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Dynamics and Limit Cycles in a Holling Type II Predator–Prey Model

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Abstract
This study explores the dynamics of a predator–prey system incorporating a Holling Type II functional response. The model is analyzed by identifying equilibrium points and examining their local stability through the eigenvalues of the Jacobian matrix. Special attention is given to the occurrence of a Hopf bifurcation at the positive equilibrium. The direction and stability of the bifurcation are investigated using center manifold theory and Lyapunov coefficient analysis. Numerical simulations complement the theoretical results, illustrating the emergence of stable or unstable limit cycles under varying parameter conditions. The findings provide insights into the mechanisms underlying oscillatory behavior in predator–prey interactions and the factors driving population cycles.

Keywords: **Predator–prey dynamics, Holling Type II functional response, Hopf bifurcation, stability analysis, limit cycles.**

1.INTRODUCTION

Mathematical ecology studies the interactions among species, populations, and their environment across temporal and spatial scales [7]. Predator-prey models, first formulated by Lotka and Volterra in the 1920s [7], have been central to this field, providing a framework to explore population dynamics, competition and resource acquisition. These models extend naturally to a variety of ecological contexts, including resource-consumer, plant-herbivore, host-parasite, and epidemiological interaction such as susceptible-infectious transmission [11].

A key refinement of classical predator-prey theory involves the incorporation of nonlinear functional responses, which more realistically describe how predator consumption saturates at high prey densities. Among these, the Holling Type II functional response is widely used due to its ecological realism and its ability to generate complex population dynamics [4,6,16]. Nonlinearities of this kind often produce oscillations, limit cycles, or even chaotic behavior, phenomena that cannot be captured by the classical Lotka-Volterra framework [4].

Bifurcation theory has become a central tool to understand such complex dynamics. In particular, Hopf bifurcation analysis explains how small changes in system parameters can lead stable equilibria to generate sustained oscillations and periodic solutions [12,14]. Numerous studies have applied this approach to predator-prey systems under extensions, including time delays [8,13], stochastic effects and fear dynamics [2], nonlinear functional responses [3,16], memory effects [6], and diffusion mechanisms [9,10]. These contributions highlight the power of bifurcation theory in uncovering the mechanisms that drive oscillatory or unstable population dynamics [15].

In this study, we focus on a Gause-type predator-prey, originally introduced by Csughley and Lawton [11], equipped with a Holling Type II functional response. Our analysis investigates the conditions under which Hopf bifurcation occurs near the interior equilibrium, examines its direction and stability, and validates the theoretical results through numerical simulations. By combining analytical and computational methods, this work extends the understanding of oscillatory behavior in predator-prey interactions, building on recent findings in [1-6,8-10,13,15,16].

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2. THE PREY PREDATOR MODEL

The prey-predator system is of the form

$$\frac{dN}{d\tau} = \gamma N\left(1 - \frac{N}{K}\right) - g(N)P$$
$$\frac{dP}{d\tau} = e\, g(N)P - mP \qquad (1)$$

Where

$$g(N) = \frac{aN}{b+N}$$

$g(N)$ is Holling type II functional response, N is the prey density and P is the predator density. The parameters are $\gamma, a, e, m, K > 0$. N (0) > 0 and P (0) > 0.

3. EQUILIBRIUM POINTS AND THEIR STABILITY

We will begin by making the following change of variables to express the model in a dimensionless form:

$$x = \frac{N}{K}, y = \frac{aP}{\gamma K}, t = \gamma\tau$$

Then (1) can be written:

$$\frac{dx}{dt} = x(1-x) - \frac{xy}{x+\alpha}$$
$$\frac{dy}{dt} = \frac{\beta xy}{x+\alpha} - \delta y \qquad (2)$$

Taking the right-hand sides of (2) equal zero, then we obtain three equilibrium points $T_0(0,0), T_1(1,0)$ and $T_*(x_*, y_*)$ where $x_* = \frac{\alpha\delta}{\beta-\delta}, y_* = (1-x_*)(\alpha + x_*)$ and and $\beta > \delta$ (1+ α) ensures system (2) has a unique positive equilibrium point $T_*(x_*, y_*)$.

Jacobian matrix of the system (1) is

$$J(x,y) = \begin{pmatrix} 1 - 2x - \frac{y}{\alpha+x} + \frac{xy}{(\alpha+x)^2} & -\frac{x}{\alpha+x} \\ \frac{\beta y}{\alpha+x} - \frac{\beta xy}{(\alpha+x)^2} & \frac{\beta x}{\alpha+x} - \delta \end{pmatrix}.$$

Theorem 3.1. For system (2), the following statements are hold.

i. For all $\alpha, \beta, \delta > 0$, the equilibrium point $T_0(0,0)$ is a saddle point which is unstable.
ii. The equilibrium point $T_1(1,0)$ is locally stable when $\frac{\beta}{\alpha+1} < \delta$ and it is unstable for $\frac{\beta}{\alpha+1} > \delta$.
iii. In the case $\frac{\beta}{\alpha+1} < \delta$, there is no limit cycle since there is no positive equilibrium.

4. BIFURCATION ANALYSIS OF THE SOLUTIONS

We focus solely on studying the Hopf bifurcation around $T_*(x_*, y_*)$. By considering α as the bifurcation parameter, we examine the presence of Hopf bifurcation for (2) and analyze the direction and stability of the bifurcation. Next, we explore the outcomes of the Hopf bifurcation for (2). First, we obtain the Jacobian matrix of equation (2) at $T_*(x_*, y_*)$.

$$J(x_*, y_*) = \begin{pmatrix} \frac{\delta}{\beta}\left(1 - \frac{\beta+\delta}{\beta-\delta}\alpha\right) & -\frac{x_*}{\alpha+x_*} \\ \frac{\alpha\beta x_* y_*}{(\alpha+x_*)^2} & 0 \end{pmatrix}.$$

The characteristic equation of J_* is

$$\lambda^2 - (trJ_*)\lambda + \det J_* = 0. \quad (3)$$

Where

$$trJ_* = \frac{\delta}{\beta}\left(1 - \frac{\beta+\delta}{\beta-\delta}\alpha\right),\ \det J_* = \frac{\alpha\beta {x_*}^2 y_*}{(\alpha+x_*)^3} > 0.$$

Let $(\tilde{x},\tilde{y})$ = (x, y) - (x_*, y_*). For convenience, we will use (x, y) to denote $(\tilde{x},\tilde{y})$. As a result, the model (2) is modified to:

$$\frac{dx}{dt} = (x + x_*)(1 - (x + x_*)) - \frac{(x+x_*)(y+y_*)}{x+x_*+\alpha}$$
$$\frac{dy}{dt} = \frac{\beta(x + x_*)(y + y_*)}{x + x_* + \alpha} - \delta(y + y_*). \quad (4)$$

Theorem 4.1 The model (2) exhibits a Hopf bifurcation at (x_*, y_*) for $\alpha = \alpha_H = \frac{\beta-\delta}{\beta+\delta}$.

Proof. Therefore, $trJ_* < 0$ or $1 - \frac{\beta+\delta}{\beta-\delta}\alpha < 0$. Alternatively, $\alpha > \frac{\beta-\delta}{\beta+\delta}$. Let $\lambda(\alpha) = \mu(\alpha) \pm i\omega(\alpha)$ represent the roots of (3) in the vicinity of α_H. Consequently, we acquire

$$\mu(\alpha) = \frac{trJ_*}{2} = \frac{\delta}{2\beta}\left(1 - \frac{\beta+\delta}{\beta-\delta}\alpha\right),$$

$$\omega(\alpha) = \frac{\alpha}{2}\sqrt{\frac{\beta\delta - \delta^2 - \delta(\beta+\delta)\alpha}{\beta-\delta} - \frac{4\delta(\beta-\delta-\alpha)}{\beta}}$$

and

$$\mu'(\alpha) = -\frac{\delta}{2\beta}\left(\frac{\beta+\delta}{\beta-\delta}\right)$$

It is evident that $trJ_*(\alpha_H) = 0$, $\det J_*(\alpha_H) > 0$, and $\mu'(\alpha_H) \neq 0$. According to the Hopf bifurcation theorem [7], the model (2) undergoes a Hopf bifurcation at (x_*, y_*, α_H). Now, a computational method is employed to ascertain whether the Hopf bifurcation is of the supercritical or subcritical nature. In order to study the system centered around the point $\alpha = \alpha_H$, we can expand the righthand side of equations (4) using the Maclaurin series. This allows us to rewrite the system (4) as follows:

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = J_* \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} F(x,y,\alpha) \\ G(x,y,\alpha) \end{pmatrix} \tag{5}$$

Where

$$F(x,y,\alpha) = \left(\frac{(\beta-\delta-\beta\delta)(\beta-\delta)\alpha}{\delta^2\alpha} - 1\right)x^2 + \frac{(\beta-\delta)^2}{2\delta^2\alpha}xy + \frac{(\beta-\delta)^3}{3\delta^3\alpha^2}x^2y + \frac{(\beta-\delta-\beta\delta)(\beta-\delta)^2}{\delta^2\alpha}x^3,$$

and

$$G(x,y,\alpha) = -\frac{\beta(\beta-\delta-\beta\delta)(\beta-\delta)}{\delta^2\alpha}x^2 + \frac{\beta(\beta-\delta)^2}{2\delta^2\alpha}xy - \frac{\beta(\beta-\delta)^3}{3\delta^3\alpha^2}x^2y + \frac{\beta(\beta-\delta-\beta\delta)(\beta-\delta)^2}{\delta^3\alpha^2}x^3.$$

Define

$$\begin{bmatrix} 1 \\ A_1 - A_2 j \end{bmatrix} \lambda = \mu(\alpha) \pm i\omega(\alpha)$$

Where

$$A = \begin{bmatrix} 1 & 0 \\ A_1 & A_2 \end{bmatrix}, A_1 = \frac{\delta}{2\beta}\left(1 - \frac{\beta+\delta}{\beta-\delta}\alpha\right), A_2 = \frac{\alpha}{2}\sqrt{\frac{\beta\delta - \delta^2 - \delta(\beta+\delta)\alpha}{\beta-\delta} - \frac{4\delta(\beta-\delta-\alpha)}{\beta}}$$

Next, we make the transformation

$$\begin{bmatrix} X \\ Y \end{bmatrix} = A^{-1}\begin{bmatrix} x \\ y \end{bmatrix}$$

Thus

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} X \\ A_1X + A_2Y \end{bmatrix}$$

Therefore, the system (5) can be transformed:

$$\begin{bmatrix} X'(t) \\ Y'(t) \end{bmatrix} = A^{-1}J_*A + A^{-1}\begin{bmatrix} F(A_1X + A_2Y, \alpha) \\ G(A_1X + A_2Y, \alpha) \end{bmatrix}$$

where

$$\mathrm{F}(\mathrm{X},\mathrm{Y}, \alpha) = M_1X^2 + M_2XY + M_3X^2Y + M_4X^3,$$

$$\mathrm{G}(\mathrm{X},\mathrm{Y}, \alpha) = N_1X^2 + N_2XY + N_3X^2Y + N_4X^3.$$

With

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$$M_1 = \left(\frac{(\beta-\delta-\beta\delta)(\beta-\delta)\alpha}{\delta^2\alpha} - 1\right)(1 + A_1 + A_1^2),\ M_2 = \frac{(\beta-\delta)^2(A_2+2A_1A_2)}{2\delta^2\alpha},$$

$$M_3 = \frac{(\beta-\delta)^3(A_2+2A_1A_2+3A_1^2A_2)}{3\delta^3\alpha^2}, M_4 = \frac{(\beta-\delta-\beta\delta)(\beta-\delta)^2(1+A_1+A_1^2+A_1^3)}{\delta^2\alpha},$$

$$N_1 = -\frac{A_1}{A_2}M_1 + \frac{\beta(\beta-\delta-\beta\delta)(1+A_1+A_1^2)}{A_2\delta^2\alpha}, N_2 = -\frac{A_1}{A_2}N_{2+}\frac{\beta(\beta-\delta)^2(1+2A_1)}{2\delta^2\alpha},$$

$$N_3 = -\frac{A_1}{A_2}M_3 - \frac{\beta(\beta-\delta)^3(1+2A_1+3A_1^2)}{3\delta^3\alpha^2}, N_4 = -\frac{A_1}{A_2}M_4 + \frac{\beta(\beta-\delta-\beta\delta)(\beta-\delta)^2(1+A_1+A_1^2+A_1^3)}{A_2\delta^3\alpha^2}.$$

In order to ascertain the stability of the periodic solution, it is necessary to calculate the sign of the coefficient σ. This coefficient is formally defined as follows:

$$\sigma = \frac{1}{16}(\mathrm{F_{XXX}} + \mathrm{F_{XYY}} + \mathrm{G_{XXY}} + \mathrm{G_{YYY}}) + \frac{1}{16\omega_0}(\mathrm{F_{XY}}(\mathrm{F_{XX}} + \mathrm{F_{YY}}) - \mathrm{G_{XY}}(\mathrm{G_{XX}} + \mathrm{G_{YY}}) - \mathrm{F_{XX}G_{XX}} + \mathrm{F_{YY}G_{YY}}).$$

To determine the partial derivative of the bifurcation at (X, Y, α) = (0,0, α_H) when ω_0 = ω (α_H), we can proceed with the following calculations:

$$\sigma = \frac{1}{8}(3M_4 + N_3) + \frac{1}{8\omega}(M_1M_2 - N_1N_2 - 2{M_1}^2{N_1}^2).$$

The explicit calculation of σ can be found in [6]. Based on Poincare-Andronow's Hopf bifurcation theory and the aforementioned calculations of σ, we can obtain the following result.

Theorem 4.2 If $\sigma < 0$, the Hopf bifurcation is supercritical, indicating that the positive equilibrium (x_*, y_*) is a stable spiral for $\alpha < \alpha_H$, but becomes unstable for $\alpha > \alpha_H$, with the presence of a stable periodic solution. If $\sigma > 0$, the Hopf bifurcation is subcritical. In this case, when $\alpha < \alpha_H$, the positive equilibrium (x_*, y_*) is stable and an unstable periodic solution exists. However, for $\alpha > \alpha_H$, (x_*, y_*) becomes unstable.

5. NUMERICAL SIMULATIONS

In this section, we present numerical simulations that further support and complement the results discussed in the previous section. Our model (2) includes three parameters: α, β, and δ. For our simulations, we set β = 0.3 and δ = 0.2. The following numerical simulations demonstrate the key theoretical findings**.**

Example 5.1. We set α = 0.2, β = 0.4, and δ = 0.35. Then $\frac{\beta}{\alpha+1} < \delta$ .With these values, model (2) does not have any positive equilibrium point. From Figure 1, we observe that T_1(1,0)is locally stable.

Example 5.2. We set α = 0.1, β = 0.4, and δ = 0.35. Then $\frac{\beta}{\alpha+1} > \delta$ and there exists unique positive equilibrium (1,0). Moreover, when β = 0.4 and δ = 0.35. From Figure 2 we can clearly see the presence of a stable periodic solution, while the positive equilibrium (1,0) is unstable.

Example 5.3. When we set β = 0.3, and δ = 0.2, we have $\alpha_H = 0.2$. When $\alpha_H = 0.2$, the system (2) exhibits a Hopf bifurcation at (x_*, y_*). Additional calculations reveal that $\sigma \approx -0.01605 < 0$, indicating that the Hopf bifurcation is subcritical. Moreover, the periodic solution of the Hopf bifurcation at (x_*, y_*) is asymptotically stable, as depicted in Fig. 3.

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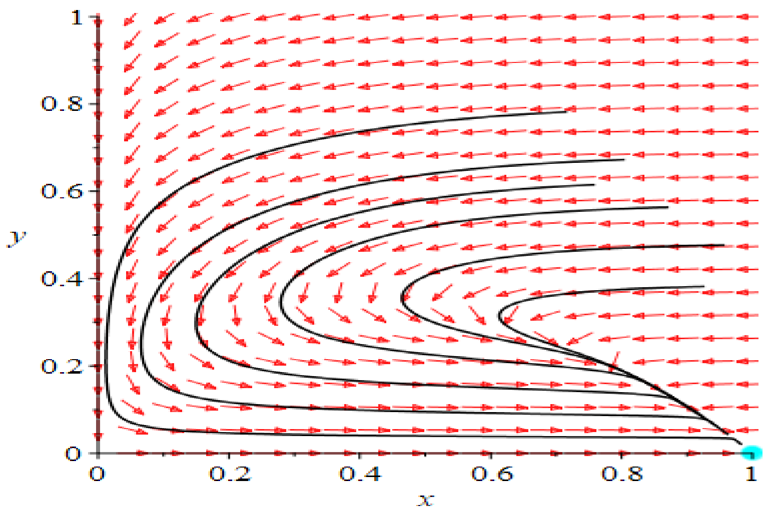

Figure 1. Equilibrium point $T_1(1,0)$ is locally stable.

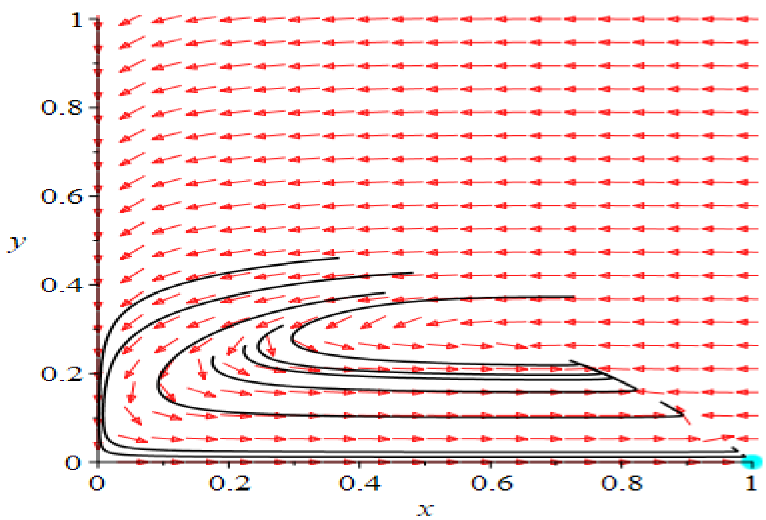

Figure 2. Equilibrium point $T_1(1,0)$ is unstable.

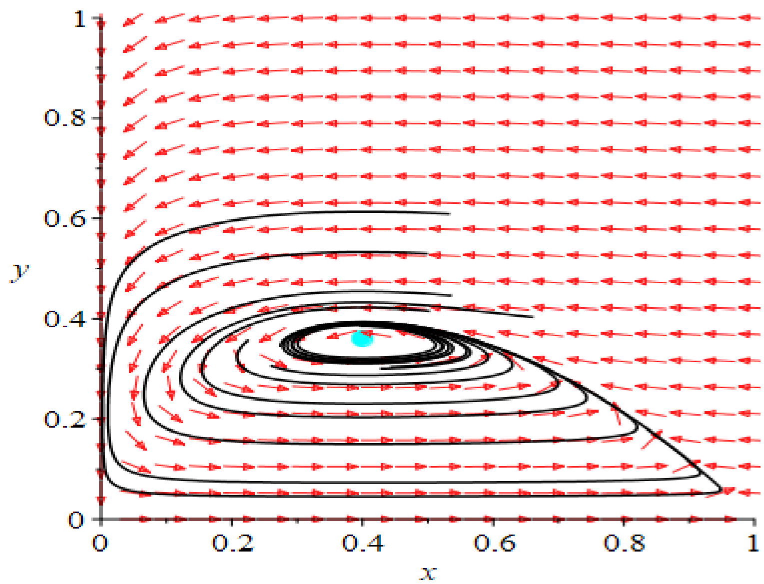

Figure 3. Hopf bifurcation at α_H.

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