Cubic Z-Ideals in Z-Algebras

S. Sowmiya
Assistant Professor, Department of Mathematics,
Sri Ramakrishna Engineering College
Vattamalaipalayam,Coimbatore-22,Tamilnadu,India

P. Jeyalakshmi
Professor and Head, Department of Mathematics,
Avinashilingam Institute for Home Science and Higher
Education for Women, Coimbatore-43,Tamilnadu,India.

***Abstract*—In this article, the notions of Cubic Z-Ideals in Z-algebras is introduced and some of their properties are investigated. The Z-homomorphic image and inverse image of cubic Z-Ideals in Z- algebras is investigated. Also, the cartesian product of cubic Z-Ideals in Z-algebras are also discussed.**

2010 Mathematics Subject Classification. 06F35, 03G25, 08A27

Keywords*—*Z-algebra, Z-ideal, Z-homomorphism, Cubic Z-ideal .

I. INTRODUCTION

Imai and Iseki [2, 3] introduced two new classes of algebras that arise from the propositional logic. In 2017, Chandramouleeswaran et al. [1] introduced the concept of Z-algebra as a new structure of algebra based on propositional logic. Zadeh [19] introduced the notion of fuzzy sets in 1965. In 1975, Zadeh [20] made an extension of the concept of fuzzy set by an interval-valued fuzzy set whose membership function is many-valued and form an interval in the membership scale. In our earlier paper [7–18] we have introduced the concept of cubic set to Z-Subalgebras in Z-algebras and the concepts of fuzzy set, interval-valued fuzzy set, intuitionistic fuzzy set, intuitionistic L-fuzzy set, interval-valued intuitionistic fuzzy set to Z-Subalgebras and Z-ideals in Z-algebras. In 2012, using a fuzzy set and an interval-valued fuzzy set, Jun et al. [6] introduced a new notion called a cubic set and investigated several properties. Meanwhile, in 2010, Jun et al. [5] introduced the notion of cubic subalgebras/cubic ideals in BCK/BCIalgebras and they investigated several properties.In 2011, Jun et al. [4] applied the notion called a cubic sets to a group and introduced the notion of cubic subgroup. In this paper, we have introduced the concept of cubic Z-Ideals of Z-algebras and investigated some of their properties.

II. PRELIMINARIES

In this section, we recall some basic definitions that are required for our work

Definition 2.1[1] A Z-algebra $(X,*,0)$ is a nonempty set X with a constant 0 and a binary operation $*$ satisfying the following conditions:

(Z1) $x*0=0$

(Z2) $0*x=x$

(Z3) $x*x=x$

(Z4) $x*y=y*x$ when $x\neq 0$ and $y\neq 0$ $\forall$ $x, y \in X$.

Definition 2.2[1] Let $(X,*,0)$ and $(Y,*',0')$ be two Z-algebras. A mapping $h:(X,*,0)\to(Y,*',0')$ is said to be a **Z-homomorphism** of Z-algebras if $h(x*y)=h(x)*'h(y)$ for all $x, y\in X$.

Definition 2.3:[6] Let X be a nonempty set . A **cubic set** A in X is a structure $A=\{\langle x,\tilde{\mu}_A(x),\omega_A(x)\rangle | x\in X\}$ briefly denoted by $A=(\tilde{\mu}_A,\omega_A)$ where $\tilde{\mu}_A(x):[\mu_A^L,\mu_A^U]:X\to D[0,1]$ is an interval-valued fuzzy set in X and $\omega_A:X\to[0,1]$ is a fuzzy set in X .

For two cubic sets $A=(\tilde{\mu}_A,\omega_A)$ and $B=(\tilde{\mu}_B,\omega_B)$ in X, we define

1. $A\subseteq B$ iff $\tilde{\mu}_A\le\tilde{\mu}_B$ and $\omega_A\ge\omega_B$

2. $A=B$ iff $A\subset B$ and $B\subset A$.

3. $A^c=\{\langle x,\omega_A(x),\tilde{\mu}_A(x)\rangle \mid x\in X\}$

4. $A\cap B=\{\langle x,\tilde{\mu}_{A\cap B}(x),\omega_{A\cup B}(x)\rangle | x\in X\}$

$=\{\langle x, r\min(\tilde{\mu}_A(x),\tilde{\mu}_B(x)),\max(\omega_A(x),\omega_B(x))\rangle | x\in X\}$

5. $A\cup B=\{\langle x,\tilde{\mu}_{A\cup B}(x),\omega_{A\cap B}(x)\rangle | x\in X\}$

$=\{\langle x, r\max(\tilde{\mu}_A(x),\tilde{\mu}_B(x)),\min(\omega_A(x),\omega_B(x))\rangle | x\in X\}$

Definition 2.4:[4] Let $A=(\tilde{\mu}_A,\omega_A)$ be a cubic set of X. For $[s_1,s_2]\in D[0,1]$ and $t\in[0,1]$, the set $U(\tilde{\mu}_A;[s_1,s_2])=\{x\in X \mid \tilde{\mu}_A(x)\ge[s_1,s_2]\}$ is called an **interval-valued upper** $[s_1,s_2]$ **-level subset** of A and $L(\omega_A;t)=\{x\in X \mid \omega_A(x)\le t\}$ is called **lower t-level subset** of A.

Definition 2.5:[4] A cubic set $A=(\tilde{\mu}_A,\omega_A)$ in a nonempty set X is said to have the **rsup-inf property** if for any subset T

International Journal of Engineering Research & Technology (IJERT)
ISSN: 2278-0181
Vol. 10 Issue 02, February-2021

of X there exists $t_0 \in T$ such that $\tilde{\mu}_A(t_0) = r\sup_{t \in T} \tilde{\mu}_A(t)$ and $\omega_A(t_0) = \inf_{t \in T} \omega_A(t)$ respectively.

Definition 2.6:[6] Consider a collection of cubic sets $A_i = \{\langle x, \tilde{\mu}_{A_i}(x), \omega_{A_i}(x)\rangle | x \in X\}$ where $i \in \Omega$,

(i) P-union and P-intersection denoted by $P\left(\bigcup_{i\in\Omega} A_i\right)$ and $P\left(\bigcap_{i\in\Omega} A_i\right)$ are defined as follows.

$$P\left(\bigcup_{i\in\Omega} A_i\right) = \left\{\left\langle x, \tilde{\mu}_{\cup_{i\in\Omega} A_i}(x), \omega_{\cup_{i\in\Omega} A_i}(x)\right\rangle \middle| x \in X\right\}$$
$$= \left\{\left\langle x, r\sup_{i\in\Omega} \tilde{\mu}_{A_i}(x), \sup_{i\in\Omega} \omega_{A_i}(x)\right\rangle \middle| x \in X\right\},$$
$$P\left(\bigcap_{i\in\Omega} A_i\right) = \left\{\left\langle x, \tilde{\mu}_{\cap_{i\in\Omega} A_i}(x), \omega_{\cap_{i\in\Omega} A_i}(x)\right\rangle \middle| x \in X\right\}$$
$$= \left\{\left\langle x, r\inf_{i\in\Omega} \tilde{\mu}_{A_i}(x), \inf_{i\in\Omega} \omega_{A_i}(x)\right\rangle \middle| x \in X\right\}$$

(ii) Union and intersection denoted by $\bigcup_{i\in\Omega} A_i$ and $\bigcap_{i\in\Omega} A_i$ are defined as follows.

$$\bigcup_{i\in\Omega} A_i = \left\{\left\langle x, \tilde{\mu}_{\cup_{i\in\Omega} A_i}(x), \omega_{\cap_{i\in\Omega} A_i}(x)\right\rangle \middle| x \in X\right\}$$
$$= \left\{\left\langle x, r\sup_{i\in\Omega} \tilde{\mu}_{A_i}(x), \inf_{i\in\Omega} \omega_{A_i}(x)\right\rangle \middle| x \in X\right\},$$
$$\bigcap_{i\in\Omega} A_i = \left\{\left\langle x, \tilde{\mu}_{\cap_{i\in\Omega} A_i}(x), \omega_{\cup_{i\in\Omega} A_i}(x)\right\rangle \middle| x \in X\right\}$$
$$= \left\{\left\langle x, r\inf_{i\in\Omega} \tilde{\mu}_{A_i}(x), \sup_{i\in\Omega} \omega_{A_i}(x)\right\rangle \middle| x \in X\right\}$$

Definition 2.7:[4] Let h be a mapping from a set X into a set Y.

(i) Let $A = (\tilde{\mu}_A, \omega_A)$ be a cubic set in X. Then the image of A under h, denoted by $h(A) = \{\langle y, \tilde{\mu}_{h(A)}(y), \omega_{h(A)}(y)\rangle | y \in Y\}$, is defined by:

$$\tilde{\mu}_{h(A)}(y) = \begin{cases} r\sup_{z\in h^{-1}(y)} \tilde{\mu}_A(z) & \text{if } \; h^{-1}(y) = \{x \mid h(x) = y\} \neq \phi \\ [0,0] & \text{otherwise} \end{cases}$$

and

$$\omega_{h(A)}(y) = \begin{cases} \inf_{z\in h^{-1}(y)} \omega_A(z) & \text{if } \; h^{-1}(y) = \{x \mid h(x) = y\} \neq \phi \\ 1 & \text{otherwise} \end{cases}$$

is a cubic set in Y.

(ii) Let $B = (\tilde{\mu}_B, \omega_B)$ be a cubic set in Y. Then the inverse image (or pre-image) of B under h, denoted by $h^{-1}(B) = \{\langle x, \tilde{\mu}_{h^{-1}(B)}(x), \omega_{h^{-1}(B)}(x)\rangle | x \in X\}$ is a cubic set in X defined by $\tilde{\mu}_{h^{-1}(B)}(x) = \tilde{\mu}_B(h(x))$ and $\omega_{h^{-1}(B)}(x) = \omega_B(h(x))$ for all $x \in X$.

Definition 2.8:[4] Let $A = (\tilde{\mu}_A, \omega_A)$ and $B = (\tilde{\mu}_B, \omega_B)$ be any two cubic sets in X. Then, the Cartesian product of cubic sets A and B is given by $A \times B = (\tilde{\mu}_{A\times B}, \omega_{A\times B})$ where $\tilde{\mu}_{A\times B} : X \times X \to D[0,1]$ and $\omega_{A\times B} : X \times X \to [0,1]$ are defined by $\tilde{\mu}_{A\times B}(x, y) = r\min\{\tilde{\mu}_A(x), \tilde{\mu}_B(y)\}$ and $\omega_{A\times B}(x, y) = \max\{\omega_A(x), \omega_B(y)\}$ for all $(x, y) \in X \times X$.

III. CUBIC Z-IDEALS IN Z-ALGEBRAS

In this section, the notion of Cubic Z-ideals in Z-algebras is defined and corresponding results are proved.

Definition 3.1: Let $(X,*,0)$ be a Z-algebra. A cubic set $A = (\tilde{\mu}_A, \omega_A)$ in X is called a **cubic Z-ideal** of X if it satisfies the following conditions:

(i) $\tilde{\mu}_A(0) \geq \tilde{\mu}_A(x)$ and $\omega_A(0) \leq \omega_A(x)$

(ii) $\tilde{\mu}_A(x) \geq r\min\{\tilde{\mu}_A(x * y), \tilde{\mu}_A(y)\}$

(iii) $\omega_A(x) \leq \max\{\omega_A(x * y), \omega_A(y)\}$, for all $x, y \in X$.

Example 3.2: Consider a Z-algebra X= {0,1,2,3} with the following Cayley table :

*	0	1	2	3
0	0	1	2	3
1	0	1	3	1
2	0	3	2	1
3	0	1	1	3

Define a cubic set A in X by $\tilde{\mu}_A(x) = [0.6, 0.8]$ and $\omega_A(x) = 0.2$, for all $x \in X$. Then, A is a cubic Z-ideal of a Z-algebra X.

International Journal of Engineering Research & Technology (IJERT)
ISSN: 2278-0181
Vol. 10 Issue 02, February-2021

Theorem 3.3: The intersection of any set of cubic Z-ideals of a Z-algebra X is also a cubic Z-ideal of X.

Proof: Let $A_i = \{\langle x, \tilde{\mu}_{A_i}(x), \omega_{A_i}(x)\rangle | x \in X\}$ where $i \in \Omega$ an index set, be a set of cubic Z-ideals of a Z-algebra X . Then for any $x, y \in X$,

$$\tilde{\mu}_{\cap A_i}(0) = r\inf \tilde{\mu}_{A_i}(0) \geq r\inf \tilde{\mu}_{A_i}(x) = \tilde{\mu}_{\cap A_i}(x)$$

$$\omega_{\cup A_i}(0) = \sup \omega_{A_i}(0) \geq \sup \omega_{A_i}(x) = \omega_{\cup A_i}(x)$$

$$\tilde{\mu}_{\cap A_i}(x) = r\inf \tilde{\mu}_{A_i}(x) \geq r\inf\{r\min\{\tilde{\mu}_{A_i}(x*y), \tilde{\mu}_{A_i}(y)\}\}$$

$$= r\min\{r\inf \tilde{\mu}_{A_i}(x*y), r\inf \tilde{\mu}_{A_i}(y)\}$$

$$= r\min\{\tilde{\mu}_{\cap A_i}(x*y), \tilde{\mu}_{\cap A_i}(y)\}$$

and $\omega_{\cup A_i}(x) = \sup \omega_{A_i}(x) \leq \sup\{\max\{\omega_{A_i}(x*y), \omega_{A_i}(y)\}\}$

$$= \max\{\sup \omega_{A_i}(x*y), \sup \omega_{A_i}(y)\}$$

$$= \max\{\omega_{\cup A_i}(x*y), \omega_{\cup A_i}(y)\}$$

Hence $\bigcap_{i\in\Omega} A_i = (\tilde{\mu}_{\cap A_i}, \omega_{\cup A_i})$ is a cubic Z-ideal of a Z-algebra X.

Theorem 3.4: Let $A_i = (\tilde{\mu}_{A_i}, \omega_{A_i})$ be a set of cubic Z-ideals of a Z-algebra X, where $i \in \Omega$ an index set. If $r\sup\{r\min\{\tilde{\mu}_{A_i}(x*y), \tilde{\mu}_{A_i}(y)\}\}$ $= r\min\{r\sup \tilde{\mu}_{A_i}(x*y), r\sup \tilde{\mu}_{A_i}(y)\}$ and $\inf\{\max\{\omega_{A_i}(x*y), \omega_{A_i}(y)\}\} = \max\{\inf \omega_{A_i}(x*y), \inf \omega_{A_i}(y)\}$, for all $x, y \in X$, then the union of A_i is again a cubic Z-ideal of X.

Theorem 3.5: Let $A_i = (\tilde{\mu}_{A_i}, \omega_{A_i})$ be a set of cubic Z-ideals of a Z-algebra X, where $i \in \Omega$ an index set. If $\inf\{\max\{\omega_{A_i}(x*y), \omega_{A_i}(y)\}\} = \max\{\inf \omega_{A_i}(x*y), \inf \omega_{A_i}(y)\}$, for all $x, y \in X$, then the P-intersection of A_i is again a cubic Z-ideal of X.

Theorem 3.6: Let $A_i = (\tilde{\mu}_{A_i}, \omega_{A_i})$ be a set of cubic Z-ideals of a Z-algebra X, where $i \in \Omega$ an index set. If $r\sup\{r\min\{\tilde{\mu}_{A_i}(x*y), \tilde{\mu}_{A_i}(y)\}\}$ $= r\min\{r\sup \tilde{\mu}_{A_i}(x*y), r\sup \tilde{\mu}_{A_i}(y)\}$, for all $x, y \in X$, then the P-union of A_i is again a cubic Z-Subalgebra of X.

Theorem 3.7: Cubic set $A = (\tilde{\mu}_A, \omega_A)$ of a Z-algebra X is a cubic Z-ideal of X where $\tilde{\mu}_A = [\mu_A^L, \mu_A^U]$ if and only if μ_A^L, μ_A^U and $(\omega_A)^c$ are fuzzy Z-ideals of X.

Analogously, the following theorems can be proved.

Theorem 3.8: Let $A = (\tilde{\mu}_A, \omega_A)$ be a cubic set in a Z-algebra X. Then A is a cubic Z-ideal of X if and only if for all $[s_1, s_2] \in D[0,1]$ and $t \in [0,1]$, the sets $U(\tilde{\mu}_A; [s_1, s_2])$ and $L(\omega_A; t)$ of A are either empty or Z-ideals of X.

Theorem 3.9: Let h be a Z-homomorphism from a Z-algebra $(X,*,0)$ onto a Z-algebra $(Y,*',0')$ and A be a cubic Z-ideal of X with rsup-inf property. Then image of A denoted by $h(A)$ is a cubic Z-ideal of Y.

Theorem 3.10: Let $h:(X,*,0) \to (Y,*',0')$ be a Z-homomorphism of Z-algebras. If B is a cubic Z-ideal of Y, then $h^{-1}(B)$ is a cubic Z-ideal of X.

Theorem 3.11: Let $h:(X,*,0) \to (Y,*',0')$ be an Z-epimorphism of Z-algebras. Let B be a cubic set of Y. If $h^{-1}(B)$ is a cubic Z-ideal of X then B is a cubic Z-ideal of Y.

Theorem 3.12: If A and B be cubic Z-ideals of Z-algebra X then $A \times B$ is a cubic Z-ideal in $X \times X$.

Theorem 3.13: Let A and B be two cubic sets of a Z-algebra X. If $A \times B$ is a cubic Z-ideal of $X \times X$, the following are true.

(i) $\tilde{\mu}_A(0) \geq \tilde{\mu}_B(y)$ and $\tilde{\mu}_B(0) \geq \tilde{\mu}_A(x)$ for all $x, y \in X$.

(ii) $\omega_A(0) \leq \omega_B(y)$ and $\omega_B(0) \leq \omega_A(x)$ for all $x, y \in X$.

Proof: Assume that $\tilde{\mu}_B(y) > \tilde{\mu}_A(0)$ and $\tilde{\mu}_A(x) > \tilde{\mu}_B(0)$ for some $x, y \in X$.

Then

$$\tilde{\mu}_{A\times B}(x,y) = r\min\{\tilde{\mu}_A(x), \tilde{\mu}_B(y)\} > r\min\{\tilde{\mu}_B(0), \tilde{\mu}_A(0)\}$$

$$= \tilde{\mu}_{A\times B}(0,0)$$

which is a contradiction.

Similarly, assume that $\omega_A(x) < \omega_B(0)$ and $\omega_B(y) < \omega_A(0)$ for some $x, y \in X$.

Then

$$\omega_{A\times B}(x,y) = \max\{\omega_A(x), \omega_B(y)\} < \max\{\omega_B(0), \omega_A(0)\}$$

$$= \omega_{A\times B}(0,0)$$

which is also a contradiction. Thus proving the result.

Theorem 3.14: Let A and B be two cubic sets of a Z-algebra X such that $A \times B$ is a cubic Z-ideal of $X \times X$. Then either A or B is a cubic Z-ideal of X.

IV CONCLUSION

In this article, we have introduced cubic Z-ideals in Z-algebras and discussed their properties. We extend this concept in our research work.

V ACKNOWLEDGMENT

Authors wish to thank Dr.M.Chandramouleeswaran, Professor and Head, PG Department of Mathematics, Sri Ramanas College of Arts and Science for Women, Aruppukottai, for his valuable suggestions to improve this paper a successful one.

REFERENCES

[1] M.Chandramouleeswaran,P.Muralikrishna,K.Sujatha,S.Sabarinathan: "A note on Z-algebra", Italian Journal of Pure and Applied Mathematics-N.38 (2017), pp.707–714.

[2] Y.Imai and K.Iseki:On axiom systems of propositional calculi XIV, Proceedings of the Japan Academy,42(1966),pp.19-22.

[3] K.Iseki:An algebra related with a propositional calculus, Proceedings of the Japan Academy, 42(1966), pp.26-29.

[4] Y.B.Jun,S.T.Jung and M.S.Kim: Cubic Subgroups,Annals of Fuzzy Mathematics and informatics,2(1)(2011),pp.9-15.

[5] Y.B.Jun,C.S.Kim and M.S.Kang:Cubic Subalgebras and Ideals of BCK/BCI-algebras,Far East Journal of Mathematical Sciences,44(2)(2010),pp.239-250.

[6] Y.B.Jun,C.S.Kim and K.O.Yang:Cubic Sets,Annals of Fuzzy Mathematics and informatics,4(1)(2012), pp.83-98.

[7] S.Sowmiya and P.Jeyalakshmi: Fuzzy Algebraic Structure in Z-algebras,World Journal of Engineering Research and Technology , 5(4)(2019),pp.74-88.

[8] S.Sowmiya and P.Jeyalakshmi: On Fuzzy Z-ideals in Z-algebras,Global Journal of Pure and Applied Mathematics,15(4), (2019), pp.505-516.

[9] S.Sowmiya and P.Jeyalakshmi:Fuzzy α-Translations and Fuzzy β-Multiplications of Z-algebras, Advances in Mathematics:Scientific Journal, 9(3)(2020), pp.1287-1292.

[10] S.Sowmiya and P.Jeyalakshmi:Z-Homomorphism and Cartesian Product on Fuzzy α-Translations and Fuzzy β- Multiplications of Z-algebras, AIP Conference Proceedings 2261 (2020), 030098-1 - 030098-5.

[11] S.Sowmiya and P.Jeyalakshmi, "Intuitionistic Fuzzy sets in Z-Algebras", Journal of Advanced Mathematical Studies, 13(3) (2020) pp.302–310.

[12] S.Sowmiya and P.Jeyalakshmi, "Intuitionistic L-Fuzzy Structures in Z-Algebras",International Journal of Engineering Research & Technology, 10(2)(2021), pp.497-501.

[13] S.Sowmiya and P.Jeyalakshmi, "On Fuzzy H-Ideals in Z-Algebras" (Submitted).

[14] S.Sowmiya and P.Jeyalakshmi, "On Fuzzy p-Ideals in Z-Algebras" (Submitted)

[15] S.Sowmiya and P.Jeyalakshmi, "On Fuzzy Implicative Ideals in Z-Algebras"(Submitted).

[16] S.Sowmiya and P.Jeyalakshmi, "Interval-Valued Fuzzy Structures in Z-Algebras" (Submitted).

[17] S.Sowmiya and P.Jeyalakshmi, "Interval-Valued Intuitionistic Fuzzy Structures in Z-Algebras"(Submitted).

[18] S.Sowmiya and P.Jeyalakshmi, "Cubic Z-Subalgebras in Z-Algebras"(Submitted).

[19] L.A.Zadeh, "Fuzzy Sets", Information and Control, 8 (1965), 338–353.

[20] L.A.Zadeh, "The concept of a linguistic variable and its application to approximate reasoning-I",Inform.Sci., 8 (1975), 199-249.