Degree of Approximation of function belonging to $Lip(\alpha, r)$ functions by Product Summability Method

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Abstract

In this paper author have been determined the degree of approximation of certain functions belonging to $Lip(\alpha, r)$ class by $(C, 1)(E, q)$ means of its Fourier series.

1 Definitions and notations

Let $f(t)$ be periodic functions with period 2π and integrable in the Lebesgue sense. The fourier series $f(t)$ is given by

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cdot \cos nt + b_n \cdot \sin nt) \tag{1}$$

A function $f \in Lip(\alpha, r)$ for $0 \le x \le 2\pi$, if

$$\left(\int_0^{2\pi} |f(x+t) - f(x)|^r dx \right)^{1/r} = O(|t|^\alpha),\ 0 < \alpha \le 1,\ r \ge 1,\ t > 0 \tag{2}$$

The degree of approximation of a function $f : R \to R$ by trigonometrical polynomial t_n of order n is defined by Zygmund [1]

$$||t_n - f||_\infty = \sup\{|t_n(x) - f(x)| : x \in R\} \tag{3}$$

If $(E, q) = E_n^q = \frac{1}{(1+q)^n} \sum_{k=0}^{n} q^{n-k} \cdot s_k \to s$ as $n \to \infty$. Then an infinite series $\sum_{k=0}^{\infty} u_k$ with the partial sums s_n is said to be summable (E, q) to the definite number s. (Hardy [4]).

The series $\sum_{k=0}^{\infty} u_k$ is said to be $(C,1)$ summable to s. If $(C,1) = \frac{1}{(n+1)} \sum_{k=0}^{n} s_k \to s$ as $n \to \infty$. The $(C,1)$ transform of the (E,q) transform defines the $(C,1)(E,q)$ transform of the partial sums s_n of the series $\sum_{k=0}^{\infty} u_k$.

Thus if

$$(CE)_n^q = \frac{1}{(n+1)} \sum_{k=0}^{n} E_k^q \to s \text{ as } n \to \infty \tag{4}$$

where E_n^q denotes the (E,q) transform of s_n, then the series $\sum_{k=0}^{\infty} u_k$ is said to be summable $(C,1)(E,q)$ means or simply summable $(C,1)(E,q)$ to s. We shall use following notation:

$$\phi(t) = f(x+t) + f(x-t) - 2f(x)$$

2 Main Theorem

In this paper we have generalized the theorem of S. Lal [12].

Theorem 2.1. If $f : R \to R$ is 2π periodic, Lebesgue integrable on $[-\pi, \pi]$ and belonging to the Lipschitz (α, r) class then the degree of approximation of f by the $(C,1)(E,q)$ product means of its Fourier series satisfies for $n = 0, 1, 2, 3, \ldots$

$$||(CE)_n^q(x) - f(x)||_\infty = O\left(\frac{1}{(n+1)^{\alpha - \frac{1}{r}}}\right) \text{ for } 0 < \alpha < 1 \text{ and } r > 1$$

3 Lemmas

For proof of our theorem, we shall use the following lemmas [12].

Lemma 1. Let

$$M_n(t) = \frac{1}{2\pi(n+1)} \sum_{k=0}^{n} \left[\frac{1}{(q+1)^k} \sum_{r=0}^{k} \binom{k}{r} q^{k-r} \frac{\sin(r+\frac{1}{2})t}{\sin\frac{t}{2}} \right]$$

then

$$M_n(t) = O(n+1) \text{ for } 0 < t < \frac{1}{n+1}$$

Lemma 2.

$$M_n(t) = O\left(\frac{1}{t}\right), \text{ for } \frac{1}{n+1} < t < \pi$$

4 Proof of the Theorem

The n$^{\text{th}}$partial sum $s_n(x)$ of the series (1) at $t = x$ is written as

$$s_n(x) = f(x) + \frac{1}{2\pi} \int_0^\pi \phi(t) \cdot \frac{\sin(n + \frac{1}{2})t}{\sin\left(\frac{t}{2}\right)} dt$$

So that (E, q) means of the series (1) are

$$\begin{aligned} E_n^q(x) &= \frac{1}{(q+1)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} s_k(x) \\ &= f(x) + \frac{1}{2\pi(q+1)^n} \int_0^\pi \frac{\phi(t)}{\sin\left(\frac{t}{2}\right)} \left(\sum_{k=0}^{n} \binom{n}{k} \sin\left(k + \frac{1}{2}\right) t \right) dt. \end{aligned}$$

Therefore $(C, 1)(E, q)$ means of the series (1) are

$$\begin{aligned} (CE)_n^q(x) &= \frac{1}{(n+1)} \sum_{k=0}^{n} E_k^q(x) \qquad (n = 0, 1, 2, 3, ...) \\ &= f(x) + \frac{1}{2\pi(n+1)} \sum_{k=0}^{n} \left\{ \frac{1}{(q+1)^k} \int_0^\pi \frac{\phi(t)}{\sin\left(\frac{t}{2}\right)} \left(\sum_{r=0}^{k} \binom{k}{r} q^{k-r} \sin\left(r + \frac{1}{2}\right) t \, dt \right) \right\} \\ &= f(x) + \int_0^\pi \phi(t) \cdot M_n(t) dt \qquad (5) \end{aligned}$$

where

$$M_n(t) = \frac{1}{2\pi(n+1)} \sum_{k=0}^{n} \left[\frac{1}{(q+1)^k} \sum_{r=0}^{k} \binom{k}{r} q^{k-r} \frac{\sin(r + \frac{1}{2})t}{\sin(t/2)} \right]$$

so

$$\begin{aligned} (CE)_n^q(x) - f(x) &= \int_0^\pi \phi(t) \cdot M_n(t) dt \\ &= \left(\int_0^{\frac{1}{n+1}} + \int_{\frac{1}{n+1}}^{\pi} \right) \phi(t) \cdot M_n(t) dt \\ &= I_1 + I_2 \qquad (6) \end{aligned}$$

Now

$$I_1 = \int_0^{\frac{1}{n+1}} \phi(t) \cdot M_n(t) dt$$

$$|I_1| \leq \left(\int_0^{\frac{1}{n+1}} [\phi(t)]^r dt \right)^{\frac{1}{r}} \cdot \left(\int_0^{\frac{1}{n+1}} [M_n(t)]^s dt \right)^{\frac{1}{s}}, \text{ using Hölder's inequality}$$

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$$|I_1| \leq O\left(\frac{1}{(n+1)^\alpha}\right) \cdot \left(\int_0^{\frac{1}{n+1}} (n+1)^s dt\right)^{\frac{1}{s}}$$

$$|I_1| \leq O\left(\frac{1}{(n+1)^\alpha}\right) \cdot \left[\frac{(n+1)^s}{n+1}\right]^{\frac{1}{s}}$$

$$|I_1| \leq O\left(\frac{1}{(n+1)^\alpha}\right) \cdot \left(\frac{1}{(n+1)^{\frac{1-s}{s}}}\right)$$

$$|I_1| \leq O\left(\frac{1}{(n+1)^{\alpha+\frac{1}{s}-1}}\right)$$

$$|I_1| \leq O\left(\frac{1}{(n+1)^{\alpha-\left(1-\frac{1}{s}\right)}}\right) \qquad \because \frac{1}{r}+\frac{1}{s}=1$$

$$|I_1| \leq O\left(\frac{1}{(n+1)^{\alpha-\frac{1}{r}}}\right)$$

Next

$$I_2 = \int_{\frac{1}{n+1}}^{\pi} \phi(t) \cdot M_n(t) dt$$

$$|I_2| = \left|\int_{\frac{1}{n+1}}^{\pi} \phi(t) \cdot M_n(t) dt\right|$$

$$|I_2| \leq \left(\int_{\frac{1}{n+1}}^{\pi} (\phi(t))^r dt\right)^{\frac{1}{r}} \left(\int_{\frac{1}{n+1}}^{\pi} (M_n(t))^s dt\right)^{\frac{1}{s}}$$

$$|I_2| \leq O\left(\frac{1}{(n+1)^\alpha}\right) \left(\int_{\frac{1}{n+1}}^{\pi} \frac{1}{t^s} dt\right)^{\frac{1}{s}}$$

$$|I_2| \leq O\left(\frac{1}{(n+1)^\alpha}\right) \left[\frac{1}{n+1}\right]^{\frac{1-s}{s}}$$

$$|I_2| \leq O\left(\frac{1}{(n+1)^{\alpha+\frac{1-s}{s}}}\right)$$

$$|I_2| \leq O\left(\frac{1}{(n+1)^{\alpha+\frac{1}{s}-1}}\right)$$

$$|I_2| \leq O\left(\frac{1}{(n+1)^{\alpha-\frac{1}{r}}}\right)$$

Then from (6) and the above inequalities we have

$$||t_n - f||_\infty = \sup\{|t_n(x) - f(x)| : x \in R\} = O\left(\frac{1}{(n+1)^{\alpha - \frac{1}{r}}}\right),\ 0 < \alpha < 1,\ r > 1.$$

This completes the Proof of the theorem.

5 Corollary

If $r \to \infty$ then degree of approximation of a function $f \in Lip\alpha$ is given by

$$||(CE)_n^q(x) - f(x)||_\infty = O\left((n+1)^{-\alpha}\right) \text{ for } 0 < \alpha < 1$$

which reduces to the theorem of S. Lal [12].

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