Convolution Theorem For Canonical Sine Transform And Its Properties

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Abstract: The Canonical Sine transform, has many applications in several areas, including signal processing and optics. In this paper we have introduced convolution theorem, linearity property, derivative property, modulation property and Parseval's identity for the generalized canonical sine transform.

Keywords: Fractional Fourier Transform, LCT.

Introduction: Linear canonical transform (LCT) is a four parameter linear transform which is a generalization of the fractional Fourier transform and is given by,

$$[LCT\, f(t)](s) =< f(t), K(t,s) >$$

where, $K_M(t,s)=\sqrt{\frac{1}{2\pi ib}}\cdot e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}\cdot e^{\frac{i}{2}\left(\frac{a}{b}\right).t^2.}\, e^{-i.\left(\frac{s}{b}t\right).}$

Hence the generalized canonical cosine transform of $f\in\mathcal{E}'(R^n)$ c)an be defined by,

$$\{\ LCT\, f(t)\ \}(s)=\sqrt{\frac{1}{2\pi ib}}\cdot e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}\int_{-\infty}^{\infty}\cdot e^{\frac{i}{2}\left(\frac{a}{b}\right).t^2.}\, e^{-i.\left(\frac{s}{b}t\right).}\, f(t)\,dt,$$

The Fractional Fourier Transform (FRFT) which is the generalization of conventional Fourier transform is a special case of LCT. Just as Fourier cosine and Fourier Sine are defined from Fourier transform similarly Canonical Cosine Transform (CCT) and Canonical Sine Transform (CST) are defined from LCT by Pei and Ding in [4]. We have already studied Operation Transform Formulae for the Generalized Half Canonical Sine Transform in [2] and analyticity theorem and operational properties of CST [3].

Convolution plays a very important role in the theory of integral transform. Almeida [1] had defined convolution for fractional Fourier transform. Zayed [5] had revised the definition in order to follow the standard Convolution theorem. This paper emphasizes on defining canonical sine transform and deriving its convolution theorem, then some properties of the canonical sine transform are discussed and finally conclusions are given.

1.1 Testing Function Space $\mathcal{E}$:

An infinitely differentiable complex valued function ϕ on R^n belongs to $\mathcal{E}$ (R^n), if for each compact set, $I\subset S_\alpha$ where $S_\alpha=\{t : t\in R^n, |t|\le\alpha, \alpha>0\}$ and for $k\in R^n$,

$$\gamma_{\mathcal{E},k}\,\varphi(t)=\sup_{t\in I}\left|D^k\varphi(t)\right|<\infty.$$

Note that space $\mathcal{E}$ is complete and a Frechet space, let $\mathcal{E}$' denotes the dual space of $\mathcal{E}$.

2.1 Definition: The generalized Canonical Sine Transform $f \in \mathcal{E}'(R^n)$ can be defined by,

{CST $f(t)$} (s) = < $f(t)$, $K_S(t, s)$ > where,

$$K_S(t,s) = (-i)\sqrt{\frac{1}{2\pi .ib}} \cdot e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2} \cdot e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} \cdot \sin\left(\frac{s}{b}t\right)$$

Hence the generalized canonical sine transform of $f \in \mathcal{E}'(R^n)$ can be defined by,

$$\{\ CST\ f(t)\ \}(s) = (-i)\sqrt{\frac{1}{2\pi ib}} \cdot e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2} \int_{-\infty}^{\infty} e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} \sin\left(\frac{s}{b}t\right) \cdot f(t)\, dt \qquad (1.1)$$

2.2 The Generalized Canonical Sine Transform of Convolution

Now we introduced a special type of convolution and product for canonical sine transform.

2.3 Definition: For any function $f(y)$, let us define the functions $\tilde{f}(y)$ and $\tilde{g}(y)$ by

$$\tilde{g}(|v-y|) = e^{\frac{i}{2}\left(\frac{a}{b}\right)(v-y)^2} g(v-y), \qquad \tilde{g}(v+y) = e^{\frac{i}{2}\left(\frac{a}{b}\right)(v+y)^2} g(v+y)\} \text{ and } \tilde{f}(y) = e^{\frac{i}{2}\left(\frac{a}{b}\right)y^2} f(y)$$

For any two functions f and g, we define the

Convolution operation $\star$ by

$$h(v) = (f \star g)(v) = \int_0^{\infty} \tilde{f}(y)(\tilde{g}(y+v) + \tilde{g}(|v-y|))dy \qquad (1.2)$$

Now we state and prove convolution theorem.

2.4 Convolution theorem for canonical Sine Transform:

If $h(s) = (f * g)(s)$ and F_S, G_S and H_S denote the Canonical Sine transform of f, g, h

respectively, then $H_S(s) = [F_S(f(y))](s)[G_S(g(t))](s) = \dfrac{e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2} e^{-\frac{i}{2}\left(\frac{a}{b}\right)v^2}}{\sqrt{2\pi ib}} \left\{F_S\left(\tilde{f} * \tilde{g}\right)(v)\right\}(s)$

Proof: From the definition of the Canonical Sine transform, we have

$$[F_s(f(y))](s)[G_s(g(t))](s) = (-i)\sqrt{\frac{2}{\pi ib}}\, e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2} \int_0^{\infty} e^{\frac{i}{2}\left(\frac{a}{b}\right)y^2} \sin\left(\frac{s}{b}y\right) f(y)\,dy\,(-i)\sqrt{\frac{2}{\pi ib}}\, e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2} \int_0^{\infty} e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} \sin\left(\frac{s}{b}t\right) g(t)\,dt$$

$$= \frac{1}{i\pi b}\, e^{i\left(\frac{d}{b}\right)s^2} \int_0^{\infty}\int_0^{\infty} e^{\frac{i}{2}\left(\frac{a}{b}\right)(y^2+t^2)} \{-2\sin\left(\frac{s}{b}y\right)\sin\left(\frac{s}{b}t\right)\}\ f(y)\,g(t)\,dy\,dt$$

$$= \frac{1}{i\pi b}\, e^{i\left(\frac{d}{b}\right)s^2} \int_0^{\infty}\int_0^{\infty} e^{\frac{i}{2}\left(\frac{a}{b}\right)(y^2+t^2)} \{\cos\left(\frac{s}{b}(y+t)\right) - \cos\left(\frac{s}{b}(y-t)\right)\}\ f(y)\,g(t)\,dy\,dt$$

$$= \frac{1}{i\pi b} e^{i\left(\frac{d}{b}\right)s^2} \int_0^\infty \int_0^\infty e^{\frac{i}{2}\left(\frac{a}{b}\right)(y^2+t^2)} \cos\left(\frac{s}{b}(y+t)\right) f(y)\, g(t)\, dy\, dt +$$

$$\frac{1}{i\pi b} e^{i\left(\frac{d}{b}\right)s^2} \int_0^\infty \int_0^\infty e^{\frac{i}{2}\left(\frac{a}{b}\right)(y^2+t^2)} \cos\left(\frac{s}{b}(y-t)\right) f(y)\, g(t)\, dy\, dt$$

$$= I_1 + I_2 \qquad (1.3)$$

For I_1, putting $y + t = v \Rightarrow dt = dv$ for limit when $t = o \Rightarrow v = y$, when $t = \infty \Rightarrow v = \infty$

For I_2, putting $y - t = -v \Rightarrow dt = dv$ for limit when $t = o \Rightarrow v = -y$, when $t = \infty \Rightarrow v = \infty$

$$(1.3) \Rightarrow [F_S(f(y))](s)[G_S(g(t))](s) = \frac{1}{i\pi b} e^{i\left(\frac{d}{b}\right)s^2} \int_{y=0}^\infty \int_{v=y}^\infty e^{\frac{i}{2}\left(\frac{a}{b}\right)y^2} e^{\frac{i}{2}\left(\frac{a}{b}\right)(v-y)^2} \cos\left(\frac{s}{b}v\right) f(y)\, g(v-y)\, dy\, dv +$$

$$\frac{1}{i\pi b} e^{i\left(\frac{d}{b}\right)s^2} \int_{y=0}^\infty \int_{v=-y}^\infty e^{\frac{i}{2}\left(\frac{a}{b}\right)y^2} e^{\frac{i}{2}\left(\frac{a}{b}\right)(v+y)^2} \cos\left(\frac{s}{b}(-v)\right) f(y)\, g(v+y)\, dy\, dv$$

$$[F_S(f(y))](s)[G_S(g(t))](s) = \frac{1}{i\pi b} e^{i\left(\frac{d}{b}\right)s^2} \int_{y=0}^\infty \int_{v=0}^\infty e^{\frac{i}{2}\left(\frac{a}{b}\right)y^2} e^{\frac{i}{2}\left(\frac{a}{b}\right)(v-y)^2} \cos\left(\frac{s}{b}v\right) f(y)\, g(v-y)\, dy\, dv$$

$$+ \frac{1}{i\pi b} e^{i\left(\frac{d}{b}\right)s^2} \int_{y=0}^\infty \int_{v=y}^0 e^{\frac{i}{2}\left(\frac{a}{b}\right)y^2} e^{\frac{i}{2}\left(\frac{a}{b}\right)(v-y)^2} \cos\left(\frac{s}{b}v\right) f(y)\, g(v-y)\, dy\, dv$$

$$+ \frac{1}{i\pi b} e^{i\left(\frac{d}{b}\right)s^2} \int_{y=0}^\infty \int_{v=0}^\infty e^{\frac{i}{2}\left(\frac{a}{b}\right)y^2} e^{\frac{i}{2}\left(\frac{a}{b}\right)(v+y)^2} \cos\left(\frac{s}{b}v\right) f(y)\, g(v+y)\, dy\, dv +$$

$$+ \frac{1}{i\pi b} e^{i\left(\frac{d}{b}\right)s^2} \int_{y=0}^\infty \int_{v=-y}^0 e^{\frac{i}{2}\left(\frac{a}{b}\right)y^2} e^{\frac{i}{2}\left(\frac{a}{b}\right)(v+y)^2} \cos\left(\frac{s}{b}v\right) f(y)\, g(v+y)\, dy\, dv$$

$$[F_S(f(y))](s)[G_S(g(t))](s) = \frac{1}{i\pi b} e^{i\left(\frac{d}{b}\right)s^2} \int_{y=0}^\infty \int_{v=0}^\infty e^{\frac{i}{2}\left(\frac{a}{b}\right)y^2} e^{\frac{i}{2}\left(\frac{a}{b}\right)(v-y)^2} \cos\left(\frac{s}{b}v\right) f(y)\, g(v-y)\, dy\, dv$$

$$- \frac{1}{i\pi b} e^{i\left(\frac{d}{b}\right)s^2} \int_{y=0}^\infty \int_{v=0}^y e^{\frac{i}{2}\left(\frac{a}{b}\right)y^2} e^{\frac{i}{2}\left(\frac{a}{b}\right)(v-y)^2} \cos\left(\frac{s}{b}v\right) f(y)\, g(v-y)\, dy\, dv$$

$$+ \frac{1}{i\pi b} e^{i\left(\frac{d}{b}\right)s^2} \int_{y=0}^\infty \int_{v=0}^\infty e^{\frac{i}{2}\left(\frac{a}{b}\right)y^2} e^{\frac{i}{2}\left(\frac{a}{b}\right)(v+y)^2} \cos\left(\frac{s}{b}v\right) f(y)\, g(v+y)\, dy\, dv +$$

$$+ \frac{1}{i\pi b} e^{i\left(\frac{d}{b}\right)s^2} \int_{y=0}^\infty \int_{v=-y}^0 e^{\frac{i}{2}\left(\frac{a}{b}\right)y^2} e^{\frac{i}{2}\left(\frac{a}{b}\right)(v+y)^2} \cos\left(\frac{s}{b}v\right) f(y)\, g(v+y)\, dy\, dv$$

$$[F_S(f(y))](s)[G_S(g(t))](s) = I_1 + I_2 + I_3 + I_4 \qquad (1.4)$$

For I_4, putting $v = -v \Rightarrow dv = -dv$ for limit when $v = -y \Rightarrow v = y$, when $v = 0 \Rightarrow v = 0$

$$I_4 = \frac{1}{i\pi b}\, e^{i\left(\frac{d}{b}\right)s^2} \int_{y=0}^{\infty}\int_{v=y}^{0} e^{\frac{i}{2}\left(\frac{a}{b}\right)y^2}\, e^{\frac{i}{2}\left(\frac{a}{b}\right)(y-v)^2} \cos\left(\frac{s}{b}(-v)\right) f(y)\, g(y-v)\, dy\,(-dv)$$

$$= \frac{1}{i\pi b}\, e^{i\left(\frac{d}{b}\right)s^2} \int_{y=0}^{\infty}\int_{v=0}^{y} e^{\frac{i}{2}\left(\frac{a}{b}\right)y^2}\, e^{\frac{i}{2}\left(\frac{a}{b}\right)(y-v)^2} \cos\left(\frac{s}{b}v\right) f(y)\, g(y-v)\, dy\, dv$$

$$(1.4) \Rightarrow [F_S(f(y))](s)[G_S(g(t))](s) = \frac{1}{i\pi b}\, e^{i\left(\frac{d}{b}\right)s^2} \int_{y=0}^{\infty}\int_{v=0}^{\infty} e^{\frac{i}{2}\left(\frac{a}{b}\right)y^2}\, e^{\frac{i}{2}\left(\frac{a}{b}\right)(v-y)^2} \cos\left(\frac{s}{b}v\right) f(y)\, g(v-y)\, dy\, dv$$

$$- \frac{1}{i\pi b}\, e^{i\left(\frac{d}{b}\right)s^2} \int_{y=0}^{\infty}\int_{v=0}^{y} e^{\frac{i}{2}\left(\frac{a}{b}\right)y^2}\, e^{\frac{i}{2}\left(\frac{a}{b}\right)(v-y)^2} \cos\left(\frac{s}{b}v\right) f(y)\, g(v-y)\, dy\, dv$$

$$+ \frac{1}{i\pi b}\, e^{i\left(\frac{d}{b}\right)s^2} \int_{y=0}^{\infty}\int_{v=0}^{\infty} e^{\frac{i}{2}\left(\frac{a}{b}\right)y^2}\, e^{\frac{i}{2}\left(\frac{a}{b}\right)(v+y)^2} \cos\left(\frac{s}{b}v\right) f(y)\, g(v+y)\, dy\, dv$$

$$+ \frac{1}{i\pi b}\, e^{i\left(\frac{d}{b}\right)s^2} \int_{y=0}^{\infty}\int_{v=0}^{y} e^{\frac{i}{2}\left(\frac{a}{b}\right)y^2}\, e^{\frac{i}{2}\left(\frac{a}{b}\right)(v-y)^2} \cos\left(\frac{s}{b}v\right) f(y)\, g(v-y)\, dy\, dv$$

$$[F_S(f(y))](s)[G_S(g(t))](s) = \frac{1}{i\pi b}\, e^{i\left(\frac{d}{b}\right)s^2} \int_{y=0}^{\infty}\int_{v=0}^{\infty} e^{\frac{i}{2}\left(\frac{a}{b}\right)y^2}\, e^{\frac{i}{2}\left(\frac{a}{b}\right)(v-y)^2} \cos\left(\frac{s}{b}v\right) f(y)\, g(v-y)\, dy\, dv$$

$$+ \frac{1}{i\pi b}\, e^{i\left(\frac{d}{b}\right)s^2} \int_{y=0}^{\infty}\int_{v=0}^{\infty} e^{\frac{i}{2}\left(\frac{a}{b}\right)y^2}\, e^{\frac{i}{2}\left(\frac{a}{b}\right)(v+y)^2} \cos\left(\frac{s}{b}v\right) f(y)\, g(v+y)\, dy\, dv$$

$$= \frac{1}{\sqrt{2i\pi b}} \frac{\sqrt{2}}{\sqrt{i\pi b}}\, e^{i\left(\frac{d}{b}\right)s^2} e^{-\frac{i}{2}\left(\frac{a}{b}\right)v^2} \int_0^{\infty}\left\{\int_0^{\infty} \tilde{f}(y)\,(\tilde{g}(y+v)+\tilde{g}(|v-y|))\,dy\right\} \cos\left(\frac{s}{b}v\right) e^{\frac{i}{2}\left(\frac{a}{b}\right)v^2}\, dv$$

$$= \frac{e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}\, e^{-\frac{i}{2}\left(\frac{a}{b}\right)v^2}}{(-i)\sqrt{2\pi ib}} \left\{(-i)\sqrt{\frac{2}{i\pi b}}\, e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2} \int_0^{\infty} e^{\frac{i}{2}\left(\frac{a}{b}\right)v^2} \sin\left(\left(\frac{\pi}{2}+\frac{s}{b}\right)v\right)\{(\tilde{f}*\tilde{g})(v)\}\,dv\right\}$$

$$[F_S(f(y))](s)[G_S(g(t))](s) = \frac{e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}\, e^{-\frac{i}{2}\left(\frac{a}{b}\right)v^2}}{\sqrt{2\pi ib}} \left\{F_S\left(\tilde{f}*\tilde{g}\right)(v)\right\}(s)$$

3. Some operational results

3.1.1 Linearity property of canonical sine transformations:

If *{CST f(t)}(s), {CST g(t)}(s)* denotes generalized canonical sine transform of *f(t),g(t)* and P_1, P_2 are constants then $\{CST\,(P_1 f(t)+P_2 g(t))\}(s) = P_1\{CST\,(f(t)\}(s) + P_2\{CST\,(g(t))\}(s)$

Proof is simple and hence omitted**.**

3.1.2 Modulation property of canonical sine transform:

If *{CST f(t)} (s)* denotes generalized canonical sine transform of *f(t) t*hen,

$$\{CST \cos zt.\, f(t)\}(s) = \frac{e^{-\frac{i}{2}(db)z^2}}{2}\left\{\left[CST\ f(t)\cdot e^{-idsz}\right]\left(\frac{s+bz}{b}\right) + \left[CST\ f(t)\cdot e^{idsz}\right]\left(\frac{s-bz}{b}\right)\right\}$$

Proof: We have,

$$\{CS\,T\ \cos zt.f(t)\}(s)=(-i)\sqrt{\frac{1}{2\pi ib}}e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}\int_{-\infty}^{\infty}e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}.\sin\left(\frac{s}{b}t\right)\cdot\cos zt\cdot f(t)\,dt$$

$$\{CS\,T\ \cos zt.f(t)\}(s)=(-i)\sqrt{\frac{1}{2\pi ib}}e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}\int_{-\infty}^{\infty}e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}\ \frac{2}{2}\sin\left(\frac{s}{b}t\right)\cdot\cos zt\cdot f(t)\,dt$$

$$\{CS\,T\ \cos zt.f(t)\}(s)=(-i)\frac{1}{2}\sqrt{\frac{1}{2\pi ib}}e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}\int_{-\infty}^{\infty}e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}.\left(\sin\left(\frac{s}{b}+z\right)t+\sin\left(\frac{s}{b}-z\right)t\right)f(t)\,dt$$

$$\{CS\,T\ \cos zt.f(t)\}(s)=(-i)\frac{1}{2}\{\sqrt{\frac{1}{2\pi ib}}\int_{-\infty}^{\infty}e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}\ e^{\frac{i}{2}\left(\frac{d}{b}\right)(s+bz)^2}\ e^{-idsz}\,e^{-\frac{i}{2}(db)z^2}\ \sin\left(\frac{s}{b}+z\right)t.\ f(t)\,dt$$

$$+\sqrt{\frac{1}{2\pi ib}}\int_{-\infty}^{\infty}e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}\ e^{\frac{i}{2}\left(\frac{d}{b}\right)(s-bz)^2}\ e^{idsz}\,e^{-\frac{i}{2}(db)z^2}\ \sin\left(\frac{s}{b}-z\right)t.f(t)\,dt\}$$

$$\{CS\,T\ \cos zt.f(t)\}(s)=\frac{1}{2}e^{-\frac{i}{2}(db)z^2}\ \{(-i)\sqrt{\frac{1}{2\pi ib}}\int_{-\infty}^{\infty}e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}\ e^{\frac{i}{2}\left(\frac{d}{b}\right)(s+bz)^2}\ \sin\left(\frac{s+bz}{b}\right)t.e^{-idsz}\ f(t)\,dt+$$

$$(-i)\sqrt{\frac{1}{2\pi ib}}\int_{-\infty}^{\infty}e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}\ e^{\frac{i}{2}\left(\frac{d}{b}\right)(s-bz)^2}\ \sin\left(\frac{s-bz}{b}\right)t.e^{idsz}.f(t)\,dt\}$$

$$\{CST\cos zt.\ f(t)\}(s)=\frac{e^{-\frac{i}{2}(db)z^2}}{2}\left\{\left[CST\ \ f(t)\cdot e^{-idsz}\right]\left(\frac{s+bz}{b}\right)+\left[CST\ \ f(t)\cdot e^{idsz}\right]\left(\frac{s-bz}{b}\right)\right\}$$

3.1.3 If *{CST f(t)} (s)* denotes generalized canonical sine transform of *f(t)* then,

$$\{CST\sin zt.\ f(t)\}(s)=(-i)\frac{e^{-\frac{i}{2}dbz^2}}{2}\left\{\left[CCT\ \ f(t)\cdot e^{-idsz}\right]\left(\frac{s+bz}{b}\right)-\left[CCT\ \ f(t)\cdot e^{idsz}\right]\left(\frac{s-bz}{b}\right)\right\}$$

Proof: We have,

$$\{CS\,T\ \sin zt.f(t)\}(s)=(-i)\sqrt{\frac{1}{2\pi ib}}e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}\int_{-\infty}^{\infty}e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}.\sin\left(\frac{s}{b}t\right)\cdot\sin zt\cdot f(t)\,dt$$

$$\{CS\,T\ \sin zt.f(t)\}(s)=\frac{1}{2}(-i)\sqrt{\frac{1}{2\pi ib}}e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}\int_{-\infty}^{\infty}e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}.\left(\cos\left(\frac{s}{b}-z\right)t-\cos\left(\frac{s}{b}+z\right)t\right)f(t)\,dt$$

$$\{CS\,T\ \sin zt.f(t)\}(s)=\frac{(-i)}{2}\{\sqrt{\frac{1}{2\pi ib}}\int_{-\infty}^{\infty}e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}\ e^{\frac{i}{2}\left(\frac{d}{b}\right)(s+bz)^2}\ e^{-idsz}\,e^{-\frac{i}{2}(db)z^2}\ \cos\left(\frac{s}{b}+z\right)t.\ f(t)\,dt$$

$$-\sqrt{\frac{1}{2\pi ib}}\int_{-\infty}^{\infty}e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}\ e^{\frac{i}{2}\left(\frac{d}{b}\right)(s-bz)^2}\ e^{idsz}\,e^{-\frac{i}{2}(db)z^2}\ \cos\left(\frac{s}{b}-z\right)t..f(t)\,dt\}$$

$$\{CS\,T\ \sin zt.f(t)\}(s)=\frac{(-i)e^{-\frac{i}{2}(db)z^2}}{2}\{\sqrt{\frac{1}{2\pi ib}}\int_{-\infty}^{\infty}e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}\ e^{\frac{i}{2}\left(\frac{d}{b}\right)(s+bz)^2}\ \cos\left(\frac{s+bz}{b}\right)t.e^{-idsz}\ f(t)\,dt$$

$$-\sqrt{\frac{1}{2\pi ib}}\int_{-\infty}^{\infty}e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}\ e^{\frac{i}{2}\left(\frac{d}{b}\right)(s-bz)^2}\ \cos\left(\frac{s-bz}{b}\right)t.e^{idsz}.f(t)\,dt\}$$

$$\{CST\sin zt.\ f(t)\}(s)=(-i)\frac{e^{-\frac{i}{2}dbz^2}}{2}\left\{\left[CCT\ \ f(t)\cdot e^{-idsz}\right]\left(\frac{s+bz}{b}\right)-\left[CCT\ \ f(t)\cdot e^{idsz}\right]\left(\frac{s-bz}{b}\right)\right\}$$

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3.1.4 Derivative (with respect to parameter) property of canonical sine transform:

If *{CST f(t)}(s)* denotes generalized canonical sine transform, then,

$$\frac{d}{ds}\left[\{CSTf(t)\}(s)\right]=i\left\{s.\left(\frac{d}{b}\right)\{CSTf(t)\}(s)-\frac{1}{b}\{CCT[t.f(t)]\}(s)\right\}$$

Proof: We have,

$$\frac{d}{ds}\{CST\ \ f(t)\}(s)=\frac{d}{ds}\left\{(-i)\sqrt{\frac{1}{2\pi ib}}\cdot e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}\int_{-\infty}^{\infty}e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}\sin\left(\frac{s}{b}t\right)f(t)dt\right\}$$

$$\frac{d}{ds}\{CST\ \ f(t)\}(s)=(-i)\sqrt{\frac{1}{2\pi ib}}\cdot\int_{-\infty}^{\infty}e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}\frac{\partial}{\partial s}\left(e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}.\sin\left(\frac{s}{b}t\right)\right)f(t)dt$$

$$=(-i)\sqrt{\frac{1}{2\pi ib}}\cdot\int_{-\infty}^{\infty}e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}\left(\left(\frac{t}{b}\right)e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}.\cos\left(\frac{s}{b}t\right)+i\left(\frac{d}{b}\right).s.e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}\sin\left(\frac{s}{b}t\right)\right)f(t)dt$$

$$=(-i)\sqrt{\frac{1}{2\pi ib}}\cdot\int_{-\infty}^{\infty}e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}\left(\frac{1}{b}\right)e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}.\cos\left(\frac{s}{b}t\right)[t.f(t)]dt+s.i.\left(\frac{d}{b}\right)(-i)\sqrt{\frac{1}{2\pi ib}}\cdot\int_{-\infty}^{\infty}e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}\sin\left(\frac{s}{b}t\right)f(t)dt$$

$$=(-i)\frac{1}{b}\{CCT[t.f(t)]\}(s)+s.i.\left(\frac{d}{b}\right)\{CSTf(t)\}(s)$$

$$\frac{d}{ds}\left[\{CSTf(t)\}(s)\right]=i\left\{s.\left(\frac{d}{b}\right)\{CSTf(t)\}(s)-\frac{1}{b}\{CCT[t.f(t)]\}(s)\right\}$$

4.1 Parseval's Identity for canonical sine transform:

If *f(t)* and *g(t)* are the inversion canonical sine transform of $F_S(s)$ and $G_S(s)$ respectively, then

(1) $\int_{0}^{\infty}f(t).\overline{g(t)}\ dt=-2\pi i\int_{-\infty}^{\infty}F_S(s).\overline{G_S(s)}\ ds$ and (2) $\int_{-\infty}^{\infty}.\left|f(t)\right|^2\ dt=-2\pi i\int_{-\infty}^{\infty}.\left|F_S(s)\right|^2\ ds$

Proof: By definition of CST,

$$\{CST\ \ g(t)\}(s)=(-i)\sqrt{\frac{1}{2\pi ib}}\cdot e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}\int_{-\infty}^{\infty}e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}\sin\left(\frac{s}{b}t\right)\cdot g(t)\,dt \qquad \text{---------------- (5.1.1)}$$

Using the inversion formula of CST

$$g(t)=(-i)\sqrt{\frac{-2\pi i}{b}}\cdot e^{-\frac{i}{2}\left(\frac{a}{b}\right)t^2}\int_{-\infty}^{\infty}e^{-\frac{i}{2}\left(\frac{d}{b}\right)s^2}\sin\left(\frac{s}{b}t\right)G_S(s)\,ds$$

Taking complex conjugate we get,

$$\overline{g(t)}=i\sqrt{\frac{2\pi i}{b}}\cdot e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}\int_{-\infty}^{\infty}e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}\sin\left(\frac{s}{b}t\right)\overline{G_S(s)}\,ds$$

$$\int_{-\infty}^{\infty}f(t).\overline{g(t)}\,dt=\int_{-\infty}^{\infty}f(t)dt\left(i\sqrt{\frac{2\pi i}{b}}\cdot e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}\int_{-\infty}^{\infty}e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}\sin\left(\frac{s}{b}t\right)\overline{G_S(s)}\,ds\right)$$

Changing the order of integration, we get,

$$\int_{-\infty}^{\infty}f(t).\overline{g(t)}\,dt=\sqrt{\frac{2\pi i}{b}}\int_{-\infty}^{\infty}\overline{G_S(s)}\,ds\frac{1}{(-1)\sqrt{\frac{1}{2\pi ib}}}\left((-i)\sqrt{\frac{1}{2\pi ib}}\ e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}\int_{-\infty}^{\infty}e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}\sin\left(\frac{s}{b}t\right)f(t)dt\right)$$

$$\int_{-\infty}^{\infty} f(t).\overline{g(t)}\, dt = -2\pi i \int_{-\infty}^{\infty} \overline{G_S(s)}\,.F_S(s)ds$$

$$\int_{-\infty}^{\infty} f(t).\overline{g(t)}\, dt = -2\pi i \int_{-\infty}^{\infty} F_S(s).\overline{G_S(s)}\, ds \qquad \text{---------------------} \ (5.1.2)$$

Hence proved

(ii) Putting $f(t) = g(t)$ in equation (5.1.2), we get

$$\int_{-\infty}^{\infty} |f(t)|^2\, dt = -2\pi i \int_{-\infty}^{\infty} |F_S(s)|^2\, ds$$

Table for canonical sine transform

S.N.	*f(t)*	$F_S(s)$
1	$\{CST\ (P_1 f(t) + P_2 g(t))\}(s)$	$P_1\{CST\ (f(t)\}(s) + P_2\{CST\ (g(t))\}(s)$
2	$\cos zt.f(t)$	$\frac{e^{-\frac{i}{2}(db)z^2}}{2}\left\{\left[CST\ f(t)\cdot e^{-idsz}\right](\frac{s+bz}{b}) + \left[CST\ f(t)\cdot e^{idsz}\right](\frac{s-bz}{b})\right\}$
3	$\sin zt.f(t)$	$(-i)\frac{e^{-\frac{i}{2}(db)z^2}}{2}\left\{\left[CCT\ f(t)\cdot e^{-idsz}\right](\frac{s+bz}{b}) - \left[CCT\ f(t)\cdot e^{idsz}\right](\frac{s-bz}{b})\right\}$

Conclusion:

The generalized canonical sine transform is developed in this paper. The convolution theorem, operation transform formulae proved in this paper can be used, when this transform is used to solve ordinary or partial differential equation.

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