International Journal of Engineering Research & Technology (IJERT)
ISSN: 2278-0181
Vol. 8 Issue 05, May-2019

Conversion of Dual Integral Equations into an Integral Equation

Anil Tiwari[1]
Ph.D. Scholar, Dept. of Mathematics
Ravindranath Tagore University
Bhopal, India

Dr. Archana Lala[2]
Department of Mathematics
S R Group of Institutions
Jhansi, India

Dr. Chitra Singh[3]
Department of Mathematics
Ravindranath Tagore University
Bhopal, India

Abstract **– The purpose of this paper is to reduce a dual integral equation into an integral equation whose kernel involves Generalized Hermite Polynomial by use of mellin transform. We believe that there is some more possible way to reduce such dual integral equations using different transform like those of Henkel, Fourier. For the purpose of illustration we consider a dual integral equation of certain type and by use of fractional operator and mellin transform reduced it to an integral equation.**

Keywords – Generalized Hermite Polynomial; Mellin Transform; Fractional operators; Fox-H function.

I. INTRODUCTION

Dual integral equations are often encountered in different branches of mathematical physics and they generally arise while solving a boundary value problem with mixed boundary conditions In the present paper, we attempt to solve the certain dual integral equations by converting them into an integral equations. Many attempts have been made in the past to solve such problems. The following integral representation is basic tool for our illustration.

$$\int_0^\infty k_1(x,u)\; A(u)\,du = \lambda(x);\; 0 \le x \le 1 \qquad (1.1)$$

$$\int_0^\infty k_2(x,u)\; A(u)\,du = \omega(x);\; x > 1 \qquad (1.2)$$

k_1 & k_2 are kernels defind over x-u plane.

$$H_n^r(x,\, a,\; p) = (-1)^r\, x^{-a} e^{px^r}\; D^n\,[x^a e^{px^r}\,]\; D = \frac{d}{dx}$$

$$a, r,\; p\;\; parameter.$$

II. THEOREM

If f is unknown function satisfying the dual integral equation.

$$\int_0^\infty (xy)^{a_1}\; e^{-(xy)^r}\; H_n^r(xy; a_1, 1)\; f(y)\,dy \;=\; h(x);\; 0 \le x < 1 \qquad (2.1)$$

$$\int_0^\infty (xy)^{a_2}\; e^{-(xy)^r}\; H_n^r(xy; a_2, 1)\; f(y)\,dy = g(x); 1 \le x < \infty \qquad (2.2)$$

When h and g are prescribed function and $a_1, a_2\; and\; r$ are parameters, then f is giving by

$$f(x) = \frac{1}{r}\int_0^\infty L(xy)\; t(y)\,dy$$

Where

$$L(x) = H_{2,1}^{1,0}\left[\, x \left|\begin{matrix}(1,1) & \left(\frac{1}{r}(a_1 - n + 1)\right), \frac{1}{r} \\ (1-n, 1) & \end{matrix}\right.\right]$$

and

$$t(x) = h(x), 0 \le x < 1$$

$$t(x) = \frac{r\, x^{-n+a_1}}{\Gamma\left(\frac{1}{r}(a_2 - a_1)\right)}$$

$$\times \int_0^\infty (v^r - x^r)^{\left(\frac{1}{r}(a_2 - a_1) - 1\right)} v^{n - a_2 + r - 1}\, g(v)\; dv; 1 \le x < \infty$$

III. MATHEMATICAL PRELIMINARY

To prove the theorems we shall use Mellin transformer and fractional integral operator.

$$f^*(s) = M[\, f(x); s] = \int_0^\infty f(x) x^{s-1}\; dx \qquad (3.1)$$

When $s = \sigma + i\tau$ is a complex variable.

The inverse melling transform f(x) of $f^*(s)$ is given by

$$M^{-1}[\, f^*(s)] = f(x) = \frac{1}{2\pi i}\int_{\sigma + i\infty}^{\sigma + i\infty} f^*(s)\, x^{-s}\, ds \qquad (3.2)$$

By convolution theorem for mellin transform

$$M\left[\int_0^\infty k(xy)\; f(y)\,dy; s\right] = k^*(s)\; f^*(1-s)$$

$$\int_0^\infty k(xy)\; f(y)\,dy = M^{-1}\left[\; k^*(s)\; f^*(1-s); s\right]$$

$$= \frac{1}{2\pi i}\int_L k^*(s)\; f^*(1-s)\, x^{-s}\, ds \qquad (3.3)$$

When L is suitable contour.

Fractional integral operator

$$\tau(\alpha;\; \beta;\; r; w(x)) = \frac{r\, x^{-r\alpha + r - \beta - 1}}{\Gamma(\alpha)}\int_0^\infty (x^r - v^r)^{\alpha - 1}\, v^\beta w(v)\,dv \qquad (3.4)$$

$$R(\,\alpha;\; \beta;\; r; w(x)) = \frac{r\, x^\beta}{\Gamma(\alpha)}\int_x^\infty (v^r - x^r)^{\alpha - 1}\, v^{-\beta - r\alpha + r - 1}\, w(v)\; dv \qquad (3.5)$$

IV. SOLUTION

Now taking

$$k_i(x) = x^{a_i}\, e^{-x^r} H_n^r(x; a_i; 1),\ i = 1, 2$$

Then from Erdeeyi [10] We get

$$k_i^*(s) = \frac{\Gamma s\, \Gamma\left(\frac{1}{r}(s-n+a_i)\right)}{r\,\Gamma(s-n)},\ i = 1, 2 \qquad (4.1)$$

Hence by use at (3.3),(2.1) & (2.2) can be written as

$$\frac{1}{2r\pi i}\int_L \frac{\Gamma s\,\Gamma\left(\frac{1}{r}(s-n+a_i)\right)}{\Gamma(s-n)} f^*(1-s)\,x^{-s}\,ds$$

$$= h(x);\ 0 \le x < 1 \qquad (4.2)$$

$$\frac{1}{2r\pi i}\int_L \frac{\Gamma(s)\,\Gamma\left(\frac{1}{r}(s-n+a_2)\right)}{\Gamma(s-n)} f^*(1-s)\,x^{-s}\,ds$$

$$= g(x);\ 1 \le x < \infty \qquad (4.3)$$

Now operating a (4.2) by the operator (3.5) we get

$$\frac{rx^\beta}{\alpha}\cdot\frac{1}{2r\pi i}\int_L \frac{\Gamma s\,\Gamma\left(\frac{1}{r}(s-n+a_2)\right)}{\Gamma(s-n)} f^*(1-s)\,x^{-s}\,ds$$

$$\int_x^\infty (v^r - x^r)^{\alpha-1} v^{-\beta-r\alpha+r-1} g(v)dv$$

$$= \frac{rx^\beta}{\Gamma\alpha}\int_x^\infty (v^r - x^r)^{\alpha-1} v^{-\beta-r\alpha+r-1} g(v)dv$$

Now putting $v^r = \frac{x^r}{t}$ and simplifying we get

$$\frac{1}{2r\pi i}\int_L \frac{\Gamma(s)\,\Gamma\left(\frac{1}{r}(s-n+a_2)\right)}{\Gamma(s-n)}\cdot\frac{x^{-s}}{\Gamma\alpha}$$

$$\int_0^1 (1-t)^{\alpha-1}.t^{\left(\frac{\beta+s}{r}-1\right)}\,dt\ \ f^*(1-s)ds$$

$$= \frac{rx^\beta}{\Gamma\alpha}\int_x^\infty (v^r - x^r)^{\alpha-1} v^{-\beta-r\alpha+r-1} g(v)dv \qquad (4.4)$$

$$\Rightarrow \frac{1}{2r\pi i}\int_L \frac{\Gamma(s)\,\Gamma\left(\frac{1}{r}(s-n+a_2)\right)}{\Gamma(s-n)}\cdot\frac{x^{-s}}{\Gamma\alpha}$$

$$\times \frac{\Gamma\left(\frac{1}{r}(\beta+S)\right)\Gamma\alpha}{\Gamma\left(\alpha+\frac{1}{r}\beta+\frac{1}{r}s\right)} f^*\ (1-s)\,ds$$

$$= \frac{rx^\beta}{\Gamma\alpha}\int_x^\infty (v^r - x^r)^{\alpha-1} v^{-\beta-r\alpha+r-1} g(v)dv$$

$$\Rightarrow \frac{1}{2r\pi i}\int_L \frac{\Gamma(s)\,\Gamma\left(\frac{1}{r}(s-n+a_2)\right)}{\Gamma(s-n)}\ x^{-s}$$

$$\times \frac{\Gamma\left(\frac{1}{r}(\beta+s)\right)}{.\Gamma\left(\alpha+\frac{1}{r}\beta+\frac{1}{r}s\right)} \times f^*(1-s)\,ds$$

$$= \frac{rx^\beta}{\Gamma\alpha}\int_x^\infty (v^r - x^r)^{\alpha-1} v^{-\beta-r\alpha+r-1} g(v)dv$$

In equation (4.4), we put $\beta = -n + a_1$ and

$\alpha = \frac{1}{r}(a_2 - a_1),$ so that (4.4) Changes to

$$\frac{1}{2r\pi i}\int_L \frac{\Gamma(s)\,\Gamma\left(\frac{1}{r}(s-n+a_2)\right)}{\Gamma(s-n)}\ x^{-s}$$

$$\times \frac{\Gamma\left(\frac{1}{r}(s-n+a_1)\right)}{\Gamma\left(\frac{a_2}{r}-\frac{a_1}{r}+\frac{1}{r}(-n+a_1)+\frac{1}{r}s\right)} \times f^*(1-s)\,ds$$

$$= \frac{r\,x^{-n+a_1}}{\Gamma\left(\frac{1}{r}(a_2-a_1)\right)}$$

$$\times\int_x^\infty (v^r - x^r)^{\left(\frac{1}{r}(a_2-a_1)-1\right)} v^{-n-a_2+r-1} g(v)dv\ \ 1 \le x \le \infty$$

$$\frac{1}{2r\pi i}\int_L \frac{\Gamma(s)\,\Gamma\left(\frac{1}{r}(s-n+a_1)\right)}{\Gamma(s-n)}\ x^{-s}\ \ f^*(1-s)ds$$

$$= \frac{r\,x^{-n+a_1}}{\Gamma\left(\frac{1}{r}(a_2-a_1)\right)}\times$$

$$\int_x^\infty (v^r - x^r)^{\left(\frac{1}{r}(a_2-a_1)-1\right)} v^{-n-a_2+r-1} g(v)dv\ \ 1 \le x \le \infty \qquad (4.5)$$

Now we write

$t(x) = h(x),\quad 0 \le x < 1$

and $t(x) = \dfrac{r\,x^{-n+a_1}}{\Gamma\left(\frac{1}{r}(a_2-a_1)\right)}$

$$\times\int_x^\infty (v^r - x^r)^{\left(\frac{1}{r}(a_2-a_1)-1\right)} v^{n-a_2+r-1} g(v)\ dv;1 \le x < \infty \qquad (4.6)$$

Now from (4.2), (4.5), (4.6) we get

$$\frac{1}{2r\pi i}\int_L \frac{\Gamma(s)\,\Gamma\left(\frac{1}{r}(s-n+a_1)\right)}{\Gamma(s-n)}\ x^{-s}\ \ f^*(1-s)ds = t(x)$$

International Journal of Engineering Research & Technology (IJERT)
ISSN: 2278-0181
Vol. 8 Issue 05, May-2019

(4.7)

Again using (3.3), (4.1) & (4.6) becomes

$$\int_0^\infty k_1(xy)\, f(y)\, dy = t(x);\ 0 \le x < \infty \quad (4.8)$$

When $k_1(x) = x^{a_1} e^{-x^r} H_x^r(x;\, a_1;1)$

Thus pair at dual integral equation (1.1) &(1.2) we have been reduced to single integral equation (4.8). Hence by mellin transform (4.8) can be written as –

$$k_1^*(s)\, f^*(1-s) = T^*(s) \quad (4.9)$$

Where $k_1^*(s) = \dfrac{\sqrt{s}\sqrt{\left(\frac{1}{r}(s-n+a_1\right)}}{\sqrt{(s-n)}}$

and $T^*(s)$ is the mellin transform of t(x).

Now replacing s by (1–s) in (4.9)

$$F^*(s) = L^*(s)\, T^*(1-s) \quad (4.10)$$

$$L^*(s) = \frac{1}{k^*(1-s)}$$

$$= \frac{\sqrt{1-s-n}}{\sqrt{(1-s)}\sqrt{\frac{1}{r}(1-s-n+a_1)}}$$

By use of definition of H – function, we get the inverse transform L(x) at $L^*(S)$ as

$$L(x) = H_{2,1}^{1,0}\left(x \,\middle|\, \begin{matrix} (1,1)\left(\frac{1}{r}\,(a_1-n+1)\right), \frac{1}{r} \\ (1-n,1) \end{matrix} \right) \quad (4.11)$$

Taking inverse mellin transform of (4.10)

$$f(x) = \int_0^\infty L(xy)\, t(y)\, dy$$

Hence using (4.11) we get

$$f(x) = \frac{1}{r}\int_0^\infty H_{2,1}^{1,0}\left(xy \,\middle|\, \begin{matrix} (1,1)\left(\frac{1}{r}\,(a_1-n+1)\right), \frac{1}{r} \\ (1-n,1) \end{matrix} \right) t(y)dy$$

When t (y) is given by (4.6).

Hence proved the theorem.

REFERENCE

[1] S. Ahdiaghdam, K. Ivaz and S. Shahmorad, “Approximate solution of dual integral equations” *Bull. Iranian Math. Soc. Vol.* 42 (2016), No. 5, pp. 1077–1086

[2] A. Chakrabarti and S. C. Martha, “Methods of solution of singular integral equations”, *Math. Sci.* (2012) 29 pages.

[3] N. A. Hoshan, “Exact solution of certain dual Integral equations involing heat conduction equation”, *Far East Journal of Applied Mathematics* 35(1)(2009) 81-88.

[4] S. R. Manam, “On the solution of dual integral equations”, *Appl. Math. Lett.* 20 (2007), no. 4, 391–395.

[5] A. Chakrabarti and G. V. Berghe, “Approximate solution of singular integral equations”, *Appl. Math. Lett.* 17 (2004), no. 5, 533–559.

[6] A. J. Jerry, “*Introduction to Integral Equations with Applications, Second Edition*”, WileyInterscience, New York, 1999

[7] C. Nasim and B. D. Aggarwala, “On some dual integral equations”, *Indian J. Pure Appl. Math.*15 (1984), no. 3, 323–340.

[8] Gould H Wand Hopper A T, “Operational formulas connected with two generalizations of Hermite polynomials”, *Duke Math. J.* Vol 29, 1962, PP.51-63.

[9] I. N. Sneddon, “The elementary solution of dual integral equations”, *Proc. Glasgow Math. Assoc.* 4 (1960) 108–110.

[10] Erdelyi A, Magnus W, Oberhettinger F and Triconi F G, “*Tables of Integral Transforms*” (New York: McGraw-Hili) Vol. I, 1954.