Dynamic Buckling of a Rod under an Oscillating Axial Load

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***Abstract*—Dynamic buckling of a rod is analyzed; the effect of axial oscillating load on a pin ended steel rod of circular cross section is modeled and numerically solved. Three cases i.e., uniform cross section; non-homogenous density distribution and variable cross section are considered. Two important parameters are found as responsible for the buckling – amplitude and the frequency of the oscillating force. Analysis is done by varying these two parameters over a fixed time period. Buckling occurs under lower amplitude of the oscillating force in case of dynamic state while in quasi-static case buckling occurs under higher amplitude of load. Instability occurs when the frequency of the oscillating force comes near to the natural frequency of the rod.**

Keywords*—dynamic buckling; axial oscillating force; forcing frequency; critical buckling load; pin ended rod.

I. INTRODUCTION

Failure of structures is a dynamic process [1], and so it is obviously more realistic to approach buckling and stability from the dynamic point of view. At the same time, it appears that a dynamic approach is necessary to define the concept of stability precisely.

Buckling from prescribed dynamic loads acting on structural element e.g. a steel rod is concerned in the present work. Difference among the several types of dynamic buckling can be made based on the physical phenomena of the buckling processes. One of the main distinctions is between buckling from oscillatory loads and buckling from transient loads consisting of a single pulse characterized by its amplitude, shape and duration. The first kind is known as vibration buckling and the later kind is known as pulse buckling.

In vibration buckling, the amplitudes of vibration caused by an oscillating load become unacceptably larger at critical combinations of load amplitude, load frequency and structure damping.

As the phenomena of dynamic buckling are very common in our everyday life, it is of a great importance to analyze and understand the behavior of it. Long, thin supports are ubiquitous in natural and engineered load buckling structures, from spider legs to the steel structure of a skyscraper [2]. A single column will buckle if too much force is applied along its axis, which can lead to the catastrophic failure of the structure. The buckling instability is seen at all sizes, from pole vaulting to protein microtubules confined in vesicles and carbon nanotube atomic force microscope probes [3].

Moreover, there are a lot of structures and bodies which go under dynamically applied load in service condition, for example in aircraft landing struts, arches and spherical caps, cylindrical shells etc. are very important both from engineering and structural viewpoints.

The reason why a simple pin ended rod is chosen for analyzing is that the simply supported rod is obviously the simplest element because of its long history in the design of rods under static loading, for which the theory reduces to very fundamental form. A lot of experimental and theoretical works have been done on this account and the findings of those researches make some interesting points which are quite influential to do analysis on this particular topic.

Historically, Abrahamson and Goodier [4], investigated the plastic buckling of columns caused by axial impact. They defined amplification factor as the ratio of the maximum amplitude of the shape imperfection associated with the nth mode to its average value and postulated that the structure buckled when the amplification factor exceeded a pre-specified value, say 10. This method has been successfully used to analyze plastic buckling of rods caused by axial impact, plastic buckling of plates due to in-plane forces, and plastic buckling of cylindrical shells under axial impact. Wang and Ru [5] used the following energy criterion for analyzing the dynamic buckling of a structure. The structure is unstable if under a kinematically admissible perturbation superimposed on the dominant motion, the work done by internal stresses is less than the work done by the external forces. The energy criterion has been applied to study the radial buckling of cylindrical shells by Gu et al [6]. Batra and Wei [7] have postulated that the buckling mode corresponds to the wavelength of the superimposed perturbation that has the maximum initial growth rate, and have used it to study the buckling of a thin anisotropic thermoviscoplastic plate due to in-plane forces.

II. MATHEMATICAL FORMULATION

The differential equation of motion of the rod under axial pulsating load (Fig.1) can be derived as follows[8]:

$$EI\frac{\partial^4 y}{\partial x^4} + (P_0 + P_t\cos\Omega t)\frac{\partial^2 y}{\partial x^2} + \rho A\frac{\partial^2 y}{\partial t^2} = 0 \quad (1)$$

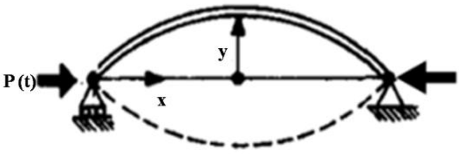


Fig.1. Pin ended rod under pulsating axial force.

Where,

$$P(t) = P_0 + P_t \cos \Omega t \qquad (2)$$

is the axial compressive load- of which $P_o$is the static load and $P_t$is the amplitude of the oscillating force, Ωis the forcing frequency, and ρAdescribes mass per unit length of the rod.

For the boundary conditions of a pin ended rod, we seek the solution in the form

$$y(x.t) = \sum_{n=1}^{\infty} f_n(t) \sin \frac{n\pi x}{l} \qquad (3)$$

Substituting into (1) yields the condition

$$\sum_{n=1}^{\infty} \sin \frac{n\pi x}{l} \left\{ \ddot{f}_n + \frac{EI}{\mu}\left(\frac{n^4\pi^4}{l^4}\right)\left[1 - \frac{l^2}{n^2\pi^2 EI}(P_0 + P_t \cos \Omega t)\right] f_n \right\} = 0 \qquad (4)$$

$$\xrightarrow{yields} \frac{d^2 f}{dt^2} + \omega^2 (1 - 2p\cos \Omega t) = 0 \qquad (5)$$

in which the following notations are made:

(6)

$$\omega^{0^2} = \frac{n^4\pi^4}{l^4}\left(\frac{EI}{\rho A}\right) \qquad (7)$$

$$P_{cr_n} = \frac{n^2\pi^2}{l^2}(\mathrm{EI}) \qquad (8)$$

$$p = \frac{P_t}{2\left(P_{cr_n} - P_o\right)} \qquad (9)$$

Here,
ω^o= free vibration frequency of the rod for no axial load
ω = free vibration frequency under static loadP_o
Pcr_n = critical load for buckling
p= excitation parameter.

III. BOUNDARY CONDITIONS

To analyze the deflection in the present work, the rod is assumed to be pin ended on both ends and so theboundary conditions are, at both ends both the deflection and bending moment is zero.

Atx=0and x=l

Deflection,y=0(10)

And, atx=0and x=l

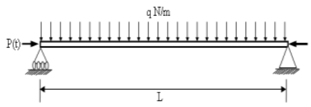
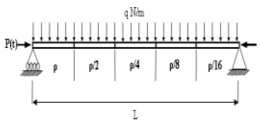
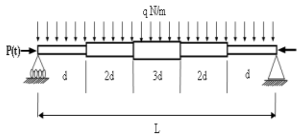
Bending moment, $EI \frac{\partial^2 y}{\partial x^2} = 0$ (11)

IV. MODELING

To study the effect of the amplitude and frequency of the oscillating force three different cases are modeled. For each cases boundary conditions, mean static load Po, time t, modulus of elasticity E, distributed load over the rod q are same; diameter of the rod d is represented in TABLE 1.

To analyze these cases finite difference method (FDM) is applied. Second order central difference formula is used for the domain while at the boundaries second order forward and second order backward difference formula (for x = 0 and x = l respectively) are used respectively, which results in a second order accurate numerical analysis.

TABLE 1. TABLE SHOWING THE DIFFERENT CASES CONSIDERED

Case	Description	Figure	Dimensions
A	Uniform cross section		L = 1m E = 207 Gpa ρ = 7810 kg/m^3 d = 0.005m q = 0.01 N P_0 = 5 N
B	Non homogenous density distribution		L = 1m Step length= 0.2m E = 207 Gpa d = 0.005m q = 0.01 N ρ =7810kg/m^3 P_0 = 5 N
C	Variable Cross Section		L = 1m step length = 0.2m E = 207 Gpa ρ =7810kg/m^3 d = 0.005m q = 0.01 N P_0 = 5 N

V. OBSERVATIONS

A. Uniform Cross Section

From (6) and (8), it is found that for this rod first critical buckling load for static case is 63N and corresponding first resonance frequency is 60.9 rad/s.

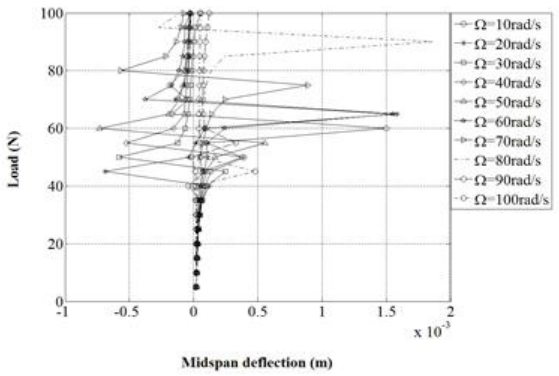


Fig.2.Load vs Midspan deflection plot under various forcing frequencies (10rad/s - 100 rad/s), dynamic case- for uniform cross section (Pcr = 60 – 65N, Ω = 60 rad/s, t = 2.5sec).

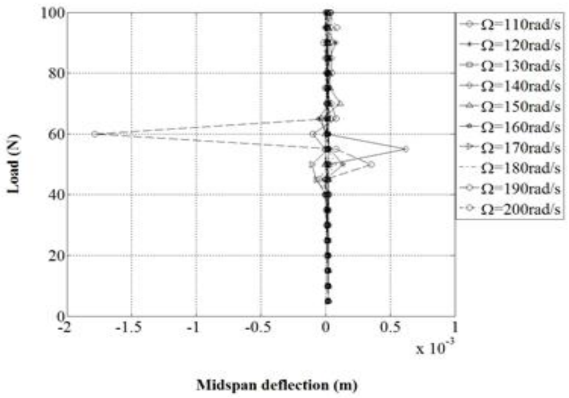


Fig. 3. Load vs Midspan plot deflection under various forcing frequencies(110 rad/s - 200 rad/s), dynamic case- for uniform cross section.

For dynamic case, from the load-deflection analysis (Fig 2. and Fig 3.) it is observed that, up to 40N applied load no buckling occurs for all the frequencies between 10 rad/s to 100 rad/s, but as the applied force crosses the limit of 40N, instability occurs and different values of buckling load for different frequencies are obtained. At 20 rad/s, buckling is observed at near 45N, similarly for 10, 30, 40, 50 rad/s buckling occurs between 50N to 60N. Among these maximum deflection occurs near at 64N at 60 rad/s ($1.6x10^{-3}$m) and at 80 rad/s ($1.8x10^{-3}$m).

B. Non Homogenous Density Distribution

The critical load and resonance frequency have been calculated using the formulas derived for the rod which has uniform density distribution and is under static condition, (6)-(8); but for the varying density distribution there are no such formulas, hence, to observe the nature of the deformation several frequencies are applied.

To compare the change in deflection nature in this specially constructed rod; different frequencies within a range of 10 to 100 rad/s have been chosen. For all these frequencies load is applied from 5N to 100N. The load vs deflection curve for frequency ranging from 10 rad/s to 100 rad/s are plotted in a single curve which shows that their nature is almost same and each experiences buckling at near 60N applied load (Fig. 4).

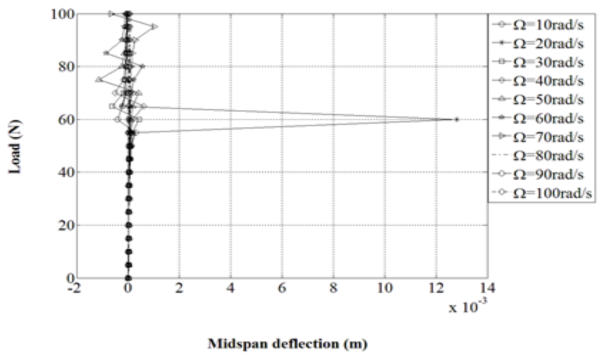


Fig. 4. Load vs Midspan deflection plot under various forcing frequencies, dynamic case, for non- homogenous density distribution. (Pcr = 60 N, Ω = 20 rad/s, t = 2.5sec)

Buckling occurs at 60N under 20 rad/s forcing frequency. The deflection at this buckling load is about 13mm, which is noticeably larger than that of the uniform-density-rod.

When analyzing each curve separately it is observed that for applied load range of 5N - 50N deflection increases almost at a steady rate and for all the frequency the deflections are in the range of 10^{-5}m to 10^{-4}m, and for 60N or above these values increase from $0.5x10^{-3}$m to $12x10^{-3}$m.

Another interesting characteristic observed is that when the load exceeds 60N or above the corresponding deflection decreases and fluctuated within $6x10^{-4}$m- $3x10^{-2}$m. This indicates that for a certain frequency the deflection reaches its maximum values at a certain load (i.e. critical load) and above that load limit do not have a regular trend.

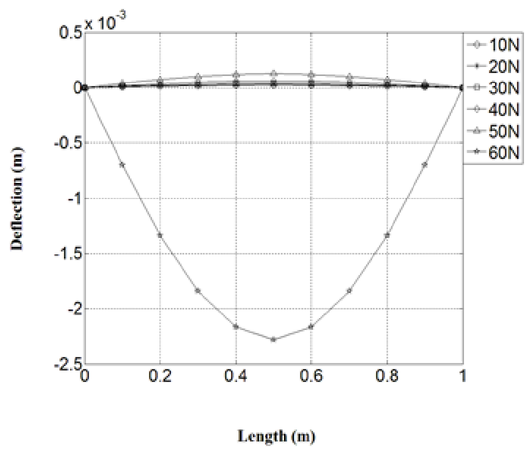


Fig. 5. Deflections under various loads along the length of the rod, Ω = 0 rad/s, quasi static case, non- homogenous density distribution.

From the static analysis the value of the buckling load is determined as 60N- the same for uniform-density-rod but the deflection is much higher as the dynamic case (Fig. 5).

C. Variable Cross Section

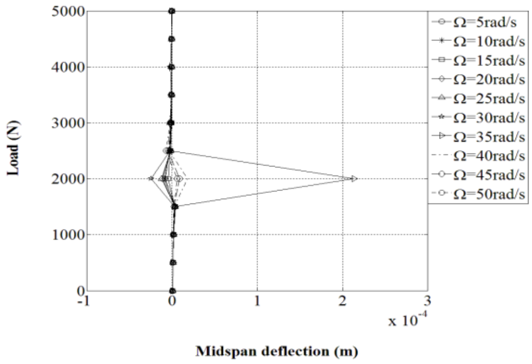


Fig. 6. Load vs Midspan plot deflection plot under various forcing frequencies, dynamic case, for variable cross section (P_{cr} = 2000 N, Ω = 35 rad/s, t = 2.5sec).

For 5N to 5000N applied load, by changing forcing frequency from 5rad/s to 50rad/s the buckling load is found as 2000N at 35 rad/s forcing frequency (Fig. 6).

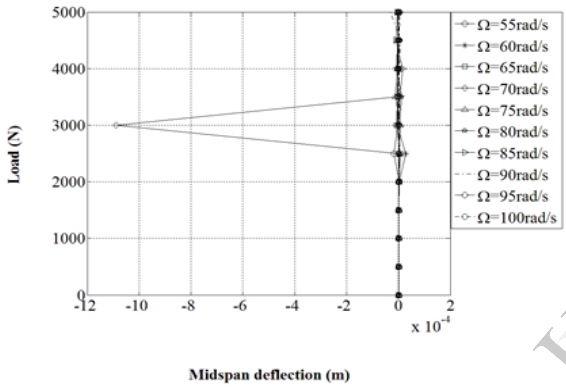


Fig. 7. Load vs Midspan deflection plot under various forcing frequencies, dynamic case, for variable cross section (P_{cr} = 3000 N, Ω = 70 rad/s, t = 2.5sec).

Again at 70 rad/s forcing frequency another buckling load is found, which is equal to 3000N (Fig. 7).

The applied force needed for buckling is much higher when the varying cross section is considered where the mid portion of the rod has diameter three times larger than that of the constant cross section rod and the forcing frequency is almost half.

TABLE 2. COMPARISON AMONG THE THREE CASES

Critical Parameters	Case A	Case B	Case C
Dynamic buckling load (N), P_t	45 - 65	60	2000; 3000
Frequency of oscillating load at buckling (rad/s), Ω	20 - 80	20	35; 70
Midspan deflection (m)	$0.7x10^{-3}$-$1.6x10^{-3}$	$12.8x10^{-3}$	$2.2x10^{-4}$

VI. VALIDATION

To check the validity of the numerical model, both 1st and 2nd critical load is calculated from the analytical formula and the values obtained by the numerical method are compared with those, the values matches for the quasi-static case. The values obtained for the dynamic case are also comparable with the analytical values, which confirms the validity of the numerical model (TABLE 3.).

In case of dynamic state the forcing frequency at which buckling occurs is almost equal to the first vibration frequency of the rod, which also confirms the validity of the model as from the analytical solution it is found that instability occurs when the forcing frequency is equal to the natural frequency of the rod.

Another important characteristic of dynamic buckling is: instability may occur at loading frequencies other than the natural frequency, which is also obtained from the analytical solution, numerical analysis also indicates the same behavior both for first and second mode of buckling (TABLE 4.).

TABLE 3. COMPARISON BETWEEN ANALYTICAL VALUES AND NUMERICAL VALUES (CRITICAL LOAD)

Critical Load	Analytical Value	Numerical Value	
		Quasi static	*Dynamic*
First critical load (N)	63	60	45 - 65
Second critical load (N)	250.68	250	225

TABLE 4. COMPARISON BETWEEN ANALYTICAL VALUES AND NUMERICAL VALUES (NATURAL FREQUENCY)

Natural Frequency	Analytical Value	Numerical Value
		Dynamic
First natural frequency (rad/s)	60.9	60 - 80
Second natural frequency (rad/s)	251.5	120

VII. CONCLUSION

At this stage only deformation and response of the oscillating force is discussed, but there remain a lot of room for studying more aspects of the dynamic buckling of an elastic body, such as the critical wavelength for the buckling instability, dynamic buckling of thin cylindrical shell, effect of damping on the dynamic stability of any arbitrary rods, frames, arches, plates, shells, thin- wall beams etc. The knowledge gained by this particular work will help to deal with more complex and important systems of motion with the effective technique to solve those.

REFERENCES

1. P.Z. Bažant, L. Ceodolin, “Stability of structures: Elastic, Inelastic, Fracture, and DamageTheories,” 2nd ed., New York, 2003, pp. 144.
2. D Wentworth Thompson, On Growth and Form, Cambridge, 1942, pp. 33.
3. M.R. Falvo, G.J. Clary, R.M. Taylor, V. Chi, F.P. Brooks, S. Washburn, R. Superfine, “Bendingand buckling of carbon nanotubes under large strain,” Nature London, vol. 389, pp. 582-584,October 1997.
4. G.R. Abrahamson, J.N. Goodier, “Dynamic flexural buckling of columns within anaxial plastic compression wave,” J Appl Mech California, vol. 33,pp.241–247, June 1966.
5. R. Wang , C.Q. Ru, “An energy criterion for dynamic plastic buckling of circularcylinders under impulsive loading” In: Reid SR, editor. Metal formingandimpact mechanics. Oxford: Pergamon Press, 1985, pp. 213–224.
6. W. Gu, W. Tang, T. Liu, “Dynamic pulse buckling of cylindrical shells subjectedto external impulsive loading,” J. Pressure Vessel Technol China, vol. 118, pp. 33–37, February 1966.
7. R.C. Batra, Z.G. Wei, “Dynamic buckling of a thin thermoviscoplastic rectangular plate,”Elsevier Virginia, vol. 43, pp. 273-290, November 2004.
8. P.Z. Bažant, L. Ceodolin, “stability of structures: Elastic, Inelastic, Fracture, and DamageTheories,” 2nd ed., New York, 2003, pp. 144-168.