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Convergence and Connectedness on Complete Measure Manifold

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***Abstract* - In this paper we study different modes of convergence and introduce some aspects of connectedness like, locally path connected μ_1 - a.e., interconnected μ_1 - a.e., maximal connected μ_1 - a.e. on a complete measure manifold of dimension n. We show that these two intrinsic topological properties remain invariant under a measurable homeomorphism and measure invariant map.**

Keywords: Convergence, connectedness, complete measure manifold, local path connected μ_1- a.e., interconnected μ_1- a.e., maximal connected μ_1 - a.e.

1. INTRODUCTION

In this paper to investigate the intrinsic properties of complete measure manifold, we have introduced the role of convergent measurable functions on measure manifold. Any point $p \in U \subset (M, \mathcal{T}, \Sigma, \mu)$ will be a limit point of the measure manifold. The point $p \in U$ of $(M, \mathcal{T}, \Sigma, \mu)$ carries additional information in terms of the ordered pair $(\{f_n\}, f)$ which induces a measurable set A_n such that the measure of such set is always positive on measure manifold. If the measure of A_n is zero and any subset $B \subset A$, also has measure zero then the measure manifold is complete. On such complete measure manifold, we have introduced different aspects of connectedness, like, locally path connected, interconnected, maximally connected and proved some important results that will generate a class of connected measure manifolds.

2. PRELIMINARIES

Some of the basic definitions referred are as follows

Definition 2.1: Measurable chart [7][8][9]

Let $(U, \mathcal{T}_1/_U, \Sigma_1/_U) \subseteq (M, \mathcal{T}_1, \Sigma_1)$ be a non-empty measurable subspace of $(M, \mathcal{T}_1, \Sigma_1)$. If there exists a map, $\phi: (U, \mathcal{T}_1/_U, \Sigma_1/_U) \rightarrow (\mathbb{R}^n, \mathcal{T}, \Sigma)$, satisfying the following conditions,

(i) ϕ is homeomorphism,

(ii) ϕ is measurable,

then the structure $((U, \mathcal{T}_1/_U, \Sigma_1/_U), \phi)$ is called as a measurable chart.

Definition 2.2: Measure Chart [7][8][9]

A measure $\mu_1/_U$ on measurable chart $((U, \mathcal{T}_1/_U, \Sigma_1/_U), \phi)$ is called as measure chart, denoted by $((U, \mathcal{T}_1/_U, \Sigma_1/_U\ \mu_1/_U), \phi)$ satisfying following conditions:

(i) ϕ is homeomorphism,

(ii) ϕ is measurable,

(iii) ϕ is measure invariant

then, the structure $((U, \mathcal{T}_1/_U, \Sigma_1/_U, \mu_1/_U), \phi)$ is called as a measure chart.

Definition 2.3: Measurable Atlas [7][8][9]

By an $\mathbb{R}^n$ measurable atlas of class C^k on M we mean a countable collection $(\mathbb{A}, \mathcal{T}_1/_{\mathbb{A}}, \Sigma_1/_{\mathbb{A}})$ of n-dimensional measurable charts $((U_i, \mathcal{T}_1/_{U_i}, \Sigma_1/_{U_i}), \phi_i)$ for all $i \in I$ on $(M, \mathcal{T}_1, \Sigma_1)$ subject to the following conditions:

(a_1) $\bigcup_{i=1}^{\infty}((U_i, \mathcal{T}_1/_{U_i}, \Sigma_1/_{U_i}), \phi_i) = M$

that is, the countable union of the measurable charts in $(\mathbb{A}, \mathcal{T}_1/_{\mathbb{A}}, \Sigma_1/_{\mathbb{A}})$ cover $(M, \mathcal{T}_1, \Sigma_1)$

(a_2)For any pair of measurable charts $((U_i, \mathcal{T}_1/_{U_i}, \Sigma_1/_{U_i}), \phi_i)$ and $((U_j, \mathcal{T}_1/_{U_j}, \Sigma_1/_{U_j}), \phi_j)$ in $(\mathbb{A}, \mathcal{T}_1/_{\mathbb{A}}, \Sigma_1/_{\mathbb{A}})$, the transition maps $\phi_i \circ \phi_j^{-1}$ and $\phi_j \circ \phi_i^{-1}$ are

(1) differentiable maps of class C^k $(K \geq 1)$ that is,

$\phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \rightarrow \phi_i(U_i \cap U_j) \subseteq (\mathbb{R}^n, \mathcal{T}, \Sigma)$

$\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \rightarrow \phi_j(U_i \cap U_j) \subseteq (\mathbb{R}^n, \mathcal{T}, \Sigma)$

are differentiable maps of class C^k $(K \geq 1)$

(2) Measurable that is, these two transition maps $\phi_i \circ \phi_j^{-1}$ and $\phi_j \circ \phi_i^{-1}$ are measurable functions if,

(i) any Borel subset $K \subseteq \phi_i(U_i \cap U_j)$ is measurable in $(\mathbb{R}^n, \mathcal{T}, \Sigma)$, then $(\phi_i \circ \phi_j^{-1})^{-1}(K) \in \phi_j(U_i \cap U_j)$ is also measurable,

(ii) any Borel subset $S \subseteq \phi_j(U_i \cap U_j)$ is measurable in $(\mathbb{R}^n, \mathcal{T}, \Sigma)$, then, $(\phi_j \circ \phi_i^{-1})^{-1}(S) \in \phi_i(U_i \cap U_j)$ is also measurable.

Definition 2.4: Measure Atlas [7][8][9][10]

By an $\mathbb{R}^n$ measure atlas of class C^k on M, we mean a countable collection $(\mathbb{A}, \mathcal{T}_1/_{\mathbb{A}}, \Sigma_1/_{\mathbb{A}}, \mu_1/_{\mathbb{A}})$ of n-dimensional measure charts $(U_i, \mathcal{T}_1/_{U_i}, \Sigma_1/_{U_i}, \mu_1/_{U_i}), \phi_i)$ for all $i \in I$ on $(M, \mathcal{T}_1, \Sigma_1, \mu_1)$ satisfying the following conditions:

(a_1) $\bigcup_{i=1}^{\infty}((U_i, \mathcal{T}_1/_{U_i}, \Sigma_1/_{U_i}, \mu_1/_{U_i}), \phi_i) = M$

that is, the countable union of the measure charts in ($\mathbb{A}$, ${\mathcal{T}_1}/_{\mathbb{A}}$, ${\Sigma_1}/_{\mathbb{A}}$, ${\mu_1}/_{\mathbb{A}}$) cover (M, $\mathcal{T}_1$, Σ_1, μ_1).

(a_2) for any pair of measure charts ((U_i, ${\mathcal{T}_1}/_{U_i}$, ${\Sigma_1}/_{U_i}$, ${\mu_1}/_{U_i}$), ϕ_i) and ((U_j, ${\mathcal{T}_1}/_{U_j}$, ${\Sigma_1}/_{U_j}$, ${\mu_1}/_{U_j}$), ϕ_j) in ($\mathbb{A}$, ${\mathcal{T}_1}/_{\mathbb{A}}$, ${\Sigma_1}/_{\mathbb{A}}$, ${\mu_1}/_{\mathbb{A}}$)

the transition maps $\phi_i \circ \phi_j^{-1}$ and $\phi_j \circ \phi_i^{-1}$ are

(1) differentiable maps of class C^k ($k \geq 1$) that is,

$\phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \rightarrow \phi_i(U_i \cap U_j) \subseteq (\mathbb{R}^n, \mathcal{T}, \Sigma, \mu)$,

$\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \rightarrow \phi_j(U_i \cap U_j) \subseteq (\mathbb{R}^n, \mathcal{T}, \Sigma, \mu)$ are differentiable maps of class C^k ($k \geq 1$),

(2) measurable and measure invariant. That is,

(i) any Borel subset $K \subseteq \phi_i(U_i \cap U_j)$ is measurable and measure invariant in ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ) then $(\phi_i \circ \phi_j^{-1})^{-1}(K) \in \phi_j(U_i \cap U_j)$ is also measurable and measure invariant, that is, $\mu((\phi_i \circ \phi_j^{-1})^{-1}(K)) = \mu(K)$.

(ii) for any Borel subset $S \subseteq \phi_j(U_i \cap U_j)$ is measurable and measure invariant in ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ) then $(\phi_j \circ \phi_i^{-1})^{-1}(S) \in \phi_i(U_i \cap U_j)$ is measurable and measure invariant, that is $\mu(K) = \mu((\phi_j \circ \phi_i^{-1})^{-1}(S))$.

(a_3) For any two measure atlases ($\mathbb{A}_1$, ${\mathcal{T}_1}/_{\mathbb{A}_1}$, ${\Sigma_1}/_{\mathbb{A}_1}$, ${\mu_1}/_{\mathbb{A}_1}$) and ($\mathbb{A}_2$, ${\mathcal{T}_1}/_{\mathbb{A}_2}$, ${\Sigma_1}/_{\mathbb{A}_2}$, ${\mu_1}/_{\mathbb{A}_2}$), $T : \mathbb{A}_1 \rightarrow \mathbb{A}_2$ is measurable and measure invariant, that is,

(i) if E is measurable in $\mathbb{A}_2$ then $T^{-1}(E)$ is measurable in $\mathbb{A}_1$

(ii) $\mu_1/\mathbb{A}_1 (T^{-1}(E)) = \mu_1/\mathbb{A}_2 (E)$, if $\mathbb{A}_1 \sim \mathbb{A}_2$.

(a_4) For any atlas $\mathbb{A}$, T, T^{-1}: $\mathbb{A} \rightarrow \mathbb{A}$ are invertible measure preserving transformations.

An $\mathbb{R}^n$, measure atlas is said to be of class C^∞ if it is of class C^k for every integer k.

Let $A^k(M)$ denotes the set of all $\mathbb{R}^n$ measure atlases of class C^k on (M, $\mathcal{T}_1$,Σ_1,μ_1).

Definition 2.5: Equivalence Relations in $A^k(M)$[7]

(i) Two measure atlases $\mathbb{A}_1$ and $\mathbb{A}_2$ in $A^k(M)$ are said to be equivalent if $\mathbb{A}_1 \cup \mathbb{A}_2$ in $A^k(M)$. In order that $\mathbb{A}_1 \cup \mathbb{A}_2$ be a member of $A^k(M)$ we require that for every measure chart ((U_i, ${\mathcal{T}_1}/_{V_i}$, ${\Sigma_1}/_{U_i}$, ${\mu_1}/_{U_i}$), ϕ_i) $\in \mathbb{A}_1$ and for every measure chart (V_j, ${\mathcal{T}_1}/_{V_j}$, ${\Sigma_1}/_{V_j}$, ${\mu_1}/_{V_j}$), Ψ_j) $\in \mathbb{A}_2$ the set of $\phi_i(U_i \cap V_j)$

and $\Psi_j(U_i \cap V_j)$ be open and measurable in ($\mathbb{R}^n$,$\mathcal{T}$, Σ, μ) and maps $\phi_i \circ \Psi_j^{-1}$ and $\Psi_j \circ \phi_i^{-1}$ be of class C^k and are measurable. The relation introduced is an equivalence relation in $A^k(M)$ and hence partitions $A^k(M)$ into disjoint equivalence classes. Each of these equivalence classes induces a differentiable structure of class C^k on (M, $\mathcal{T}_1$, Σ_1, μ_1). Any two atlases are compatible, that is, $\mathbb{A}_1 \sim \mathbb{A}_2$ if ($\mathbb{A}_1 \cup \mathbb{A}_2$) in $A^k(M)$.

(ii) Also any two atlases are compatible, that is, $\mathbb{A}_1 \sim \mathbb{A}_2$ if $\mu_1(\mathbb{A}_1) = \mu_1(\mathbb{A}_2)$.

Definition 2.6:

A topological measure space (M, $\mathcal{T}_1$, Σ_1, μ_1) that is measurable homeomorphism and measure invariant to a measure space ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ) endowed with two structural relations between any two atlases $\mathbb{A}_1$ and $\mathbb{A}_2 \in A^k(M)$:

i) $\mathbb{A}_1 \sim \mathbb{A}_2$, if $\mathbb{A}_1 \cup \mathbb{A}_2 \in A^k(M)$

ii) $\mathbb{A}_1 \sim \mathbb{A}_2$ if $\mu_1(\mathbb{A}_1) = \mu_1(\mathbb{A}_2)$

then, (M, $\mathcal{T}_1$, Σ_1, μ_1) is called as a measure manifold.

Definition 2.7: Complete measure space ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ) [4][5][12]

Let ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ) be a measure space of dimension n. Suppose that for every Borel subset $U \subseteq (\mathbb{R}^n, \mathcal{T}, \Sigma, \mu)$, $\mu(U) = 0$ and every $V \subseteq U$, $\mu(V)=0$ then ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ) is a complete measure space.

3. CONVERGENCE ON MEASURE SPACE ($\mathbb{R}^n$,,Σ, μ)

Let us consider a measure space ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ) and now we shall discuss some modes of convergence that arise from measure theory.

Let f_n, $f : (\mathbb{R}^n, \mathcal{T}, \Sigma, \mu) \rightarrow \mathbb{R}$, be measurable functions on measure space ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ).

The following modes of convergence on measure space are discussed in [1], [4], [5], [12]:

(1) We say that f_n converges to f point wise if, for every $x \in X$, $f_n(x)$ converges to f(x). In other words, for every $\epsilon > 0$ and $x \in X$, there exists N (that depends both ϵ and x) such that $|f_n(x) - f(x)| \leq \epsilon$ whenever $n \geq N$.

(2) We say that f_n converges to f uniformly, for every $\epsilon > 0$, there exists N such that, for every $n \geq N$, $|f_n(x) - f(x)| \leq \epsilon$, for every $x \in X$. The difference between uniform convergence and point wise convergence is that with the former, the time N at which $f_n(x)$ must be permanently ϵ – close to f(x) is not permitted to depend on x, but must instead be chosen uniformly in x.

Now, we discuss some of the modes of convergence on measure space ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ):

(1) We say that f_n converges to f point wise almost everywhere if, for (μ-a.e.) almost everywhere $x \in (\mathbb{R}^n, \mathcal{T},\Sigma, \mu)$, $f_n(x)$ converges to f(x).

(2) We say that f_n converges to f point wise almost uniformly if, for every $\epsilon > 0$, there exists an exceptional set $E \in \Sigma$ for measure $\mu(E) \leq \epsilon$ such that, f_n converges uniformly to f on the complement of E.

(3) We say that f_n converges to f in measure if, for every $\epsilon > 0$, the measures $(\{x \in (\mathbb{R}^n, \mathcal{T}, \Sigma, \mu) : |f_n(x) - f(x)| \geq \epsilon \})$ converge to zero as $n \rightarrow \infty$.

According to Christos Papachristodoulous [1], each pair ($\{f_n\}$, f) induces a double sequence of measurable sets $A_n(\{f_n\},f)$ or simply A_n, $n \in N$ where $A_n = \{ x \in (\mathbb{R}^n, \mathcal{T}, \Sigma, \mu) : |f_n(x) - f(x)| \geq \epsilon \}$, determining the behavior of the pair ($\{f_n\}$, f) with respect to convergence. More precisely, some results are as follows:

(1) $f_n \xrightarrow{\mu} f \Leftrightarrow$ for each $j \in N$, $\lim_{n\rightarrow\infty} \mu(A_n) = 0$.

(2) $\{\{ x \in (\mathbb{R}^n, \mathcal{T}, \Sigma, \mu) : f_n(x) \nrightarrow f(x) \} = \bigcup_{j=1}^{\infty}(\bigcap_{n=1}^{\infty} E_n)$, where

$E_n = \bigcup_{k=n}^{\infty} A_n = \{x \in (\mathbb{R}^n, \mathcal{T}, \Sigma, \mu) : \exists\, k \geq n : |f_k(x) - f(x)| \geq \epsilon\}$

(3) $f_n \xrightarrow{\mu-ae} f \Leftrightarrow$ for each $j \in N$, $\lim_{n\rightarrow\infty} \mu(\bigcap_{n=1}^{\infty} E_n) = 0$.

If $\{f_n\}$ is a sequence of measurable functions on ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ) converging to f point wise almost everywhere on ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ), then the ordered pair ($\{f_n\}$, f) induces a Borel subset A_n satisfying the following conditions:
$A_n = \{x \in (\mathbb{R}^n, \mathcal{T}, \Sigma, \mu) : | f_n(x) - f(x) | < \epsilon \} \forall n \in N$, where,
(i) $\mu(A_n) > 0$, if $| f_n(x) - f(x) | < \epsilon$, $\forall n \in N$,
(ii) $\mu(A_n) = 0$, if $| f_n(x) - f(x) | \geq \epsilon$, $\forall n \geq N$,
that is, $\mu(A_n) = 0$ as $n \to \infty$.

Definition 3.1: Dark point of ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ)
The point $f(x) \in A_n$ is called as a dark point, if $f_n(x) \nrightarrow f(x)$ in A_n and $\mu(A_n) = 0$.

If ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ) be a measure space and $\{f_n\}$ be a sequence of measurable functions and we say that f_n converges to f a.e. in measure on ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ), and the ordered pair ($\{f_n\}$,f) induces a Borel subsets A_n, satisfying the following conditions:
For any n, $A_n = \{x \in (\mathbb{R}^n, \mathcal{T}, \Sigma, \mu): | f_n(x) - f(x) | < \epsilon\}, \forall n \in N\}$, such that,
(i) $\mu(A_n) > 0$, if $| f_n(x) - f(x) | < \epsilon$, $\forall n \in N$,
(ii) $\mu(A_n) = 0$, if $| f_n(x) - f(x) | \geq \epsilon$, $\forall n \geq N$, that is $\mu(A_n) = 0$ as $n \to \infty$.
Suppose $(A_n) = 0$, then $f_n(x) \nrightarrow f(x)$ on ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ) then $\exists$ $k \geq n \geq N$ such that $f_k(x) \nrightarrow f(x)$, $\forall$ k. This implies, the measure of $E_n = \bigcup_{k=n}^{\infty} A_k$ is zero, $\forall$ $k \geq n$, and the region E_n in ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ) is designated as a dark region. To study this dark region where $\mu(E_n) = 0$, we consider the intersection of E_n whose measure can also be zero.
That is, we undertake the study of $\mu(E_n)$ and $\mu(\bigcap_{n=1}^{\infty} E_n)$.
If $E_n = \bigcup_{k=n}^{\infty} A_k = \{ x \in (\mathbb{R}^n, \mathcal{T}, \Sigma, \mu) : \exists\ k \geq n \geq N : | f_k(x) - f(x) | \geq \epsilon\}$ with $\mu(E_n) = 0$ and $\bigcap_{n=1}^{\infty} E_n = \bigcap_{n=1}^{\infty} (\bigcup_{k=n}^{\infty} A_k)$, $\mu(\bigcap_{n=1}^{\infty} E_n) = 0$.

Definition 3.2:
The Borel set $\bigcap_{n=1}^{\infty} E_n$ is called as the dark region of ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ) if,
$\mu(\bigcap_{n=1}^{\infty} E_n) = 0$.
Now, we shall extend the study of above modes of convergence of ($\mathbb{R}^n$,$\mathcal{T}$,Σ,μ) to a measure manifold (M, $\mathcal{T}_1$, Σ_1, μ_1) which is measurable homeomorphic to ($\mathbb{R}^n$,$\mathcal{T}$,Σ,μ) that is ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ) is a complete measure space.

Definition 3.3: Convergence point wise almost everywhere on (M,$\mathcal{T}_1$,Σ_1,μ_1)
Let $f_n \to f$ point wise almost everywhere in ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ) and if any measure manifold (M, $\mathcal{T}_1$, Σ_1, μ_1) is measurable homeomorphic to ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ) then for every $x \in A_n \in$ ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ) $\exists$ $\phi^{-1}(x) = p \in \phi^{-1}(A_n)$ denoted by,
$S = \phi^{-1}(A_n) \in (U, \phi) \in (M,\mathcal{T}_1,\Sigma_1, \mu_1)$ such that $(f_n \circ \phi) \to f \circ \phi$ point wise a.e. in $S \in (U, \phi) \in (M, \mathcal{T}_1, \Sigma_1, \mu_1)$ such that
$S = \phi^{-1}(A_n) = \{\phi^{-1}(x) = p \in (M,\mathcal{T}_1,\Sigma_1,\mu_1): | (f_n \circ \phi)(p) - (f \circ \phi)(p)| < \epsilon, \forall n \in N\}$ on the chart (U, ϕ) satisfying the following conditions:
(i) $\mu_1(S) > 0$, if $| (f_n \circ \phi)(p) - (f \circ \phi)(p) | < \epsilon$, $\forall n \in N$,
(ii) $\mu_1(S) = 0$, if $| (f_n \circ \phi)(p) - (f \circ \phi)(p) | \geq \epsilon$, $\forall n \geq N$,
that is, $\mu_1(S) = 0$ as $n \to \infty$.

Note: Now onwards we denote the Borel subset $\phi^{-1}(A_n)$ of (U, ϕ) by S.

Definition 3.4:
The point $(f \circ \phi)(p) \in S \in (U, \phi)$ is called as a dark point in the chart (U, ϕ) if
$(f_n \circ \phi)(p) \nrightarrow (f \circ \phi)(p)$ in S and $\mu_1(S) = 0$.

Definition 3.5: Convergence μ_1-a.e. on (M, $\mathcal{T}_1$, Σ_1, μ_1)
Let $f_n \xrightarrow{\mu_1-a.e} f$ in $A_n \in$ ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ) and if any measure manifold (M,$\mathcal{T}_1$,Σ_1,μ_1) is measurable homeomorphic to ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ) then for every $A_n \in$ ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ) $\exists$ $\phi^{-1}(A_n) = S \in (U,\phi) \in (M,\mathcal{T}_1,\Sigma_1,\mu_1)$ such that,
$(f_n \circ \phi) \xrightarrow{\mu_1-a.e} (f \circ \phi)$, $\forall$ x on $S \in (U,\phi) \in (M, \mathcal{T}_1, \Sigma_1, \mu_1)$ and
$S = \{ p \in (M, \mathcal{T}_1, \Sigma_1, \mu_1) : | (f_n \circ \phi)(p) - (f \circ \phi)(p) | < \epsilon, \forall n \in N\}$ with
(i) $\mu_1(S) > 0$, if $| (f_n \circ \phi)(p) - (f \circ \phi)(p) | < \epsilon$, $\forall n \in N$,
(ii) $\mu_1(S) = 0$, if $| (f_n \circ \phi)(p) - (f \circ \phi)(p) | \geq \epsilon$, $\forall n \geq N$,that is, $\mu_1(S) = 0$.

Definition 3.6: Complete Measure Manifold
If (M,$\mathcal{T}_1$,Σ_1,μ_1) is a measure manifold of dimension n and suppose that for every measure chart $(U,\phi) \subseteq (M,\mathcal{T}_1,\Sigma_1,\mu_1)$, $\mu_1(U)=0$ and every $V \subset (U,\phi)$, if $\mu_1(V) = 0$, then (M,$\mathcal{T}_1$,Σ_1,μ_1) is called as a complete measure manifold.

4. DIFFERENT ASPECTS OF CONNECTEDNESS μ_1-a.e. PROPERTY ON COMPLETE MEASURE MANIFOLD

In this section, we study the concept of connectedness μ_1-a.e. like locally path connectedness μ_1-a.e., interconnected μ_1-a.e. and maximally path connected μ_1-a.e. on complete measure manifold introduced by S. C. P. Halakatti ([7],[8],[9],[10],[11]).
Let (M, $\mathcal{T}_1$, Σ_1, μ_1) be a complete measure manifold of dimension n which is measurable homeomorphic to a measure space ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ). Let $\{f_n\}$, $\{g_n\}$ be measurable real valued functions converging to f and g respectively in ($\mathbb{R}^n$, $\mathcal{T}$,Σ,μ).
Since ϕ is measurable homeomorphism from (M, $\mathcal{T}_1$, Σ_1, μ_1) to ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ) for every $\{f_n\}$ and $\{g_n\}$ on ($\mathbb{R}^n$, $\mathcal{T}$, Σ, μ) there exist corresponding measurable real valued functions $\{f_n \circ \phi\}$ and $\{g_n \circ \phi\}$ converging to $f \circ \phi$ and $g \circ \phi$ on (M, $\mathcal{T}_1$, Σ_1, μ_1).
The ordered pair ($\{f_n \circ \phi\}$, $f \circ \phi$) induces a Borel subset $S \in (U, \phi) \in (M, \mathcal{T}_1, \Sigma_1, \mu_1)$ satisfying the following condition:
$S = \{p \in (M, \mathcal{T}_1, \Sigma_1, \mu_1): | (f_n \circ \phi)(p) - (f \circ \phi)(p) | < \epsilon, \forall n \in N\}$ on the chart (U, ϕ) for which $\mu_1(S) > 0$.

Definition 4.1: Locally path connected μ_1 – a.e. on complete measure manifold

The Borel subset S is locally path connected μ_1 –a.e. if ∃ a C^∞- map $\gamma : [0, 1] \to S \in (U, \phi)$ such that

$\gamma(0) = p \in S$,

$\gamma(1) = q \in S$, such that $\mu_1(S) > 0$.

That is, p is locally path connected μ_1 –a.e. to q in $S \subset (U, \phi) \in \mathbb{A} \in A^k(M)$.

That is, locally path connectedness μ_1 –a.e. is between two points in the same chart $(U,\phi) \in \mathbb{A} \in A^k(M)$.

If $\mu_1(S) = 0$, then there does not exist a path γ between p and q.

Definition 4.2:

If $\mu_1(S) = 0$ where $(f_n \circ \phi) \nrightarrow f \circ \phi$, then $S \subset (U, \phi) \subset (M, \mathcal{T}_1, \Sigma_1, \mu_1)$ is called as a dark region in the chart (U, ϕ).

Let $(M, \mathcal{T}_1,\Sigma_1,\mu_1)$ be a complete measure manifold on which $\{f_n \circ \phi\}$ and $\{g_n \circ \phi\}$ are sequence of real valued measurable functions converging to $f \circ \phi$ and $g \circ \phi$ pointwise μ_1 – a.e. on (U, ϕ) and (V, ψ) belonging to the atlas $\mathbb{A}$ respectively. The ordered pairs $(\{f_n \circ \phi\}, f \circ \phi)$ and $(\{g_n \circ \phi\}, g \circ \phi)$ induce two Borel subsets $S \in (U, \phi) \in \mathbb{A} \in A^k(M)$ and $R \in (V, \psi) \in \mathbb{A} \in A^k(M)$ satisfying the following condition:

$S = \{p \in (M, \mathcal{T}_1, \Sigma_1, \mu_1): | (f_n \circ \phi)(p) - (f \circ \phi)(p) | < \epsilon, \forall n \in N\}$ on the chart (U, ϕ) for which $\mu_1(S) > 0$.

$R = \{q \in (M, \mathcal{T}_1, \Sigma_1, \mu_1): | (g_n \circ \phi)(q) - (g \circ \phi)(q) | < \epsilon, \forall n \in N\}$ on the chart (V, ψ) for which $\mu_1(R) > 0$.

Note: We denote the Borel subsets $\phi^{-1}(A_n) = S \in (U,\phi) \in (M,\mathcal{T}_1,\Sigma_1,\mu_1)$ and $\phi^{-1}(B_n) = R \in (V, \psi) \in (M, \mathcal{T}_1, \Sigma_1, \mu_1)$.

Definition 4.3: Interconnected μ_1 – a.e on complete measure manifold

The Borel subset $S \in (U, \phi) \in \mathbb{A} \in (M, \mathcal{T}_1, \Sigma_1, \mu_1)$ is interconnected to the Borel subset $R \in (V, \psi) \in \mathbb{A} \in (M, \mathcal{T}_1, \Sigma_1, \mu_1)$ μ_1 –a.e. if ∃ a C^∞- map $\gamma : [0, 1] \to S \cup R \in \mathbb{A} \in A^k(M)$ such that

$\gamma(0) = p \in S \in \mathbb{A}$,

$\gamma(1) = q \in S \in \mathbb{A}$, such that $\mu_1(S) > 0$ and $\mu_1(R) > 0$.

That is, p is interconnected μ_1 – a.e. to q in $S \cup R \in \mathbb{A} \in A^k(M)$.

That is, interconnectedness μ_1 – a.e. is between two charts in the same atlas $\in A^k(M)$.

If $\mu_1(S) = 0$ and $\mu_1(R) = 0$ then ∄ a path between p and q.

Definition 4.4:

If $\mu_1(S)=0$ where $\{f_n \circ \phi\} \nrightarrow f \circ \phi$ in (U,ϕ) and $\mu_1(R)=0$ where $\{g_n \circ \phi\} \nrightarrow g \circ \phi$ in (V,ψ), then S is called as dark region in the chart (U, ϕ) and R is called as dark region in the chart (V, ψ) belonging to the same atlas $\mathbb{A}$ in $A^k(M)$.

Let $(\mathbb{R}^n, \mathcal{T}, \Sigma, \mu)$ be a measure space and $\{f_n\}$, $\{g_n\}$and $\{h_n\}$are sequences of measurable functions on $(\mathbb{R}^n, \mathcal{T}, \Sigma, \mu)$ converging to f, g and h point wise almost everywhere on $(\mathbb{R}^n, \mathcal{T}, \Sigma, \mu)$. The ordered pairs $(\{f_n\},f)$, $(\{g_n\}, g)$ and $(\{h_n\}, h)$ induce the following Borel subsets A_n, B_n and C_n. We define Borel subsets

$A_n = \{ x \in (\mathbb{R}^n, \mathcal{T}, \Sigma, \mu) : | f_n(x) - f(x) | < \epsilon \}, \forall n \in N$, where

(i) $\mu(A_n) > 0$, if $|f_n(x) - f(x)| < \epsilon$, $\forall n \in N$,

(ii) $\mu(A_n) = 0$, if $|f_n(x) - f(x)| \geq \epsilon$, $\forall n \geq N$,

that is, $\mu(A_n) = 0$ as $n \to \infty$.

Similarly,

(i)$B_n = \{ y \in (\mathbb{R}^n, \mathcal{T}, \Sigma, \mu) : | g_n(y) - g(y) | < \epsilon \}, \forall n \in N$, where

$\mu(B_n) > 0$, if $| g_n(y) - g(y) | < \epsilon, \forall n \in N$,

(ii)$\mu(B_n) = 0$, if $| g_n(y) - g(y) | \geq \epsilon, \forall n \geq N$,

that is, $\mu(B_n) = 0$ as $n \to \infty$ and

$C_n = \{ z \in (\mathbb{R}^n, \mathcal{T}, \Sigma, \mu) : | h_n(z) - h(z) | < \epsilon \}, \forall n \in N$, where

(i) $\mu(C_n) > 0$, if $| h_n(z) - h(z) | < \epsilon$, $\forall n \in N$,

(ii) $\mu(C_n) = 0$, if $| h_n(z) - h(z) | \geq \epsilon$, $\forall n \geq N$,

that is, $\mu(C_n) = 0$ as $n \to \infty$.

If $(M, \mathcal{T}_1, \Sigma_1, \mu_1)$ is a complete measure manifold that is measurable homeomorphic and measure invariant to $(\mathbb{R}^n, \mathcal{T}, \Sigma, \mu)$. Then ∃ a measurable homeomorphism and measure invariant transformation

$\phi: (M, \mathcal{T}_1, \Sigma_1, \mu_1) \to (\mathbb{R}^n, \mathcal{T}, \Sigma, \mu)$, such that, $\{f_n \circ \phi\}$, $\{g_n \circ \phi\}$ and $\{h_n \circ \phi\}$ are sequences of real valued measurable functions converging to $f \circ \phi$, $g \circ \phi$ and $h \circ \phi$ point wise μ_1-a.e. on (U,ϕ), (V, ψ) and (W, χ) belonging to the atlases $\mathbb{A}_i$, $\mathbb{A}_j$, $\mathbb{A}_l$ respectively. Also, for every induced Borel subsets A_n, B_n and C_n in $(\mathbb{R}^n, \mathcal{T}, \Sigma, \mu)$, ∃ the corresponding induced Borel subsets namely

$S \in (U, \phi) \in \mathbb{A}_i \in A^k(M)$, $R \in (V, \psi) \in \mathbb{A}_j \in A^k(M)$ and $Q \in (W, \chi) \in \mathbb{A}_l \in A^k(M)$ on $(M, \mathcal{T}_1, \Sigma_1, \mu_1)$.

Now, we define S, R and Q as follows:

$S = \{p \in (M, \mathcal{T}_1, \Sigma_1, \mu_1): | (f_n \circ \phi)(p) - (f \circ \phi)(p) | < \epsilon, \forall n \in N\}$ on the chart $(U, \phi) \in \mathbb{A}_i \in A^k(M)$, for which $\mu_1(S) > 0$,

$R = \{q \in (M, \mathcal{T}_1, \Sigma_1, \mu_1): | (g_n \circ \phi)(q) - (g \circ \phi)(q) | < \epsilon, \forall n \in N\}$ on the chart $(V, \psi) \in \mathbb{A}_j \in A^k(M)$, for which $\mu_1(R) > 0$ and

$Q = \{r \in (M, \mathcal{T}_1, \Sigma_1, \mu_1): | (h_n \circ \phi)(r) - (h \circ \phi)(r) | < \epsilon, \forall n \in N\}$ on the chart $(W, \chi) \in \mathbb{A}_l \in A^k(M)$, for which $\mu_1(Q) > 0$.

Note: We denote the Borel subsets $\phi^{-1}(A_n) = S \in (U, \phi) \in \mathbb{A}_i \in A^k(M)$, $\phi^{-1}(B_n) = R \in (V, \psi) \in \mathbb{A}_j \in A^k(M)$ and $\phi^{-1}(C_n) = Q \in (W, \chi) \in \mathbb{A}_l \in A^k(M)$.

Definition 4.5: Maximal connected μ_1-a.e on complete measure manifold

Let $(M, \mathcal{T}_1, \Sigma_1, \mu_1)$ be a complete measure manifold and let $\mathbb{A}_i$, $\mathbb{A}_j$ and $\mathbb{A}_l \in A^k(M)$ be atlases on $(M, \mathcal{T}_1, \Sigma_1, \mu_1)$. Let S, R and Q be Borel subsets of $\mathbb{A}_i$, $\mathbb{A}_j$ and $\mathbb{A}_l$. Then, we say that $A^k(M) \in (M, \mathcal{T}_1, \Sigma_1, \mu_1)$ is maximally connected if ∃ a map $\gamma:[0,1] \to S \cup R \cup Q \in \mathbb{A}_i \cup \mathbb{A}_j \cup \mathbb{A}_l \in A^k(M)$ such that,

$\gamma(0) = p \in S \in (U, \phi) \in \mathbb{A}_i \in A^k(M)$ for which $\mu_1(S) > 0$,

$\gamma(\frac{1}{2}) = q \in R \in (V, \psi) \in \mathbb{A}_j \in A^k(M)$ for which $\mu_1(R) > 0$ and

$\gamma(1) = r \in Q \in (W, \chi) \in \mathbb{A}_l \in A^k(M)$ for which $\mu_1(Q) > 0$.

That is, for each $p \in (U, \phi) \in \mathbb{A}_i$ is path connected to each $q \in (V, \psi) \in \mathbb{A}_j$, for $\mathbb{A}_i \cup \mathbb{A}_j \in A^k(M)$, $\mu_1(\mathbb{A}_i \cup \mathbb{A}_j) > 0$ for each $q \in (V, \psi) \in \mathbb{A}_j$ is path connected to each $r \in (W, \chi) \in \mathbb{A}_l \in A^k(M)$ and for $\mathbb{A}_j \cup \mathbb{A}_l \in A^k(M)$, $\mu_1(\mathbb{A}_j \cup \mathbb{A}_l) > 0$. Then, if for each $p \in (U, \phi) \in \mathbb{A}_i$ is path connected to each r ∈ (W,

$\chi) \in \mathbb{A}_l \in A^k(M)$ and for $\mathbb{A}_i \cup \mathbb{A}_l \in A^k(M)$, $\mu_1(\mathbb{A}_i \cup \mathbb{A}_l) > 0$ then $(\mathbb{A}_i \cup \mathbb{A}_j \cup \mathbb{A}_l) \in A^k(M) \in (M, \mathcal{T}_1, \Sigma_1, \mu_1)$ is maximally path connected if $\mu_1(\mathbb{A}_i \cup \mathbb{A}_j \cup \mathbb{A}_l) > 0$ on complete measure manifold.

If $\mu_1(S) = 0$ and $\mu_1(R) = 0$ then there does not exist a path γ between $p \in S$ and $q \in R$.

Definition 4.6:

If $\mu_1(S) = 0$ where $\{f_n \circ \phi\} \nrightarrow f \circ \phi$ in (U, ϕ) and $\mu_1(R)=0$ where $\{g_n \circ \phi\} \nrightarrow g \circ \phi$ in (V, ψ) and $\mu_1(Q) = 0$ where $\{h_n \circ \phi\} \nrightarrow h \circ \phi$ in (W, χ), then S is called as dark region in the chart $(U, \phi) \in \mathbb{A}_i$, R is called as dark region in the chart $(V, \psi) \in \mathbb{A}_j$ and Q is called as dark region in the chart $(W, \chi) \in \mathbb{A}_l$ in $A^k(M)$.

Now, we show that locally path connectedness is invariant with respect to measurable homeomorphism and measure invariant on a complete measure manifold if $\mu_1(S) > 0$.

Theorem 4.7: Let $(M_1, \mathcal{T}_1, \Sigma_1, \mu_1)$ and $(M_2, \mathcal{T}_2, \Sigma_2, \mu_2)$ be complete measure manifolds of dimension n and m respectively. If M_1 is locally path connected μ_1-a.e in $S \subset M_1$, $\mu_1(S) > 0$, then $\exists$ a measurable homeomorphism and measure invariant map $F: M_1 \longrightarrow M_2$ such that M_2 is also locally path connected μ_2-a.e in $F(S) \subset M_2$ with $\mu_2(F(S)) > 0$.

Proof: Let $(M_1, \mathcal{T}_1, \Sigma_1, \mu_1)$ be a complete measure manifold and it is locally path connected. Let (U_1, ϕ_1) be a measure chart in $(M_1, \mathcal{T}_1, \Sigma_1, \mu_1)$ that is, $S \subset M_1$ and $\mu_1(S) > 0$.

Consider a measurable homeomorphism and measure invariant map

$F: (M_1, \mathcal{T}_1, \Sigma_1, \mu_1) \longrightarrow (M_2, \mathcal{T}_2, \Sigma_2, \mu_2)$ defined for $S \subset M_1$, $F(S) \subset M_2$.

To prove that $(M_2, \mathcal{T}_2, \Sigma_2, \mu_2)$ is locally path connected, let us consider

$S = \{p \in (M_1, \mathcal{T}_1, \Sigma_1, \mu_1): |(f_n \circ \phi)(p) - (f \circ \phi)(p)| < \epsilon, \forall n \in N\}$ on the chart (U, ϕ) for which $\mu_1(S) > 0$.

Now, for every $S \in (M_1, \mathcal{T}_1, \Sigma_1, \mu_1)$, $F(S) \in (M_2, \mathcal{T}_2, \Sigma_2, \mu_2)$: $|F(f_n \circ \phi)(p) - F(f \circ \phi)(p)| < \epsilon, \forall n \in N\}$ for which $\mu_2(F(S)) > 0$.

Let (U, ϕ), (V, ψ) be charts in M_1 and M_2 respectively and $F(p_1), F(p_2) \in F(S) \in (M_2, \mathcal{T}_2, \Sigma_2, \mu_2)$.

Since F is measurable homeomorphism and measure invariant map, there exists F^{-1} such that for every measure chart $(V, \psi) \in (M_2, \mathcal{T}_2, \Sigma_2, \mu_2)$ such that $\mu_2(F(S)) > 0$.

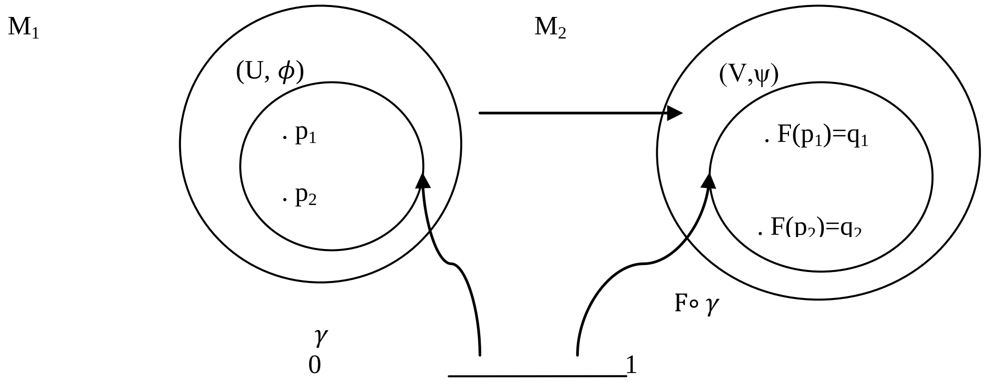

Fig. 1

That is, for every $F(p_1), F(p_2) \in (V, \psi) \in (M_2, \mathcal{T}_2, \Sigma_2, \mu_2)$, there exists,

$F^{-1}(F(p_1)) = p_1 \in (U, \phi) \in (M_1, \mathcal{T}_1, \Sigma_1, \mu_1)$.

$F^{-1}(F(p_2)) = p_2 \in (U, \phi) \in (M_1, \mathcal{T}_1, \Sigma_1, \mu_1)$.

But $(M_1, \mathcal{T}_1, \Sigma_1, \mu_1)$ is locally path connected

Therefore, $\exists$ a C^∞ map $\gamma : [0,1] \longrightarrow (M_1, \mathcal{T}_1, \Sigma_1, \mu_1)$ such that

$\gamma(0) = p_1 \in S \in (U, \phi)$, $\mu_1(S) > 0$

$\gamma(1) = p_2 \in S \in (U, \phi)$, $\mu_1(S) > 0$.

If $\mu_1(S) = 0$ then p_1 is not locally path connected to p_2.

Now, since F is homeomorphism $\exists$ a map $F \circ \gamma : [0, 1] \longrightarrow (M_2, \mathcal{T}_2, \Sigma_2, \mu_2)$

such that, $F \circ \gamma(0) = F(p_1) = q_1 \in F(S) \in (V, \psi)$, $\mu_2(F(S)) > 0$.

$F \circ \gamma(1) = F(p_2) = q_2 \in F(S) \in (V, \psi)$, $\mu_2(F(S)) > 0$.

If $\mu_2(F(S)) = 0$ then q_1 is not maximally path connected to q_2.

Therefore, q_1 is locally path connected μ_1-a.e to q_2 by $F \circ \gamma$ in $F(S) \in (V, \psi) \in (M_2, \mathcal{T}_2, \Sigma_2, \mu_2)$. If q_1 is locally path connected μ_1-a.e. to q_2 by $F \circ \gamma$ in $F(S) \in (V,\psi) \in (M_2, \mathcal{T}_2, \Sigma_2, \mu_2)$ then $(M_2, \mathcal{T}_2, \Sigma_2, \mu_2)$ is locally path connected.

Therefore, if $(M_1, \mathcal{T}_1, \Sigma_1, \mu_1)$ is locally path connected μ_1-a.e. then $(M_2, \mathcal{T}_2, \Sigma_2, \mu_2)$ is also locally path connected μ_2-a.e.

Hence, local path connectedness is invariant with respect to measurable homeomorphism and measure invariant map, if $\mu_1(S) > 0$, $\mu_2(F(S)) > 0$. ■

Now, we show that inter connectedness is invariant with respect to measurable homeomorphism and measure invariant on a complete measure manifold if $\mu_1(S) > 0$ and $\mu_1(R) > 0$.

Theorem 4.8:

Let $(M_1, \mathcal{T}_1, \Sigma_1, \mu_1)$ and $(M_2, \mathcal{T}_2, \Sigma_2, \mu_2)$ be complete measure manifolds of dimension n and m respectively. If $(M_1, \mathcal{T}_1, \Sigma_1, \mu_1)$ is interconnected μ_1-a.e. in $S \cup R \in \mathbb{A} \in$

$A^k(M)$ with $\mu_1(S \cup R)>0$ then ∃ a measurable homeomorphism and measure invariant map $F:(M_1,\mathcal{T}_1,\Sigma_1,\mu_1)\longrightarrow (M_2,\mathcal{T}_2,\Sigma_2,\mu_2)$ such that M_2 is also interconnected
μ_1-a.e. in $F(S \cup R) \in \mathbb{B} \in A^k(M)$ with μ_2 $F(S \cup R) > 0$.
Proof: Let $(M_1, \mathcal{T}_1, \Sigma_1, \mu_1)$ and $(M_2, \mathcal{T}_2, \Sigma_2, \mu_2)$ be complete measure manifolds of dimension n and m respectively.
Suppose $(M_1, \mathcal{T}_1, \Sigma_1, \mu_1)$ is interconnected μ_1-a.e. in $\mathbb{A} \subset (M_1, \mathcal{T}_1, \Sigma_1, \mu_1)$ with $\mu_1(S) > 0$, $\mu_1(R) > 0$.
Consider a measurable homeomorphism and measure invariant map
$F: (M_1, \mathcal{T}_1, \Sigma_1, \mu_1) \longrightarrow (M_2, \mathcal{T}_2, \Sigma_2, \mu_2)$ defined as follows:
for any two Borel subsets
$S = \{p_1 \in (M_1, \mathcal{T}_1, \Sigma_1, \mu_1): |(f_n \circ \phi)(p_1) - (f \circ \phi)(p_1)| < \epsilon, \forall n \in N\}$ on the chart (U_1, ϕ_1) for which $\mu_1(S) > 0$
and $R = \{p_2 \in (M_1, \mathcal{T}_1, \Sigma_1, \mu_1): |(f_n \circ \phi)(p_2) - (f \circ \phi)(p_2)| < \epsilon, \forall n \in N\}$ on the chart (U_2, ϕ_2) for which $\mu_1(R) > 0$.

If $\mu_1(S) = 0$ and $\mu_1(R) = 0$ then p_1 is not interconnected to p_2. There exist
$F(S) = \{q_1 \in (M_2, \mathcal{T}_2, \Sigma_2, \mu_2) : |F(f_n \circ \phi)(q_1) - F(f \circ \phi)(q_1)| < \epsilon, \forall n \in N\}$ on the chart (V_1, ψ_1) for which $\mu_2(F(S)) > 0$ and
$F(R) = \{q_2 \in (M_2, \mathcal{T}_2, \Sigma_2, \mu_2) : |F(f_n \circ \phi)(q_2) - F(f \circ \phi)(q_2)| < \epsilon, \forall n \in N\}$ on the chart (V_2, ψ_2) for which $\mu_2(F(R)) > 0$.
As $(M_1, \mathcal{T}_1, \Sigma_1, \mu_1)$ is interconnected μ_1-a.e then ∃ a C^∞-map $\gamma: [0,1]\rightarrow \mathbb{A}$ such that, for every $S, R \in (M_1,\mathcal{T}_1, \Sigma_1,\mu_1)$ ∃ $F(S), F(R)$ as ascribed in $(M_2,\mathcal{T}_2,\Sigma_2, \mu_2)$ such that ∃ $F\circ\gamma: [0,1] \rightarrow (M_2, \mathcal{T}_2, \Sigma_2, \mu_2)$ such that
$F\circ\gamma(0) = q_1 \in (V_1, \psi_1) \in \mathbb{B} \in (M_2, \mathcal{T}_2, \Sigma_2, \mu_2)$, $\mu_2(F(S)) > 0$,
$F\circ\gamma(1) = q_2 \in (V_2, \psi_2) \in \mathbb{B} \in (M_2, \mathcal{T}_2, \Sigma_2, \mu_2)$, $\mu_2(F(R)) > 0$.
This implies q_1 is interconnected to q_2 in $(M_2, \mathcal{T}_2, \Sigma_2, \mu_2)$.

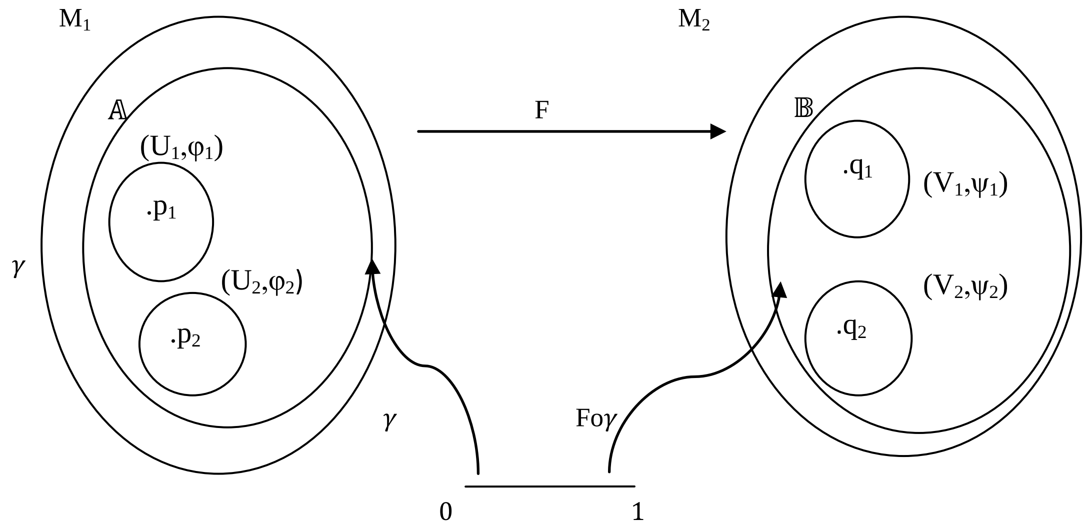

Fig 2

If $\mu_2(F(S)) = 0$ and $\mu_2(F(R)) = 0$ then q_1 is not interconnected to q_2.
Therefore, if $(M_1, \mathcal{T}_1, \Sigma_1, \mu_1)$ is interconnected μ_1-a.e. then $(M_2, \mathcal{T}_2,\Sigma_2, \mu_2)$ is interconnected μ_1-a.e.
Hence, interconnectedness is invariant with respect to measurable homeomorphism and measure invariant transformation if
$\mu_1(S) > 0$, $\mu_1(R) > 0$ and $\mu_2(F(S)) > 0$, $\mu_2(F(R)) > 0$. ■

Now, we show that maximal path connectedness is invariant with respect to measurable homeomorphism and measure invariant on complete measure manifold if $\mu_1(S) > 0$ and $\mu_1(R) > 0$.
Theorem 4.9:
Let $(M_1, \mathcal{T}_1, \Sigma_1, \mu_1)$ and $(M_2, \mathcal{T}_2, \Sigma_2, \mu_2)$ be complete measure manifolds of dim n and m respectively. If $(M_1, \mathcal{T}_1, \Sigma_1, \mu_1)$ is maximally path connected μ_1 - a.e. in $\mathbb{A}_i \cup \mathbb{A}_j \cup \mathbb{A}_l$ of $A^k(M_1)$ such that $\mu_1(S)> 0$, $\mu_1(R)>0$ and $\mu_1(Q)>0$ and if ∃ a measurable homeomorphism and measure invariant map $F: M_1 \longrightarrow M_2$ then M_2 is also maximally path connected μ_2-a.e. in $\mathbb{B}_i \cup \mathbb{B}_j \cup \mathbb{B}_l$ of $A^k(M_2)$ such that $\mu_2(F(S)) > 0$, $\mu_2(F(R)) > 0$ and $\mu_2(F(Q)) > 0$

Proof: Let $(M_1, \mathcal{T}_1, \Sigma_1, \mu_1)$ and $(M_2, \mathcal{T}_2, \Sigma_2, \mu_2)$ be complete measure manifolds of dimension n and m respectively.
Suppose $(M_1, \mathcal{T}_1, \Sigma_1, \mu_1)$ is maximally path connected μ_1-a.e. in $\mathbb{A}_i \cup \mathbb{A}_j \cup \mathbb{A}_l \in A^k(M_1)$ with $\mathbb{A}_i, \mathbb{A}_j, \mathbb{A}_l \in A^k(M_1)$: $\mu_1(S) > 0$, $\mu_1(R) > 0$ and $\mu_1(Q)> 0$.
Consider a measurable homeomorphism $F:(M_1,\mathcal{T}_1,\Sigma_1,\mu_1)\longrightarrow(M_2,\mathcal{T}_2,\Sigma_2,\mu_2)$. Since $(M_1, \mathcal{T}_1,\Sigma_1,\mu_1)$ is maximally path connected μ_1-a.e. in $\mathbb{A}_i \cup \mathbb{A}_j \cup \mathbb{A}_l \in A^k(M)$, then ∃ three Borel subsets S, R and Q such that,
$S = \{p_1 \in (M_1, \mathcal{T}_1, \Sigma_1, \mu_1): |(f_n \circ \phi)(p_1) - (f \circ \phi)(p_1)| < \epsilon, \forall n \in N\}$ on the chart $(U_1, \phi_1) \in \mathbb{A}_i \in A^k(M)$, for which $\mu_1(S) > 0$ and
$R = \{p_2 \in (M_1, \mathcal{T}_1, \Sigma_1, \mu_1): |(g_n \circ \phi)(p_2) - (g \circ \phi)(p_2)| < \epsilon, \forall n \in N\}$ on the chart $(U_2, \phi_2) \in \mathbb{A}_j \in A^k(M)$, for which $\mu_1(R) > 0$ and
$Q = \{p_3 \in (M_1, \mathcal{T}_1, \Sigma_1, \mu_1): |(h_n \circ \phi)(p_3) - (h \circ \phi)(p_3)| < \epsilon, \forall n \in N\}$ on the chart $(U_3, \phi_3) \in \mathbb{A}_l \in A^k(M)$, for which $\mu_1(Q) > 0$.
Then, there exist a path $\gamma:[0,1]\rightarrow S \cup R \cup Q \in \mathbb{A}_i \cup \mathbb{A}_j \cup \mathbb{A}_l \in A^k(M)$
in a complete measure manifold $(M_1, \mathcal{T}_1, \Sigma_1, \mu_1)$ such that
$\gamma(0) = p_1 \in S \in (U_1, \phi_1) \in \mathbb{A}_i \in A^k(M)$ for which $\mu_1(S) > 0$,

$\gamma(\frac{1}{2}) = p_2 \in R \in (U_2, \phi_2) \in \mathbb{A}_j \in A^k(M)$ for which $\mu_1(R) > 0$ and
$\gamma(1) = p_3 \in Q \in (U_3, \phi_3) \in \mathbb{A}_l \in A^k(M)$ for which $\mu_1(Q) > 0$.
where each $p_1 \in S \in (U_1, \phi_1) \in \mathbb{A}_i$ is maximally path connected to each $p_2 \in R \in (U_2, \phi_2) \in \mathbb{A}_j$ and each $p_2 \in R \in (U_2, \phi_2) \in \mathbb{A}_j$ is maximally path connected to each $p_3 \in Q \in (U_3, \phi_3) \in \mathbb{A}_l$ in $\mathbb{A}_i \cup \mathbb{A}_j \cup \mathbb{A}_l \in A^k(M)$.
If $\mu_1(S) = 0$, $\mu_1(R) = 0$ and $\mu_1(Q) = 0$ then p_1 is not maximally path connected to p_2 and p_2 is not maximally path connected to p_3.
Since F is measurable homeomorphism and measure invariant, for $p_1 \in S \in (U_1, \phi_1) \in \mathbb{A}_i \in (M_1, \mathcal{T}_1, \Sigma_1, \mu_1)$ there exist $q_1 \in F(S) \in (V_1, \psi_1) \in \mathbb{B}_i \in (M_2, \mathcal{T}_2, \Sigma_2, \mu_2)$ such that,
$F(S) = \{ F(p_1) = q_1 \in (M_2, \mathcal{T}_2, \Sigma_2, \mu_2): | F(f_n \circ \phi)(p_1) - F(f \circ \phi)(p_1) | < \epsilon$,
$\forall n \in N\}$ on the chart $(V_1, \Psi_1) \in \mathbb{B}_i \in A^k(M)$, for which $\mu_2(F(S)) > 0$.
Similarly, for every $p_2 \in R \in (U_2, \phi_2) \in \mathbb{A}_2$ in $(M_1, \mathcal{T}_1, \Sigma_1, \mu_1)$ there exist $q_2 \in F(R) \in (V_2, \psi_2) \in \mathbb{B}_j \in (M_2, \mathcal{T}_2, \Sigma_2, \mu_2)$ such that,
$F(R) = \{ F(p_2) = q_2 \in (M_2, \mathcal{T}_2, \Sigma_2, \mu_2): | F(g_n \circ \phi)(p_2) - F(g \circ \phi)(p_2) | < \epsilon$,
$\forall n \in N\}$ on the chart $(V_2, \Psi_2) \in \mathbb{B}_j \in A^k(M)$, for which $\mu_2(F(R)) > 0$

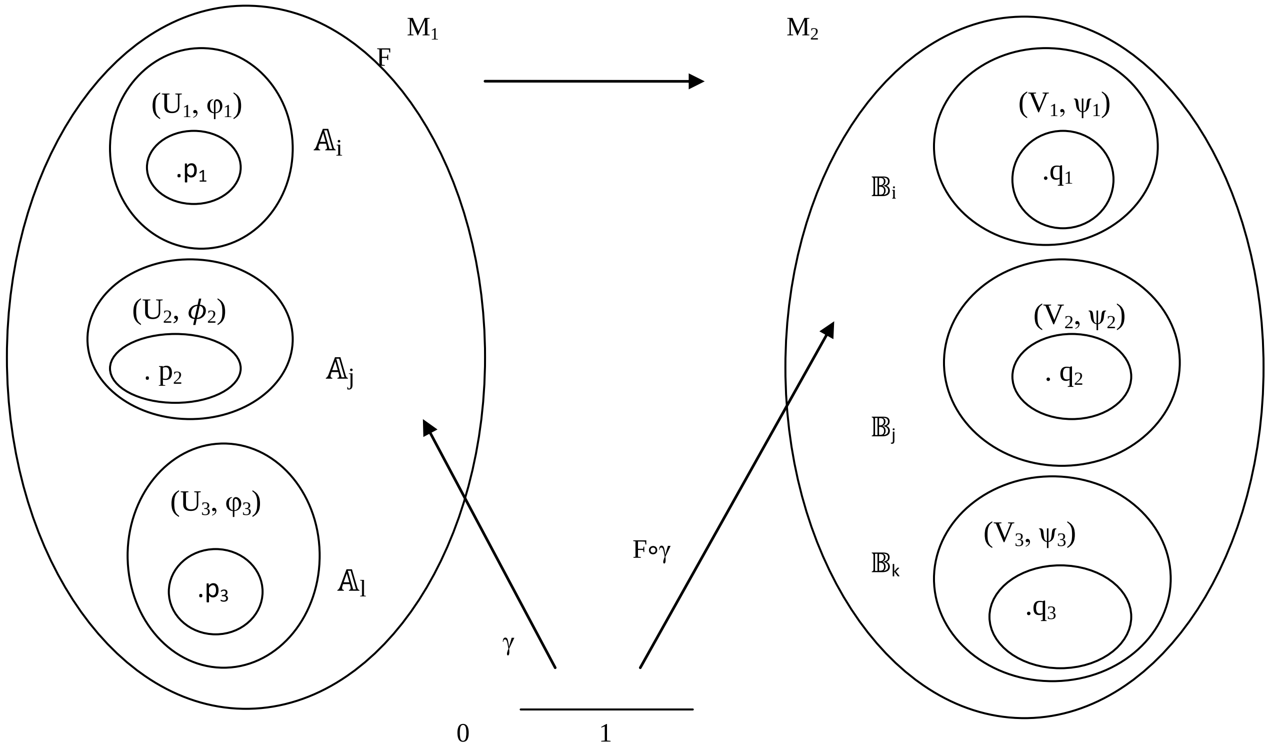

Fig. 3

and similarly, for every $p_3 \in Q \in (U_3, \phi_3) \in \mathbb{A}_j$ in $(M_1, \mathcal{T}_1, \Sigma_1, \mu_1)$ there exist $q_3 \in F(Q) \in (V_3, \psi_3) \in \mathbb{B}_l \in (M_2, \mathcal{T}_2, \Sigma_2, \mu_2)$ such that,
$F(Q) = \{ F(p_3) = q_3 \in (M_2, \mathcal{T}_2, \Sigma_2, \mu_2): |F(h_n \circ \phi)(p_3) - F(h \circ \phi)(p_3)| < \epsilon, \forall n \in N\}$ on the chart $(V_3, \Psi_3) \in \mathbb{B}_l \in A^k(M)$, for which $\mu_2(F(Q)) > 0$.
Since F is measurable homeomorphism and measure invariant, for every $\gamma:[0,1] \to S \cup R \cup Q \in \mathbb{A}_i \cup \mathbb{A}_j \cup \mathbb{A}_l \in A^k(M)$ in $(M_1, \mathcal{T}_1, \Sigma_1, \mu_1)$ which connects p_1, p_2 and p_3 maximally, there exist a corresponding path $F\circ\gamma: [0, 1] \to F(S) \cup F(R) \cup F(Q)$ in $(M_2, \mathcal{T}_2, \Sigma_2, \mu_2)$ which connects maximally $F(p_1) = q_1$ to $F(p_2) = q_2$ and $F(p_2) = q_2$ to $F(p_3) = q_3$ in
$B_i \cup B_j \cup B_l \cup \in A^k(M_2)$ satisfying $\mu_2(F(S)) > 0$, $\mu_2(F(R)) > 0$ and $\mu_2(F(Q)) > 0$ then q_1 is maximally path connected to q_2 and q_2 is maximally path connected to q_3.

Therefore, we have shown that if $(M_1, \mathcal{T}_1, \Sigma_1, \mu_1)$ is maximally path connected then $(M_2, \mathcal{T}_2, \Sigma_2, {}_2)$ is also maximally path connected.

Hence, maximal path connectedness μ_1-a.e. is invariant with respect to measurable homeomorphism and measure invariant function F. ■

The above theorem shows that if $\mu_1(S) = 0$, $\mu_1(R) = 0$ and $\mu_1(Q) = 0$ are dark regions in the respective charts (U_1,ϕ_1), (U_2,ϕ_2) and (U_3,ϕ_3) in $(M_1,\mathcal{T}_1,\Sigma_1,\mu_1)$ then $\mu_2(F(S)) = 0$, $\mu_2(F(R)) = 0$ and $\mu_2(F(Q)) = 0$ are dark regions in the corresponding charts (V_1,Ψ_1), (V_2, Ψ_2) and (V_3, Ψ_3) in $(M_2, \mathcal{T}_2, \Sigma_2, \mu_2)$.

5 CONCLUSION

In this paper S.C.P. Halakatti has investigated two intrinsic properties on complete measure manifold. We have shown that locally path connected μ_1- a.e. property, interconnected μ_1- a.e. property and maximally path connected μ_1- a.e. on complete measure manifold are invariant under measurable homeomorphism and measure invariant map.

The above study on different aspects of connectedness on complete measure manifold vindicates that, the local path connectedness, the inter connectedness and the maximally path connectedness are invariant under measurable homeomorphism and measure invariant function F.

One can show that if F: $(M_1,\mathcal{T}_1,\Sigma_1,\mu_1) \longrightarrow (M_2,\mathcal{T}_2,\Sigma_2,\mu_2)$, G:$(M_2,\mathcal{T}_2, \Sigma_2, \mu_2) \longrightarrow (M_3,\mathcal{T}_3,\Sigma_3,\mu_3)$, then the composite function $G \circ F$: $(M_1,\mathcal{T}_1,\Sigma_1, \mu_1) \longrightarrow (M_3,\mathcal{T}_3,\Sigma_3,\mu_3)$, satisfies equivalence relation on a complete measure manifold paving a way for a new manifold called network manifold. We carry the study on such network manifolds in our future work.

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