Edge-Odd Gracefulness of Middle Graphs and Total Graphs of Paths and Cycles

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Abstract

Graph theory is rapidly moving into the mainstream of mathematics mainly because of its applications in diverse fields which include biochemistry (genomics), electrical engineering (communication network and coding theory), computer science (algorithms and computations) and operations research (scheduling). The recent applications of graph theory are DNA sequencing (SNP assembly problem), computer network (worm propagation and assignment of frequencies in GSM mobile networks [6]. For a simple, finite, undirected and connected graph G(V,E), an Edge Odd Graceful labeling (EOGL) is an injective function f : E(G) → {1,3,5,...,2q−1} such that the induced function f° : V(G) → {0,1,2,...,2k−1} defined by f°(x) = \sum_{y \in G} f(xy) (mod 2k) is injective where k=max{p,q}. A graph which admits an Edge Odd Graceful labeling is called an Edge Odd Graceful Graph (EOGG) [2]. In this paper, the Edge Odd Gracefulness of Middle Graphs and Total graphs of Paths and Cycles are obtained.

Keywords: Graceful graph, Edge odd graceful labelling, Edge odd graceful graph, Middle graph, Total graph.

AMS Classification Number: 05C78

1. Introduction

In Graph theory, labelled graphs are the most useful mathematical models in addressing and identification system on a communication network to a larger network obtained by introducing new user terminals and new communication links between the new terminals of the original network. Also labelled graphs serve as useful models for coding theory, X-Ray, radars, astronomy, circuit design, crystallography, network addressing, database management, secret sharing schemes, etc..[7]. A. Solairaju and K. Chitra [2] introduced and obtained edge-odd graceful labelling of some graphs related to paths. The results on edge-odd graceful labelling are given in Gallian survey [5].

All the graphs considered in this paper are simple, undirected, finite and connected. The terms not defined here may be found in [4].

Definition 1.1: Graceful Graph: [1][5]

A (p,q) connected graph G is said to be a graceful graph if there exists an injective function f from the vertex set of G to the set \{0, 1, 2, ..., q\} such that when each edge uv is assigned the label |f(u)−f(v)| and the resulting edge labels are distinct.

Definition 1.2. Edge odd graceful graph: [2][3]

A (p,q) connected graph G is said to be an edge-odd graceful graph if there exists an injective function f : E(G) → \{1,3,5,...,2q−1\} such that the induced function f° : V(G) → \{0,1,2,3...2k−1\}, defined by f°(x) = \sum_{y \in G} f(xy) (mod 2k) where k = max{p,q} is also injective.
Definition 1.3. Middle graph: [9]

Let G be a graph with vertex set V(G) and edge set E(G). The middle graph of G, denoted by M(G) is defined as follows. The vertex set of M(G) is V(G) U E(G). Two vertices x, y in the vertex set of M(G) are adjacent in M(G) in case one of the following holds: (i) x, y are in E(G) and x, y are adjacent in G. (ii) x is in V(G), y is in E(G) and x, y are incident in G.

Definition 1.4. Total Graph: [9]

Let G be a graph with vertex set V(G) and edge set E(G). The total graph of G, denoted by T(G) is defined as follows. The vertex set of T(G) is V(G) U E(G). Two vertices x, y in the vertex set of T(G) are adjacent in T(G) in case one of the following holds: (i) x, y are in V(G) and x is adjacent to y in G. (ii) x, y are in E(G) and x, y are adjacent in G. (iii) x is in V(G), y is in E(G) and x, y are incident in G.

2. Results on Middle Graphs of Paths and Cycles:

Theorem 2.1. The Middle Graph of path P_n for n = 3, 4, 5, 6, 7, 8 and n = 2 (mod 4) is edge odd graceful.

Proof: From the diagram, the middle graph of the path P_3 is EOG.

The total number of edges in M(P₃) is (3n – 4).

For n > 3 and n = 2 (mod 4), define a function f : E(M(Pₙ)) → {1,3,6n-9} such that the edges are labeled as

\[ f((v_i, u_j)) = (6i-5) \mod (6n-8) \]

for 1 ≤ i ≤ n-1,
\[ f((v_i, u_{i+1})) = (6i-1) \mod (6n-8) \]

for 1 ≤ i ≤ n-2,
\[ f((u_i, v_{i+1})) = (6n-9) \mod (6n-8) \]

and
\[ f((u_i, u_{i+1})) = (6i-3) \mod (6n-8) \]

for 1 ≤ i ≤ n-2.

Define \( f^* \) : V(M(Pₙ)) → {0,1,2,3,6n-9} such that the vertices are labeled as

\[ f^*(v_i) = 1, f^*(u_{i}) = (6n-9) \mod (6n-8), \]
\[ f^*(v_i) = 6i \mod (6n-8) \]

for 2 ≤ i ≤ n-1,
\[ f^*(u_{i}) = (24i-18) \mod (6n-8) \]

for 2 ≤ i ≤ n-2,
\[ f^*(u_{i}) = 9 \] and \( f^*(u_{i+1}) = (6n-19) \) for n ≥ 4

Hence the induced map \( f^* \) provides the distinct labels for vertices. Also edge labels are distinct.

Thus M(Pₙ) for n = 3, 4, 5, 6, 7, 8 and n = 2 (mod 4) is Edge Odd Graceful.

Theorem 2.2. For all n ≥ 3, the middle graph of cycle Cₙ for odd n is Edge Odd graceful.

Proof: From the diagram, the middle graph of C₃ is EOG.

The Total number of edges in M(Cₙ) = 3n.

For all odd n > 3, define a function f : E(M(Cₙ)) → {1,3,...,6n-1} such that the edges are labeled as

\[ f((u_i, v_{i})) = 1, \]
\[ f((u_i, v_{i+1})) = (6i+1) \mod 6n \]

for 1 ≤ i ≤ (n-1),
\[ f((v_i, u_i)) = (6i-3) \mod 6n \]

for 1 ≤ i ≤ n,
\[ f((u_i, v_{i+1})) = (6i+5) \mod 6n \]

for 1 ≤ i ≤ (n-1) and
\[ f((u_i, u_{i+1})) = 5. \]

Define \( f^* \) : V(M(Cₙ)) → {0,1,2,3,...,6n-1} such that the vertices are labeled as

\[ f^*(v_i) = (12i-8) \mod 6n \]

for 1 ≤ i ≤ n,
\[ f^*(u_i) = (24i+2) \mod 6n \]

for 1 ≤ i ≤ n.

Hence the induced map \( f^* \) provides the distinct labels for vertices. Also edge labels are distinct.

Thus for all odd n, M(Cₙ) is Edge Odd graceful.
3. Results on Total graphs of Paths and Cycles:

Theorem 3.1. For all $n \geq 3$, the total graph of a path $P_n$ is Edge Odd Graceful.

Proof:
The Total number of edges in $T(P_n) = 4n - 5$.
For $n \geq 3$, define a function $f : E(T(P_n)) \rightarrow \{1, 3, \ldots, 8n - 11\}$ such that the edges are labeled as

- $f(v_i, u_i) = (8i - 7) \mod (8n - 10)$ for $1 \leq i \leq (n - 1)$,
- $f(u_i, u_{i+1}) = (8i - 3) \mod (8n - 10)$ for $1 \leq i \leq (n - 1)$,
- $f(v_i, v_{i+1}) = (8i - 5) \mod (8n - 10)$ for $1 \leq i \leq (n - 1)$ and
- $f(u_i, u_{i+1}) = (8i - 1) \mod (8n - 10)$ for $1 \leq i \leq (n - 2)$

Hence the induced map $f$ provides the distinct labels for vertices. Also edge labels are distinct.

Thus for all $n \geq 3$, the total graph of $P_n$, $T(P_n)$ is Edge Odd Graceful.

Example 3.2.
The total graph of a path $P_4$ is EOG.

Define $f^* : V(T(P_n)) \rightarrow \{0, 1, 2, 3, \ldots, 8n - 11\}$ such that the vertices are labeled as

- $f^*(v_1) = 4$,
- $f^*(v_i) = (32i - 36) \mod (8n - 10)$ for $2 \leq i \leq (n - 1)$,
- $f^*(u_1) = (32i - 20) \mod (8n - 10)$ for $2 \leq i \leq (n - 2)$,
- $f^*(u_i) = (8n - 23) \mod (8n - 10)$ for $n \geq 3$ and
- $f^*(u_1) = 13$.

Hence the induced map $f^*$ provides the distinct labels for vertices. Also edge labels are distinct.

Thus for all $n \geq 3$, the total graph of $P_n$, $T(P_n)$ is Edge Odd Graceful.

Theorem 3.3. For a given odd integer $n \geq 3$, the total graph of cycle $C_n$ is Edge Odd Graceful.

Proof:
The Total number of edges in $T(C_n) = 4n$.
For $n \geq 3$, define a function $f : E(T(C_n)) \rightarrow \{1, 3, \ldots, 8n - 1\}$ such that the edges are labeled as

- $f(u_i, v_i) = 1$, $f(v_i, v_{i+1}) = 8n - 1$, $f(u_i, u_{i+1}) = 5$,
- $f(v_i, v_{i+1}) = (8i - 1) \mod 8n$ for $1 \leq i \leq (n - 1)$,
- $f(u_i, u_{i+1}) = (8i + 1) \mod 8n$ for $1 \leq i \leq (n - 1)$,
- $f(v_i, u_i) = (8i - 5) \mod 8n$ for $1 \leq i \leq n$ and

$\text{f}(u_i, u_{i+1}) \equiv (8i + 5) \mod 8n$ for $1 \leq i \leq (n - 1)$

Define $f^* : V(T(C_n)) \rightarrow \{0, 1, 2, 3, \ldots, 8n - 1\}$ such that the vertices are labeled as

- $f^*(v_i) = (32i - 22) \mod 8n$ for $1 \leq i \leq n$,
- $f^*(u_1) = (32i - 2) \mod 8n$ for $2 \leq i \leq (n - 1)$ and
- $f^*(u_1) = (8n - 2) \mod 8n$.

Hence the induced map $f^*$ provides the distinct labels for vertices. Also edge labels are distinct.

Thus for all odd $n$, the total graph of $C_n$, $T(C_n)$ is Edge Odd Graceful.

Example 3.4.
The total graph of a cycle $C_7$ is EOG.

Figure 4. Total graph of $C_7$

Acknowledgement

The authors are thankful to the anonymous referee for valuable comments and kind suggestions.

References


