Convergence and Connectedness on Complete Measure Manifold

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Abstract - In this paper we study different modes of convergence and introduce some aspects of connectedness like, locally path connected μ1 - a.e., interconnected μ1 - a.e., maximal connected μ1 - a.e. on a complete measure manifold of dimension n. We show that these two intrinsic topological properties remain invariant under a measurable homeomorphism and measure invariant map.

Keywords: Convergence, connectedness, complete measure manifold, local path connected μ1 - a.e., interconnected μ1 - a.e., maximal connected μ1 - a.e.

1. INTRODUCTION
In this paper to investigate the intrinsic properties of complete measure manifold, we have introduced the role of convergent measurable functions on measure manifold. Any point p ∈ Uc(M, 𝒯, Σ, μ) will be a limit point of the measure manifold. The point p ∈ U of (M, 𝒯, Σ, μ) carries additional information in terms of the ordered pair (⁡[fσ], f) which induces a measurable set Aσ such that the measure of such set is always positive on measure manifold. If the measure of Aσ is zero and any subset B ⊆ Aσ also has measure zero then the measure manifold is complete. On such complete measure manifold, we have introduced different aspects of connectedness, like, locally path connected, interconnected, maximally connected and proved some important results that will generate a class of connected measure manifolds.

2. PRELIMINARIES
Some of the basic definitions referred are as follows
Definition 2.1: Measurable chart [7][8][9]
Let (U, 𝒯/U, Σ1/U) ⊆ (M, 𝒯, Σ) be a non-empty measurable subspace of (M, 𝒯, Σ). If there exists a map, ϕ: (U, 𝒯/U, Σ1/U) → (ℝn, 𝒯, Σ), satisfying the following conditions,
(i) ϕ is homeomorphism,
(ii) ϕ is measurable,
then the structure ((U, 𝒯/U, Σ1/U), ϕ) is called as a measurable chart.

Definition 2.2: Measure Chart [7][8][9]
A measure μ1|U on measurable chart ((U, 𝒯/U, Σ1/U), ϕ) is called as measure chart, denoted by ((U, 𝒯/U, Σ1/U, μ1|U), ϕ) satisfying following conditions:
(i) ϕ is homeomorphism,
(ii) ϕ is measurable,
(iii) ϕ is measure invariant
then, the structure ((U, 𝒯/U, Σ1/U, μ1|U), ϕ) is called as a measure chart.

Definition 2.3: Measurable Atlas [7][8][9]
By an 𝕀n measurable atlas of class 𝕀k on M we mean a countable collection (An, 𝒯n/An, Σn/An) of n-dimensional measurable charts ((Ui, 𝒯i/Ui, Σi/Ui, μi), fi) for all i ∈ 1 on (M, 𝒯, Σ) subject to the following conditions:
(a1) Ui = Ui, Σi/Ui, μi = M
that is, the countable union of the measurable charts in (An, 𝒯n/An, Σn/An) cover (M, 𝒯, Σ)
(a2) For any pair of measurable charts ((Ui, 𝒯i/Ui, Σi/Ui, μi), fi) and ((Uj, 𝒯j/Uj, Σj/Uj, μj), fj) in
(An, 𝒯n/An, Σn/An), the transition maps fi ◦ fj−1 and fj ◦ fi−1 are
(1) differentiable maps of class 𝕀k (K ≥ 1) that is,
ϕ ◦ fj−1: Ui ∩ Uj → Ui, Σi/Ui, μi = Φ(Ui ∩ Uj) ⊆ ℝn, 𝒯, Σ
ϕ ◦ fi−1: Uj ∩ Ui → Uj, Σj/Uj, μj = Φ(Ui ∩ Uj) ⊆ ℝn, 𝒯, Σ
are differentiable maps of class 𝕀k (K ≥ 1)
(2) Measurable that is, these two transition maps φi ◦ fi−1 and φj ◦ fj−1 are measurable functions if,
(i) any Borel subset K ⊆ φi(Ui ∩ Uj) is measurable in (ℝn, 𝒯, Σ), then (φj ◦ fi−1)(K) ∈ φi(Ui ∩ Uj) is also measurable,
(ii) any Borel subset S ⊆ φj(Ui ∩ Uj) is measurable in (ℝn, 𝒯, Σ), then, (φi ◦ fi−1)(S) ∈ φj(Ui ∩ Uj) is also measurable.

Definition 2.4: Measure Atlas [7][8][9][10]
By an 𝕀n measure atlas of class 𝕀k on M, we mean a countable collection (An, 𝒯n/An, Σn/An, μn/An) of n-dimensional measure charts (Uv/Uv, Σv/Uv, μv|Uv, φv) for all i ∈ 1 on (M, 𝒯, Σ, μ1) satisfying the following conditions:
(a1) Uv = Uv, Σv/Uv, μv = M

that is, the countable union of the measure charts in $(\mathcal{A}, \mathcal{T}_i / \mathbb{R}, \Sigma_i / \mathbb{R}, \mu_i / \mathbb{R})$ cover $(M, \mathcal{T}, \Sigma, \mu)$.

(a2) For any pair of measure charts

\[(U, \mathcal{T}_i / \mathbb{R}, \Sigma_i / \mathbb{R}, \mu_i / \mathbb{R}, \phi_i) \quad \text{and} \quad ((U, \mathcal{T}_j / \mathbb{R}, \Sigma_j / \mathbb{R}, \mu_j / \mathbb{R}, \phi_j)) \quad \text{in} \quad (\mathcal{A}, \mathcal{T}_i / \mathbb{R}, \Sigma_i / \mathbb{R}, \mu_i / \mathbb{R}) \]

the transition maps $\phi_i \circ \phi_j^{-1}$ are

(1) differentiable maps of class $C^k$ ($k \geq 1$) that is,

$\phi_i \circ \phi_j^{-1} : \phi_i(U \cap U_j) \rightarrow \phi_j(U \cap U_i) \subseteq (\mathbb{R}^n, \mathcal{T}, \Sigma, \mu)$,

$\phi_i \circ \phi_j^{-1} : \phi_i(U \cap U_j) \rightarrow \phi_j(U \cap U_i) \subseteq (\mathbb{R}^n, \mathcal{T}, \Sigma, \mu)$ are differentiable maps of class $C^k$ ($k \geq 1$),

(2) measurable and measure invariant. That is,

(i) Two measure atlases

\[\mathcal{A}_1 \sim \mathcal{A}_2 \quad \text{if} \quad \forall (\mathcal{A}, \mathcal{T}, \Sigma, \mu) \in \Lambda^\mu(M) \rightarrow \mathcal{A}_1 \rightarrow \mathcal{A}_2 \text{ is measurable and measure invariant, that is,} \]

$\mu((\phi \circ \phi_i^{-1} \circ \phi_j^{-1}(K)) = \mu(K)$.

(iii) Let $\mathcal{A}_1, \mathcal{A}_2, \mathcal{T}_i \rightarrow \mathcal{A}_1, \mathcal{T}_j \rightarrow \mathcal{A}_2, \Sigma_i \rightarrow \mathcal{A}_1, \Sigma_j \rightarrow \mathcal{A}_2$ and $T : \mathcal{A}_1 \rightarrow \mathcal{A}_2$ be invertible measure preserving transformations.

A topological measure space $(M, \mathcal{T}, \Sigma, \mu)$ is measurable homeomorphism and measure invariant to a measure space $(\mathbb{R}^n, \mathcal{T}, \Sigma, \mu)$ endowed with two structural relations between any two atlases $\mathcal{A}_1$ and $\mathcal{A}_2 \in \Lambda^\mu(M)$:

(i) $\mathcal{A}_1 \sim \mathcal{A}_2$, if $\mathcal{A}_1 \cup \mathcal{A}_2 \in \Lambda^\mu(M)$

(ii) $\mathcal{A}_1 \sim \mathcal{A}_2$ if $\mu(\mathcal{A}_1) = \mu(\mathcal{A}_2)$

then, $(M, \mathcal{T}, \Sigma, \mu)$ is called as a measure manifold.

Definition 2.7: Complete measure space $(\mathbb{R}^n, \mathcal{T}, \Sigma, \mu)$

Let $(\mathbb{R}^n, \mathcal{T}, \Sigma, \mu)$ be a measure space of dimension $n$.

Suppose that for every Borel subset $U \subseteq (\mathbb{R}^n, \mathcal{T}, \Sigma, \mu)$, $\mu(U) = 0$ and every $V \subseteq U$, $\mu(V) = 0$ then $(\mathbb{R}^n, \mathcal{T}, \Sigma, \mu)$ is a complete measure space.

3. CONVERGENCE ON MEASURE SPACE $(\mathbb{R}^n, \mathcal{T}, \Sigma, \mu)$

Let us consider a measure space $(\mathbb{R}^n, \mathcal{T}, \Sigma, \mu)$ and now we shall discuss some modes of convergence that arise from measure theory.

Let $f_n, f : (\mathbb{R}^n, \mathcal{T}, \Sigma, \mu) \rightarrow \mathbb{R}$, be measurable functions on measure space $(\mathbb{R}^n, \mathcal{T}, \Sigma, \mu)$.

The following modes of convergence on measure space are discussed in [1], [4], [5], [12]:

(1) We say that $f_n$ converges to $f$ point wise if, for every $x \in \mathbb{R}$, $f_n(x)$ converges to $f(x)$. In other words, for every $\epsilon > 0$ and $x \in \mathbb{R}$, there exists $N$ (that depends both $\epsilon$ and $x$) such that $|f_n(x) - f(x)| \leq \epsilon$ whenever $n \geq N$.

(2) We say that $f_n$ converges to $f$ uniformly, for every $\epsilon > 0$, there exists $N$ such that, for every $n \geq N$, $|f_n(x) - f(x)| \leq \epsilon$, for every $x \in \mathbb{R}$. The difference between uniform convergence and point wise convergence is that with the former, the time $N$ at which $f_n(x)$ must be permanently $\epsilon$ close to $f(x)$ is not permitted to depend on $x$, but must instead be chosen uniformly in $x$.

Now, we discuss some of the modes of convergence on measure space $(\mathbb{R}^n, \mathcal{T}, \Sigma, \mu)$:

(1) We say that $f_n$ converges to $f$ point wise almost everywhere if, for $(\mu-a.e.)$ almost everywhere $x \in (\mathbb{R}^n, \mathcal{T}, \Sigma, \mu)$, $f_n(x)$ converges to $f(x)$.

(2) We say that $f_n$ converges to $f$ uniformly almost if, for every $\epsilon > 0$, there exists an exceptional set $E \subseteq \Sigma$ for measure $\mu(E) \leq \epsilon$ such that, $f_n$ converges uniformly to $f$ on the complement of $E$.

(3) We say that $f_n$ converges to $f$ in measure if, for every $\epsilon > 0$, the measures $\{\{ x \in (\mathbb{R}^n, \mathcal{T}, \Sigma, \mu) : |f_n(x) - f(x)| \geq \epsilon \} \}$ converge to zero as $n \rightarrow \infty$.

According to Christos Papachristodoulou [1], each pair $(f_n, f)$ induces a double sequence of measurable sets $A_n(f_n, f)$ or simply $A_n, n \in \mathbb{N}$ where $A_n = \{ x \in (\mathbb{R}^n, \mathcal{T}, \Sigma, \mu) : |f_n(x) - f(x)| \geq \epsilon \}$, determining the behavior of the pair $(f_n, f)$ with respect to convergence. More precisely, some results are as follows:

(1) $f_n \rightarrow f \iff$ for each $j \in \mathbb{N}, \lim_{n \rightarrow \infty} \mu(A_n) = 0$.

(2) $\{ \{ x \in (\mathbb{R}^n, \mathcal{T}, \Sigma, \mu) : f_n(x) \rightarrow f(x) \} \} = \bigcup_{n \in \mathbb{N}} \bigcap_{n \in \mathbb{N}} E_n$, where $E_n = \bigcup_{k \geq n} A_n = \{ x \in (\mathbb{R}^n, \mathcal{T}, \Sigma, \mu) : E_k \geq n \}$.

(3) $f \mu \rightarrow f \iff$ for each $j \in \mathbb{N}, \lim_{n \rightarrow \infty} \mu(\bigcap_{n \in \mathbb{N}} E_n) = 0$.
If \( \{ f_n \} \) is a sequence of measurable functions on \((\mathbb{R}^n, \mathcal{T}, \Sigma, \mu)\) converging to \( f \) pointwise almost everywhere on \((\mathbb{R}^n, \mathcal{T}, \Sigma, \mu)\), then the ordered pair \((\{ f_n \}, f)\) induces a Borel subset \( A_\mu \) satisfying the following conditions:
\[
A_\mu = \{ x \in (\mathbb{R}^n, \mathcal{T}, \Sigma, \mu) : | f_n(x) - f(x) | < \epsilon \} \quad \forall \ n \in \mathbb{N},
\]
where,
(i) \( \mu (A_\mu) > 0 \), if \( | f_n(x) - f(x) | < \epsilon, \forall \ n \in \mathbb{N} \),
(ii) \( \mu (A_\mu) = 0 \), if \( | f_n(x) - f(x) | \geq \epsilon, \forall \ n \in \mathbb{N} \),
that is, \( \mu (A_\mu) = 0 \) as \( n \to \infty \).

Definition 3.1: Dark point of \((\mathbb{R}^n, \mathcal{T}, \Sigma, \mu)\)
The point \( f(x) \in A_\mu \) is called as a dark point, if \( f_n(x) \to f(x) \) in \( A_\mu \) and \( \mu (A_\mu) = 0 \).

If \((\mathbb{R}^n, \mathcal{T}, \Sigma, \mu)\) be a measure space and \( \{ f_n \} \) be a sequence of measurable functions and we say that \( f_n \) converges to \( f \) a.e. in measure on \((\mathbb{R}^n, \mathcal{T}, \Sigma, \mu)\), and the ordered pair \((\{ f_n \}, f)\) induces a Borel subsets \( A_\mu \), satisfying the following conditions:
For any \( n \), \( A_n = \{ x \in (\mathbb{R}^n, \mathcal{T}, \Sigma, \mu) : | f_n(x) - f(x) | < \epsilon \} \), \( \forall \ n \in \mathbb{N} \), such that,
(i) \( \mu (A_n) > 0 \), if \( | f_n(x) - f(x) | < \epsilon, \forall \ n \in \mathbb{N} \),
(ii) \( \mu (A_n) = 0 \), if \( | f_n(x) - f(x) | \geq \epsilon, \forall \ n \in \mathbb{N} \),
that is, \( \mu (A_n) = 0 \) as \( n \to \infty \).

Suppose \( (\mathcal{T}, \Sigma, \mu) \) is a measure manifold \((\mathcal{T}, \Sigma, \mu)\) and if any measure manifold \((\mathcal{T}, \Sigma, \mu)\) then for every \( \mu = (\mathcal{T}, \Sigma, \mu) \) \( \exists \) \( \phi^{-1}(A_n) = S \in (U, \phi) \in (\mathcal{T}, \Sigma, \mu) \) such that,
\[
\mu (\phi^{-1}(A_n)) = 0, \quad \forall \ x \in S \in (U, \phi) \in (\mathcal{T}, \Sigma, \mu)
\]
and
\[
S = \{ p \in (M, \mathcal{T}, \Sigma, \mu) : | f_n(x) - f(x) | < \epsilon, \forall \ n \in \mathbb{N} \}.
\]

Definition 3.3: Convergence point wise almost everywhere
Now, we shall extend the study of above modes of convergence of \((\mathbb{R}^n, \mathcal{T}, \Sigma, \mu)\) to a measure manifold \((M, \mathcal{T}, \Sigma, \mu)\) that is \((\mathbb{R}^n, \mathcal{T}, \Sigma, \mu)\) complete a measure space.

Definition 3.5: Convergence \( \mu \)-a.e. on \((\mathcal{T}, \Sigma, \mu)\)
Let \( f_n, f \to f \) in \( A_\mu \in (\mathbb{R}^n, \mathcal{T}, \Sigma, \mu) \) and if any measure manifold \((\mathcal{T}, \Sigma, \mu)\) is measurable homeomorphic to \((\mathbb{R}^n, \mathcal{T}, \Sigma, \mu)\) then \( \forall \ x \in (U, \phi) \in (\mathcal{T}, \Sigma, \mu) \) and
\[
\mu (\phi^{-1}(A_n)) > 0, \quad \forall \ n \in \mathbb{N},
\]
that is, \( \mu (A_n) = 0 \) as \( n \to \infty \).

Definition 3.6: Complete Measure Manifold
If \((\mathcal{T}, \Sigma, \mu)\) is a measure manifold of dimension \( n \) and suppose that for every measure chart \((U, \phi) \subseteq (\mathcal{T}, \Sigma, \mu)\), \( \mu (U) = 0 \) and every \( V \subseteq (U, \phi) \), if \( \mu (V) = 0 \), then \((\mathcal{T}, \Sigma, \mu)\) is called as a complete measure manifold.

4. DIFFERENT ASPECTS OF CONNECTEDNESS \( \mu \)-a.e. PROPERY ON COMPLETE MEASURE MANIFOLD
In this section, we study the concept of connectedness \( \mu \)-a.e. like locally path connectedness \( \mu \)-a.e., interconnected \( \mu \)-a.e. and maximally path connected \( \mu \)-a.e. on complete measure manifold introduced by S. C. P. Halakatti (17],[8],[9],[10],[11]).

Let \((M, \mathcal{T}, \Sigma, \mu)\) be a complete measure manifold of dimension \( n \) which is measurable homeomorphic to a measure space \((\mathbb{R}^n, \mathcal{T}, \Sigma, \mu)\). Let \( \{ f_n \}, \{ g_n \} \) be measurable real valued functions converging to \( f \) and \( g \) respectively in \((\mathbb{R}^n, \mathcal{T}, \Sigma, \mu)\).
Since \( \phi \) is measurable homeomorphism from \((M, \mathcal{T}, \Sigma, \mu)\) to \((\mathbb{R}^n, \mathcal{T}, \Sigma, \mu)\) for every \( \{ f_n \} \) and \( \{ g_n \} \) on \((\mathbb{R}^n, \mathcal{T}, \Sigma, \mu)\) there exist corresponding measurable real valued functions \( \{ f_n \circ \phi \} \) and \( \{ g_n \circ \phi \} \) converging to \( f \circ \phi \) and \( g \circ \phi \) on \((M, \mathcal{T}, \Sigma, \mu)\).

The ordered pair \((f_n \circ \phi, f \circ \phi)\) induces a Borel subset \( S \subseteq (U, \phi) \subseteq (M, \mathcal{T}, \Sigma, \mu) \) satisfying the following condition:
\[
S = \{ p \in (M, \mathcal{T}, \Sigma, \mu) : | f_n(x) - f(x) | < \epsilon, \forall \ n \in \mathbb{N} \} \quad \text{on the chart} \quad (U, \phi) \quad \text{for which} \quad \mu_1 (S) > 0.
\]
Definition 4.1: Locally path connected $\mu_1$ a.e. on complete measure manifold

The Borel subset $S$ is locally path connected $\mu_1$ – a.e. if $\exists$ a C$^\infty$ map $\gamma: [0, 1] \to S \in (U, \phi)$ such that

$\gamma(0) = p \in S$, $\gamma(1) = q \in S$, such that $\mu_1(S) > 0$.

That is, $p$ is locally path connected $\mu_1$ – a.e. to $q$ in $S \subset (U, \phi) \in A(M)$.

That is, locally path connectedness $\mu_1$ – a.e. is between two points in the same chart $(U, \phi) \in A(M)$.

If $\mu_1(S) = 0$, then there does not exist a path $\gamma$ between $p$ and $q$.

Definition 4.2:

If $\mu_1(S) = 0$ where $(f_n \circ \phi) \to f \circ \phi$, then $S \subset (U, \phi) \subset (M, T_1, \Sigma, \mu)$ is called as a dark region in the chart $(U, \phi)$.

Let $(M, T_1, \Sigma, \mu_1)$ be a complete measure manifold on which $(f_n \circ \phi)$ and $(g_n \circ \phi)$ are sequence of real valued measurable functions converging to $f \circ \phi$ and $g \circ \phi$ pointwise $\mu_1$ – a.e. on $(U, \phi)$ and $(V, \psi)$ belonging to the atlas $A$ respectively. The ordered pairs $((f_n \circ \phi), f \circ \phi)$ and $((g_n \circ \phi), g \circ \phi)$ induce two Borel subsets $S \in (U, \phi) \in A(M)$ and $R \in (V, \psi) \in A(M)$ satisfying the following condition:

$S = \{p \in (M, T_1, \Sigma, \mu_1): |(f_n \circ \phi)(p) - (f \circ \phi)(p)| < \epsilon \forall n \in \mathbb{N}\}$ on the chart $(U, \phi)$ for which $\mu_1(S) > 0$.

$R = \{q \in (M, T_1, \Sigma, \mu_1): |(g_n \circ \phi)(q) - (g \circ \phi)(q)| < \epsilon \forall n \in \mathbb{N}\}$ on the chart $(V, \psi)$ for which $\mu_1(R) > 0$.

Note: We denote the Borel subsets $\phi^{-1}(A_n) = S(U, \phi) \in (M, T_1, \Sigma, \mu_1)$ and $\phi^{-1}(B_n) = R(V, \psi) \in (M, T_1, \Sigma, \mu_1)$.

Definition 4.3: Interconnected $\mu_1$ – a.e on complete measure manifold

The Borel subset $S \in (U, \phi) \in A(M, T_1, \Sigma, \mu_1)$ is interconnected to the Borel subset $R \in (V, \psi) \in A(M, T_1, \Sigma, \mu_1)$ if $\exists$ a C$^\infty$ map $\gamma: [0, 1] \to S \cup R \in A(M)$ such that $\gamma(0) = p \in S \in A(M)$, $\gamma(1) = q \in S \in A(M)$, such that $\mu_1(S) > 0$ and $\mu_1(R) > 0$.

That is, p is interconnected $\mu_1$ – a.e. to q in S U R \in A(M).

That is, interconnectedness $\mu_1$ – a.e. is between two charts in the same atlas $A \in A(M)$.

If $\mu_1(S) = 0$ and $\mu_1(R) = 0$ then $\exists$ a path between p and q.

Definition 4.4:

If $\mu_1(S) = 0$ where $(f_n \circ \phi) \to f \circ \phi$ in $(U, \phi)$ and $\mu_1(R) = 0$ where $(g_n \circ \phi) \to g \circ \phi$ in $(V, \psi)$, then $S$ is called as dark region in the chart $(U, \phi)$ and $R$ is called as dark region in the chart $(V, \psi)$ belonging to the same atlas $A \in A(M)$.

Let $(\mathbb{R}^n, \mathcal{T}, \Sigma, \mu)$ be a measure space and $(f_n), (g_n)$ and $(h_n)$ are sequences of measurable functions on $(\mathbb{R}^n, \mathcal{T}, \Sigma, \mu)$ converging to $f, g$ and $h$ point wise almost everywhere on $(\mathbb{R}^n, \mathcal{T}, \Sigma, \mu)$. The ordered pairs $((f_n)_1, f), ((g_n)_1, g)$ and $((h_n)_1, h)$ induce the following Borel subsets $A_n, B_n$ and $C_n$. We define Borel subsets $A_n = \{ x \in (\mathbb{R}^n, \mathcal{T}, \Sigma, \mu) : |f_n(x) - f(x)| < \epsilon \}, \forall n \in \mathbb{N}$, where

(i) $\mu(A_n) > 0$, if $|f_n(x) - f(x)| < \epsilon, \forall n \in \mathbb{N}$,
(ii) $\mu(A_n) = 0$, if $|f_n(x) - f(x)| \geq \epsilon, \forall n \in \mathbb{N}$, that is, $\mu(A_n) = 0$ as $n \to \infty$.

Similarly,

(i) $B_n = \{ y \in (\mathbb{R}^n, \mathcal{T}, \Sigma, \mu) : |g_n(y) - g(y)| < \epsilon \}, \forall n \in \mathbb{N}$, where

(ii) $\mu(B_n) = 0$, if $|g_n(y) - g(y)| < \epsilon, \forall n \in \mathbb{N}$,
(iii) $\mu(B_n) = 0$, if $|g_n(y) - g(y)| \geq \epsilon, \forall n \in \mathbb{N}$, that is, $\mu(B_n) = 0$ as $n \to \infty$.

$C_n = \{ z \in (\mathbb{R}^n, \mathcal{T}, \Sigma, \mu) : |h_n(z) - h(z)| < \epsilon \}, \forall n \in \mathbb{N}$, where

(i) $\mu(C_n) > 0$, if $|h_n(z) - h(z)| < \epsilon, \forall n \in \mathbb{N}$,
(ii) $\mu(C_n) = 0$, if $|h_n(z) - h(z)| \geq \epsilon, \forall n \in \mathbb{N}$, that is, $\mu(C_n) = 0$ as $n \to \infty$.

If $(M, T_1, \Sigma, \mu)$ is a complete measure manifold that is measurable homeomorphic and measure invariant to $(\mathbb{R}^n, \mathcal{T}, \Sigma, \mu)$. Then $\exists$ a measurable homeomorphism and measure invariant transformation $\phi: (M, T_1, \Sigma, \mu) \to (\mathbb{R}^n, \mathcal{T}, \Sigma, \mu)$, such that, $f_n \circ \phi$, $(g_n \circ \phi)$ and $(h_n \circ \phi)$ are sequences of real valued measurable functions converging to $f \circ \phi$, $g \circ \phi$ and $h \circ \phi$ point wise $\mu_1$ – a.e. on $(U, \phi), (V, \psi)$ and $(W, \chi)$ belonging to the atlases $A_1$, $A_2$ respectively. Also, for every induced Borel subsets $A_n, B_n$ and $C_n$ in $(\mathbb{R}^n, \mathcal{T}, \Sigma, \mu)$ the corresponding induced Borel subsets $\phi^{-1}(A_n) \in \mathcal{A}(M), \phi^{-1}(B_n) \in \mathcal{A}(M)$ and $\phi^{-1}(C_n) = Q \subset (W, \chi) \in \mathcal{A}(M)$.

Definition 4.5: Maximal connected $\mu_1$-a.e on complete measure manifold

Let $(M, T_1, \Sigma, \mu)$ be a complete measure manifold and let $A_n, A_1$ and $A_2$ in $\mathcal{A}(M)$ be atlases on $(M, T_1, \Sigma, \mu)$. Let $S, R$ and $Q$ be Borel subsets of $A_n, A_1$ and $A_2$. Then, we say that $\mathcal{A}(M)$ in $(M, T_1, \Sigma, \mu)$ is maximally connected if $\exists$ a map $\gamma: [0, 1] \to S \cup R \cup Q \in A_1 \cup A_2 \cup A_3 \in \mathcal{A}(M)$ such that $\gamma(0) = \gamma(1) = p \in S \in \mathcal{A}(M)$, $\gamma(1/2) = q \in R \in \mathcal{A}(M)$ for which $\mu_1(Q) > 0$ and $\gamma(1/2) = q \in Q \in \mathcal{A}(M)$, such that $\mu_1(Q) > 0$ and $\gamma(1) = r \in Q \in \mathcal{A}(M)$, such that $\mu_1(Q) > 0$ and $\gamma(1) = r \in Q \in \mathcal{A}(M)$, such that $\mu_1(Q) > 0$. That is, for each $p \in (U, \phi) \in A_2$ is path connected to each q $\in (V, \psi) \in A_2$ for $A_2 \cup A_3 \in \mathcal{A}(M)$, $\mu(A_2 \cup A_3) > 0$ for each $q \in (V, \psi) \in A_2$ is path connected to each $r \in (W, \chi) \in A_2 \cup A_3 \in \mathcal{A}(M)$ and for $A_2 \cup A_3 \in \mathcal{A}(M)$, $\mu(A_2 \cup A_3) > 0$. If for each $p \in (U, \phi) \in A_2$ is path connected to each $r \in (W, \chi)$.

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χ) ∈ A, ∈ A₁(M) and for A, U A₁ ∈ A₁(M), μ₁(A∪U A₁) > 0 then (A, U A₁, U A₁) ∈ A₁(M) ∈ (M, ℑ₁, ℑ₁, 𝜇₁) is maximally path connected if μ₁(A∪U A₁, U A₁) > 0 on complete measure manifold.

If μ₁(S) = 0 and μ₁(R) = 0 then there does not exist a path γ between p ∈ S and q ∈ R.

Definition 4.6:
If μ₁(S) = 0 where \( f_o \phi \to f \circ \phi \) in (U, φ) and μ₁(R)=0 where \( g_o \phi \to g \circ \phi \) in (W, χ), then S is called as dark region in the chart (U, φ) ∈ A. R is called as dark region in the chart (V, ψ) and Q is called as dark region in the chart (W, χ) ∈ A in A₁(M).

Now, we show that locally path connectedness is invariant with respect to measurable homeomorphism and measure invariant on a complete measure manifold if μ₁(S) > 0.

Theorem 4.7: Let (M₁, ℑ₁, ℑ₁, 𝜇₁) and (M₂, ℑ₂, ℑ₂, 𝜇₂) be complete measure manifolds of dimension n and m respectively. If M₁ is locally path connected μ₁-a.e in S ∈ M₁, then M₂ is also locally path connected μ₂-a.e in F(S) ∈ M₂ with μ₂(F(S)) > 0.

Proof: Let (M₁, ℑ₁, ℑ₁, 𝜇₁) be a complete measure manifold and it is locally path connected. Let (U₁, φ₁) be a measure chart in (M₁, ℑ₁, ℑ₁, 𝜇₁) that is, S ∈ M₁ and μ₁(S) > 0.

Consider a measurable homeomorphism and measure invariant map F: (M₁, ℑ₁, ℑ₁, 𝜇₁) → (M₂, ℑ₂, ℑ₂, 𝜇₂) defined for S ∈ M₁, F(S) ∈ M₂. To prove that (M₂, ℑ₂, ℑ₂, 𝜇₂) is locally path connected, let us consider S = \( \{ p ∈ (M₁, ℑ₁, ℑ₁, 𝜇₁): \mid (f_o \phi \circ \phi) - (f \circ \phi) \mid < \epsilon \), \( \forall \: n ∈ N \) on the chart (U, φ) for which μ₁(S) > 0.

Now, for every S ∈ (M₁, ℑ₁, ℑ₁, 𝜇₁), F(S) ∈ (M₂, ℑ₂, ℑ₂, 𝜇₂) such that F(S) ∈ (M₂, ℑ₂, ℑ₂, 𝜇₂).

Let (U, φ), (V, ψ) be charts in M₁ and M₂ respectively and F(p₁), F(p₂) ∈ F(S) ∈ (M₂, ℑ₂, ℑ₂, 𝜇₂).

Since F is measurable homeomorphism and measure invariant map, there exists \( F^{-1} \) such that for every measure chart (V, ψ) ∈ (M₂, ℑ₂, ℑ₂, 𝜇₂) such that μ₂(F(S)) > 0.

That is, for every F(p₁), F(p₂) ∈ (V, ψ) ∈ (M₂, ℑ₂, ℑ₂, 𝜇₂), there exists, \( F^{-1}(F(p₁)) = p₁ \in (U, φ) \) and \( F^{-1}(F(p₂)) = p₂ \in (U, φ) \).

But (M₁, ℑ₁, ℑ₁, 𝜇₁) is locally path connected.

Therefore, there exists a C∞ map γ : [0,1] → (M₁, ℑ₁, ℑ₁, 𝜇₁) such that γ(0) = p₁ ∈ S in (U, φ), μ₁(S) > 0.

γ(1) = p₂ ∈ S in (U, φ), μ₁(S) > 0.

Hence, if μ₁(S) = 0 then p₁ is not locally path connected to p₂.

Now, since F is homeomorphism there exists a map \( F \circ γ : [0, 1] → (M₂, ℑ₂, ℑ₂, 𝜇₂) \)

such that, \( F \circ γ(0) = F(p₁) = q₁ \in F(S) ∈ (V, ψ), \mu₂(F(S)) > 0 \).

\( \mu₂(F(S)) = 0 \) then q₁ is not maximally path connected to q₂.

Therefore, q₁, is locally path connected μ₁-a.e to q₂ by \( F \circ γ \) in F(S) ∈ (V, ψ) ∈ (M₂, ℑ₂, ℑ₂, 𝜇₂).

If q₁ is locally path connected μ₁-a.e. to q₂ by \( F \circ γ \) in F(S) ∈ (V, ψ) ∈ (M₂, ℑ₂, ℑ₂, 𝜇₂).

Then \( F \circ γ(1) = F(p₂) = q₂ \in F(S) ∈ (V, ψ), \mu₂(F(S)) > 0 \).

If μ₂(F(S)) = 0 then q₁ is not maximally path connected to q₂.

Therefore, q₁ is locally path connected μ₁-a.e to q₂ by \( F \circ γ \) in F(S) ∈ (V, ψ) ∈ (M₂, ℑ₂, ℑ₂, 𝜇₂).
\( A^1(M) \) with \( \mu_j(S \cup R) > 0 \) then \( \exists \) a measurable homeomorphism and measure invariant map \( F: (M_1, T_1, \Sigma_1, \mu_1) \rightarrow (M_2, T_2, \Sigma_2, \mu_2) \) such that \( M_2 \) is also interconnected.

**Proof:** Let \( (M_1, T_1, \Sigma_1, \mu_1) \) be complete measure manifolds of dimension \( n \) and \( m \) respectively. If \( (M_1, T_1, \Sigma_1, \mu_1) \) is interconnected, then \( \mu_1(S) > 0 \), \( \mu_1(R) > 0 \).

Consider a measurable homeomorphism and measure invariant map

\[
F: (M_1, T_1, \Sigma_1, \mu_1) \rightarrow (M_2, T_2, \Sigma_2, \mu_2)
\]

defined as follows: for any two Borel subsets

\[
S = \{ p_1 \in (M_1, T_1, \Sigma_1, \mu_1) : \left| \left( f_n \circ \phi \right)(p_1) - \left( f \circ \phi \right)(p_1) \right| < \epsilon, \forall n \in \mathbb{N} \}
\]

on the chart \((U_1, \phi_1)\), for which \( \mu_1(S) > 0 \) and

\[
R = \{ p_2 \in (M_1, T_1, \Sigma_1, \mu_1) : \left| \left( f_n \circ \phi \right)(p_2) - \left( f \circ \phi \right)(p_2) \right| < \epsilon, \forall n \in \mathbb{N} \}
\]

on the chart \((U_2, \phi_2)\), for which \( \mu_2(R) > 0 \).

If \( \mu_2(F(S)) = 0 \) and \( \mu_2(F(R)) = 0 \) then \( q_1 \) is not interconnected to \( q_2 \).

Therefore, if \( (M_1, T_1, \Sigma_1, \mu_1) \) is interconnected, then \( (M_2, T_2, \Sigma_2, \mu_2) \) is interconnected.

Hence, interconnectedness is invariant with respect to measurable homeomorphism and measure invariant transformation if

\[
\mu_1(S) > 0, \mu_1(R) > 0 \text{ and } \mu_2(F(S)) > 0, \mu_2(F(R)) > 0.
\]

Now, we show that maximal path connectedness is invariant with respect to measurable homeomorphism and measure invariant on complete measure manifold if \( \mu_1(S) > 0 \) and \( \mu_1(R) > 0 \).

**Theorem 4.9:**

Let \( (M_1, T_1, \Sigma_1, \mu_1) \) and \( (M_2, T_2, \Sigma_2, \mu_2) \) be complete measure manifolds of dim \( n \) and \( m \) respectively. If \( (M_1, T_1, \Sigma_1, \mu_1) \) is maximally path connected \( \mu_1\text{-a.e.} \) in \( A^1(M_1) \) such that \( \mu_1(S) > 0, \mu_1(R) > 0 \) and \( \mu_1(Q) > 0 \), then there exist three Borel subsets \( S, R \) and \( Q \) such that

\[
S = \{ p_1 \in (M_1, T_1, \Sigma_1, \mu_1) : \left| \left( f_n \circ \phi \right)(p_1) - \left( f \circ \phi \right)(p_1) \right| < \epsilon, \forall n \in \mathbb{N} \}
\]

on the chart \((U_1, \phi_1)\), for which \( \mu_1(S) > 0 \) and

\[
R = \{ p_2 \in (M_1, T_1, \Sigma_1, \mu_1) : \left| \left( f_n \circ \phi \right)(p_2) - \left( f \circ \phi \right)(p_2) \right| < \epsilon, \forall n \in \mathbb{N} \}
\]

on the chart \((U_2, \phi_2)\), for which \( \mu_1(R) > 0 \) and \( \mu_1(Q) > 0 \).

Consider a measurable homeomorphism \( F: (M_1, T_1, \Sigma_1, \mu_1) \rightarrow (M_2, T_2, \Sigma_2, \mu_2) \). Since \( (M_1, T_1, \Sigma_1, \mu_1) \) is maximally path connected \( \mu_1\text{-a.e.} \) in \( A^1(M_1) \), then \( \exists \) three Borel subsets \( S, R \) and \( Q \) such that

\[
S = \{ p_1 \in (M_1, T_1, \Sigma_1, \mu_1) : \left| \left( f_n \circ \phi \right)(p_1) - \left( f \circ \phi \right)(p_1) \right| < \epsilon, \forall n \in \mathbb{N} \}
\]

on the chart \((U_1, \phi_1)\), for which \( \mu_1(S) > 0 \) and

\[
R = \{ p_2 \in (M_1, T_1, \Sigma_1, \mu_1) : \left| \left( f_n \circ \phi \right)(p_2) - \left( f \circ \phi \right)(p_2) \right| < \epsilon, \forall n \in \mathbb{N} \}
\]

on the chart \((U_2, \phi_2)\), for which \( \mu_1(R) > 0 \) and \( \mu_1(Q) > 0 \).

Then, there exist a path \( \gamma: [0, 1] \rightarrow S \cup R \cup Q \) such that \( \gamma(0) = p_1 \in S \in (U_1, \phi_1) \in A^1(M_1) \) for which \( \mu_1(S) > 0 \).
\( \gamma(\frac{1}{2}) = p_2 \in R \in (U_2, \phi_2) \in \mathcal{A}_j \in \Lambda^h(M) \) for which \( \mu_1(R) > 0 \) and

\( \gamma(1) = p_3 \in Q \in (U_3, \phi_3) \in \mathcal{A}_l \in \Lambda^h(M) \) for which \( \mu_1(Q) > 0 \),

where each \( p_1 \in S \in (U_1, \phi_1) \in \mathcal{A}_i \) is maximally path connected to each \( p_2 \in R \in (U_2, \phi_2) \in \mathcal{A}_j \) and each \( p_3 \in R \in (U_3, \phi_3) \in \mathcal{A}_l \) is maximally path connected to each \( p_3 \in Q \in (U_3, \phi_3) \in \mathcal{A}_l \).

If \( \mu_1(S) = 0 \), \( \mu_1(R) = 0 \) and \( \mu_1(Q) = 0 \) then \( p_1 \) is not maximally path connected to \( p_2 \) and \( p_2 \) is not maximally path connected to \( p_3 \).

Since \( F \) is measurable homeomorphism and measure invariant, for

\[ p_1 \in S \in (U_1, \phi_1) \in \mathcal{A}_i \in (M_1, \mathcal{T}_1, \Sigma_1, \mu_1) \] there exist \( q_1 \in F(S) \in (V_1, \psi_1) \in \mathcal{B}_i \in (M_2, \mathcal{T}_2, \Sigma_2, \mu_2) \) such that,

\[ F(S) = \{ F(p_1) = q_1 \in (M_2, \mathcal{T}_2, \Sigma_2, \mu_2) \mid |F(h_0 \circ \phi)(p_1) - F(h \circ \phi)(p_1)| < \epsilon, \forall n \in N \} \]

on the chart \( (V_1, \psi_1) \in \mathcal{B}_i \in \Lambda^h(M_2) \), for which \( \mu_2(F(S)) > 0 \).

Similarly, for every \( p_2 \in R \in (U_2, \phi_2) \in \mathcal{A}_j \) in \( (M_1, \mathcal{T}_1, \Sigma_1, \mu_1) \) there exist \( q_2 \in F(R) \in (V_2, \psi_2) \in \mathcal{B}_j \in (M_2, \mathcal{T}_2, \Sigma_2, \mu_2) \) such that,

\[ F(R) = \{ F(p_2) = q_2 \in (M_2, \mathcal{T}_2, \Sigma_2, \mu_2) \mid |F(g_0 \circ \phi)(p_2) - F(g \circ \phi)(p_2)| < \epsilon, \forall n \in N \} \]

on the chart \( (V_2, \psi_2) \in \mathcal{B}_j \in \Lambda^h(M_2) \), for which \( \mu_2(F(R)) > 0 \).

Therefore, we have shown that if \((M_1, \mathcal{T}_1, \Sigma_1, \mu_1)\) is maximally path connected then \((M_2, \mathcal{T}_2, \Sigma_2, \mu_2)\) is also maximally path connected.

Hence, maximal path connectedness \( \mu_1 \text{-a.e.} \) is invariant with respect to measurable homeomorphism and measure invariant function \( F \).

The above theorem shows that if \( \mu_1(S) = 0 \), \( \mu_1(R) = 0 \) and \( \mu_1(Q) = 0 \) are dark regions in the respective charts \((U_1, \phi_1), (U_2, \phi_2)\) and \((U_3, \phi_3)\) in \((M_1, \mathcal{T}_1, \Sigma_1, \mu_1)\) then \( \mu_2(F(S)) = 0 \), \( \mu_2(F(R)) = 0 \) and \( \mu_2(F(Q)) = 0 \) are dark regions in the corresponding charts \((V_1, \psi_1), (V_2, \psi_2)\) and \((V_3, \psi_3)\) in \((M_2, \mathcal{T}_2, \Sigma_2, \mu_2)\).
5 CONCLUSION

In this paper S.C.P. Halakatti has investigated two intrinsic properties on complete measure manifold. We have shown that locally path connected $\mu_1$ a.e. property, interconnected $\mu_1$ a.e. property and maximally path connected $\mu_1$ a.e. on complete measure manifold are invariant under measurable homeomorphism and measure invariant map.

The above study on different aspects of connectedness on complete measure manifold vindicates that, the local path connectedness, the interconnectedness and the maximally path connectedness are invariant under measurable homeomorphism and measure invariant function $F$. One can show that if $F$; $(M_1, \mathcal{F}_1, \Sigma_1, \mu_1) \to (M_2, \mathcal{F}_2, \Sigma_2, \mu_2)$, $G;(M_2, \mathcal{F}_2, \Sigma_2, \mu_2) \to (M_3, \mathcal{F}_3, \Sigma_3, \mu_3)$, then the composite function $G \circ F$; $(M_1, \mathcal{F}_1, \Sigma_1, \mu_1) \to (M_3, \mathcal{F}_3, \Sigma_3, \mu_3)$, satisfies equivalence relation on a complete measure manifold paving a way for a new manifold called network manifold. We carry the study on such network manifolds in our future work.

REFERENCE


