1,5,7},{7,9,11} and {3,5,11}.Now the last set is now forced to be as {1,3,9}. Then both {1,3,9} and {1,5,7} are from *case* 1, which is a contradiction to the fact that there should be only one set from each case.
Case 1.3.1: {5,9,11},{1,3,11} and {1,7,9}.Now the last set is now forced to be as {3,5,7}. Then both {1,3,11} and {3,5,7} are from *case* 2, which is a contradiction to the fact that there should be only one set from each case.
Case 1.3.2: {5,9,11},{1,3,11} and {3,7,9}.Now the last set is now forced to be as {1,5,7}. Then both {1,5,7} and {5,9,11} are from *case* 1, which is a contradiction to the fact that there should be only one set from each case.
Case 1.3.3: {5,9,11},{3,5,7} and {1,7,9}.Now the last set is now forced to be as {1,3,11}. Then both {1,3,11} and {3,5,7} are from *case* 2, which is a contradiction to the fact that there should be only one set from each case.
Case 1.3.4: {5,9,11},{3,5,7} and {1,7,11}.Now the last set is now forced to be as {1,3,9}. Then both {1,3,9} and {5,9,11} are from *case* 1, which is a contradiction to the fact that there should be only one set from each case.
Case 1.3.5: {5,9,11},{1,7,9} and {1,3,5}.Now the last set is now forced to be as {3,7,11}. Then both {3,7,11} and {1,3,5} are from *case* 5, which is a contradiction to the fact that there should be only one set from each case.
Case 1.3.6: {5,9,11},{3,7,9} and {1,3,5}.Now the last set is now forced to be as {1,7,11}. Then both {1,7,11} and {3,7,9} are from *case* 4, which is a contradiction to the fact that there should be only one set from each case.
Case 1.3.7: {5,9,11},{1,7,11} and {1,3,5}.Now the last set is now forced to be as {3,7,9}. Then both {3,7,9} and {1,7,11} are from *case* 4, which is a contradiction to the fact that there should be only one set from each case.
From the above cases we can conclude that a vertex with sum S = 1 is never possible. Therefore *case* 1 is eliminated.

Now we use *case* 3.
Case 3.1.1: {1,5,11},{3,5,7} and {3,9,11}. Now the last set is forced to be as {1,7,9}. Then both {1,7,9} and {1,5,11} are from *case* 3, which is a contradiction to the fact that there should be only one set from each case.
Case 3.1.2: {1,5,11},{7,9,11} and {1,3,7}. Now the last set is forced to be as {3,5,9}. Then both {1,5,11} and {3,5,9} are from *case* 3, which is a contradiction to the fact that there should be only one set from each case.
Case 3.1.3: {1,5,11},{5,7,9} and {1,3,7}. Now the last set is forced to be as {3,9,11}. Then both {3,9,11} and {1,3,7} are from *case* 6, which is a contradiction to the fact that there should be only one set from each case.

There exists no unique pattern different from the previously known patterns. Therefore no correct pattern containing the set {1,7,9} exists.

There exists no unique pattern different from the previously known patterns. Therefore no correct pattern containing the set {3,5,9} exists.

From the above cases we can conclude that a vertex with sum S = 5 is never possible. Therefore *case* 3 is eliminated.

Now we use *case* 4.
Case 4.1.1: {1,7,11},{1,5,9} and {3,9,11}. Now the last set is forced to be as {3,5,7}. Then both {3,5,7} and {1,5,9} are from *case* 2, which is a contradiction to the fact that there should be only one set from each case.
Case 4.2.1: {3,5,11},{1,5,9} and {1,3,7}. Now the last set is forced to be as {7,9,11}. Then both {1,5,9} and {7,9,11} are from *case* 2, which is a contradiction to the fact that there should be only one set from each case.
Case 4.3.1: {3,7,9},{1,3,11} and {5,7,11}. Now the last set is forced to be as {1,5,9}. Then both {1,5,9} and {1,3,11} are from *case* 2, which is a contradiction to the fact that there should be only one set from each case.

From the above cases, we can conclude that a vertex with sum S = 7 is never possible. Therefore *case* 4 is eliminated.

Remaining sums are only in 3 cases which are not sufficient to label 4 vertices which is a contradiction.
Therefore there exists no edge-odd graceful labelling for $W_{1,3}$.

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