Degree of Approximation of function belonging to $\text{Lip}(\alpha, r)$ functions by Product Summability Method

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Abstract

In this paper author have been determined the degree of approximation of certain functions belonging to $\text{Lip}(\alpha, r)$ class by $(C, 1)(E, q)$ means of its Fourier series.

1 Definitions and notations

Let $f(t)$ be periodic functions with period $2\pi$ and integrable in the Lebesgue sense. The Fourier series $f(t)$ is given by

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cdot \cos nt + b_n \cdot \sin nt)$$  \hspace{1cm} (1)

A function $f \in \text{Lip}(\alpha, r)$ for $0 \leq x \leq 2\pi$, if

$$\left( \int_0^{2\pi} |f(x + t) - f(x)|^r dx \right)^{1/r} = O(|t|^\alpha), \quad 0 < \alpha \leq 1, \quad r \geq 1, \quad t > 0$$  \hspace{1cm} (2)

The degree of approximation of a function $f : R \to R$ by trigonometrical polynomial $t_n$ of order $n$ is defined by Zygmund [1]

$$||t_n - f||_\infty = \sup \{ |t_n(x) - f(x)| : x \in R \}$$  \hspace{1cm} (3)

If $(E, q) = E^q_n = \frac{1}{(1 + q)^n} \sum_{k=0}^{n} q^{n-k} \cdot s_k \to s$ as $n \to \infty$. Then an infinite series $\sum_{k=0}^{\infty} u_k$ with the partial sums $s_n$ is said to be summable $(E, q)$ to the definite number $s$. (Hardy [4]).
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The series $\sum_{k=0}^{\infty} u_k$ is said to be $(C, 1)$ summable to $s$. If $(C, 1) = 1 \frac{1}{n+1} \sum_{k=0}^{n} s_k \rightarrow s$ as $n \rightarrow \infty$. The $(C, 1)$ transform of the $(E, q)$ transform defines the $(C, 1)(E, q)$ transform of the partial sums $s_n$ of the series $\sum_{k=0}^{\infty} u_k$.

Thus if $(CE)_n^q = \frac{1}{n+1} \sum_{k=0}^{n} E_k^q \rightarrow s$ as $n \rightarrow \infty$ (4)

where $E_n^q$ denotes the $(E, q)$ transform of $s_n$, then the series $\sum_{k=0}^{\infty} u_k$ is said to be summable $(C, 1)(E, q)$ means or simply summable $(C, 1)(E, q)$ to $s$. We shall use following notation:

$$\phi(t) = f(x+t) + f(x-t) - 2f(x)$$

2 Main Theorem

In this paper we have generalized the theorem of S. Lal [12].

Theorem 2.1. If $f : R \rightarrow R$ is $2\pi$ periodic, Lebesgue integrable on $[-\pi, \pi]$ and belonging to the Lipschitz $(\alpha, r)$ class then the degree of approximation of $f$ by the $(C, 1)(E, q)$ product means of its Fourier series satisfies for $n = 0, 1, 2, 3, ...$

$$||(CE)_n^q f(x) - f(x)||_{\infty} = O\left( \frac{1}{(n+1)^{\alpha-\frac{1}{r}}} \right) \text{ for } 0 < \alpha < 1 \text{ and } r > 1$$

3 Lemmas

For proof of our theorem, we shall use the following lemmas [12].

Lemma 1. Let

$$M_n(t) = \frac{1}{2\pi(n+1)} \sum_{k=0}^{n} \left[ \frac{1}{(q+1)^k} \sum_{r=0}^{k} \binom{k}{r} q^{k-r} \sin(r+\frac{1}{2})t \right]$$

then

$$M_n(t) = O(n+1) \text{ for } 0 < t < \frac{1}{n+1}$$

Lemma 2.

$$M_n(t) = O\left( \frac{1}{t} \right), \text{ for } \frac{1}{n+1} < t < \pi$$
4 Proof of the Theorem

The $n^{th}$ partial sum $s_n(x)$ of the series (1) at $t = x$ is written as

$$ s_n(x) = f(x) + \frac{1}{2\pi} \int_0^\pi \phi(t) \cdot \frac{\sin(n + \frac{1}{2})t}{\sin(\frac{t}{2})} dt $$

So that $(E, q)$ means of the series (1) are

$$ E^q_n(x) = \frac{1}{(q+1)^n} \sum_{k=0}^n \left( \frac{n}{k} \right) q^{n-k} s_k(x) $$

$$ = f(x) + \frac{1}{2\pi(q+1)^n} \int_0^\pi \phi(t) \left( \sum_{k=0}^n \left( \frac{n}{k} \right) \sin \left( k + \frac{1}{2} \right) t \right) dt. $$

Therefore $(C, 1)(E, q)$ means of the series (1) are

$$ (CE)^q_n(x) = \frac{1}{(n+1)} \sum_{k=0}^n E^q_k(x) \quad (n = 0, 1, 2, 3, \ldots) $$

$$ = f(x) + \frac{1}{2\pi(n+1)} \sum_{k=0}^n \left\{ \frac{1}{(q+1)^k} \int_0^\pi \phi(t) \left( \sum_{r=0}^k \left( \frac{k}{r} \right) q^{k-r} \sin \left( r + \frac{1}{2} \right) t \right) dt \right\} $$

$$ = f(x) + \int_0^\pi \phi(t) \cdot M_n(t) dt $$

(5)

where

$$ M_n(t) = \frac{1}{2\pi(n+1)} \sum_{k=0}^n \left[ \frac{1}{(q+1)^k} \sum_{r=0}^k \left( \frac{k}{r} \right) q^{k-r} \sin \left( r + \frac{1}{2} \right) t \right] $$

so

$$ (CE)^q_n(x) - f(x) = \int_0^\pi \phi(t) \cdot M_n(t) dt $$

$$ = \left( \int_0^\frac{1}{n+1} \phi(t) \cdot M_n(t) dt + \int_\frac{1}{n+1}^\pi \phi(t) \cdot M_n(t) dt \right) $$

$$ = I_1 + I_2 $$

(6)

Now

$$ I_1 = \int_0^\frac{1}{n+1} \phi(t) \cdot M_n(t) dt $$

$$ |I_1| \leq \left( \int_0^\frac{1}{n+1} [\phi(t)]^r dt \right)^\frac{1}{r} \cdot \left( \int_0^\frac{1}{n+1} [M_n(t)]^s dt \right)^\frac{1}{s} $$

using Hölder’s inequality
\[ |I_1| \leq O \left( \frac{1}{(n+1)^a} \right) \cdot \left( \int_0^{\frac{1}{n+1}} (n+1)^s \, dt \right)^{\frac{1}{2}} \]
\[ |I_1| \leq O \left( \frac{1}{(n+1)^a} \right) \cdot \left[ \frac{(n+1)^s}{n+1} \right]^{\frac{1}{2}} \]
\[ |I_1| \leq O \left( \frac{1}{(n+1)^a} \right) \cdot \left( \frac{1}{(n+1)^{\frac{1-s}{2}}} \right) \]
\[ |I_1| \leq O \left( \frac{1}{(n+1)^{a+\frac{1}{2}-1}} \right) \]
\[ |I_1| \leq O \left( \frac{1}{(n+1)^{a-(1-\frac{1}{2})}} \right) \quad \therefore \frac{1}{r} + \frac{1}{s} = 1 \]
\[ |I_1| \leq O \left( \frac{1}{(n+1)^{a-\frac{1}{2}}} \right) \]

Next
\[ I_2 = \int_{\frac{1}{n+1}}^{\pi} \phi(t) \cdot M_n(t) \, dt \]
\[ |I_2| = \left| \int_{\frac{1}{n+1}}^{\pi} \phi(t) \cdot M_n(t) \, dt \right| \]
\[ |I_2| \leq \left( \int_{\frac{1}{n+1}}^{\pi} (\phi(t))^r \, dt \right)^{\frac{1}{2}} \left( \int_{\frac{1}{n+1}}^{\pi} (M_n(t))^s \, dt \right)^{\frac{1}{2}} \]
\[ |I_2| \leq O \left( \frac{1}{(n+1)^a} \right) \left( \int_{\frac{1}{n+1}}^{\pi} \frac{1}{t^s} \, dt \right)^{\frac{1}{2}} \]
\[ |I_2| \leq O \left( \frac{1}{(n+1)^a} \right) \left[ \frac{1}{n+1} \right]^{\frac{1-s}{2}} \]
\[ |I_2| \leq O \left( \frac{1}{(n+1)^{a+\frac{1}{2}}} \right) \]
\[ |I_2| \leq O \left( \frac{1}{(n+1)^{a+\frac{1}{2}-1}} \right) \]
\[ |I_2| \leq O \left( \frac{1}{(n+1)^{a-\frac{1}{2}}} \right) \]
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Then from (6) and the above inequalities we have

$$||t_n - f||_\infty = \sup \{|t_n(x) - f(x)| : x \in R\} = O \left(\frac{1}{(n+1)^{\alpha-\frac{1}{r}}}\right), \quad 0 < \alpha < 1, \quad r > 1.$$  

This completes the Proof of the theorem.

5 Corollary

If $r \to \infty$ then degree of approximation of a function $f \in \text{Lip}_\alpha$ is given by

$$||(CE)^n_q(x) - f(x)||_\infty = O \left((n+1)^{-\alpha}\right)\quad \text{for } 0 < \alpha < 1$$

which reduces to the theorem of S. Lal [12].

References


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