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Computation of the State Transition Matrix for General Linear Time-Varying Systems

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Abstract

This paper introduces a method to develop the state transition matrix for n-dimensional linear, continuous time-varying systems. The state transition matrix is essential for determining the complete solution, stability, controllability and observability of linear time-varying systems. Cayley-Hamilton technique for finding the solution of linear-time invariant systems is extended to find the state transition matrix of general, n-dimensional continuous time-varying systems. The method gives a general procedure to find the state transition matrix for n-dimensional linear time-varying systems and is very useful in the study of time-varying systems.

Keywords: Time-varying, State-transition matrix

1. INTRODUCTION

Applications of linear time-varying systems include rocket dynamics, time-varying linear circuits, satellite systems and pneumatic actuators. Linear time-varying structure is also often assumed in adaptive and standard gain scheduled control systems.

In particular, we are interested in linear time-varying systems of the form

$$\dot{x}\ t\ = A\ t\ x\ t\ + B\ t\ u\ t$$
$$y\ t\ = C\ t\ x\ t\ + D\ t\ u\ t \qquad (1)$$

where $x\ t\ = \Re^n$ is the state vector, $u\ t\ = \Re^m$ is the control input and $y\ t\ = \Re^p$ is the system output. The state transition matrix is the unique solution to

$$\frac{\partial \phi(t,t_0)}{\partial t} = A(t)\phi(t,t_0)\ , \qquad \phi(t_0,t_0) = I_n \qquad (2)$$

where I_n is the identity matrix. The state transition matrix is essential in determining the complete solution, stability, controllability and observability of (1). It is also useful in the design of controllers and observers.

Several formulations exist to find the state transition matrix of continuous time-varying systems [1,2,3] but no method is developed to find the state transition matrix for a linear time-varying system of nth order. In [1], Peano Baker series method has been used to define the transition matrix for time-varying systems but it says that computation of solutions via the Peano-Baker series is a frightening prospect, though calm calculations is profitable in the simplest cases. Floquet theorem [4] is also used to find the fundamental matrix of homogeneous linear systems with periodic coefficients. In [5], five general classes of methods have been described to compute the exponential of a matrix but the systems considered are defined by linear, constant coefficient ordinary differential equations.

In this paper the well known Cayley-Hamilton technique that is used for the systems described by the linear, constant coefficient ordinary differential equations has been extended to find the state transition matrix for the general linear time-varying systems. It has been shown that this methodology is very versatile and works for periodic coefficients also.

2. STATE TRANSITION MATRIX PROPERTIES

The state transition matrix is an integral component in the study of linear-time-varying systems of the form given by (1). It is used for determining the complete solution, stability, controllability and observability of the system. It can also be used in the design of controllers and observers for equation (1). In this section we will discuss some of the properties of the state transition matrix.

The state transition matrix which satisfies

$$\frac{\partial \phi(t,t_0)}{\partial t} = A(t)\phi(t,t_0) \tag{3}$$

has the following important properties [6]

$$\phi(t,t) = I_n \tag{4}$$

$$\phi(t_0,t_1) = \phi^{-1}(t_1,t_0) \tag{5}$$

$$\phi(t_2,t_0) = \phi(t_2,t_1)\phi(t_1,t_0) \tag{6}$$

Stability of the homogenous system

$$\dot{x}\;t\; = A\;t\;\;x\;t \tag{7}$$

whose solution is given by

$$x\;t\; = \phi(t,t_0)x_0 \tag{8}$$

where $x_0 = x(t_0)$, can be determined from the state transition matrix, according to well known stability theorem [6]. The necessary and sufficient conditions on $\phi(t,t_0)$ for stability are summarized in [6].

It is easy to verify that the solution to the non-homogenous system (1) is given by

$$\begin{aligned} x(t) &= \phi(t,t_0)x_0 + \int_{t_0}^{t} \phi(t,\tau)B(\tau)u(\tau)d\tau \\ y(t) &= C(t)\phi(t,t_0)x_0 + C(t)\int_{t_0}^{t} \phi(t,\tau)B(\tau)u(\tau)d\tau + D(t)u(t) \end{aligned} \tag{9}$$

To guarantee that the system can be driven from one state x_0 to another state x_1 with an input $u(t)$, it is necessary to show that the system is controllable. The linear time-varying system (1) is said to be controllable if any given x_0 there exists an input $u(t)[t_0,t_1]$ such that $x(t_1) = 0$. Controllability of (1) can be determined from the state transition matrix according to well known theorem [1].

To guarantee that the system $x(t)$ can be estimated from the system output $y(t)$, it is necessary to show that the system is observable. The linear time-varying system (1) is said to be observable on $[t_0,t_1]$ if the initial state x_0 is uniquely determined by the output $y(t)$ for $t \in [t_0,t_1]$. Observability of (1) can be determined from the state transition matrix according to a well known theorem [1]. The controllability and observability Gramians can also be used in the design of controllers and observers for (1).

It is clear that the state transition matrix is important for studying stability, controllability and observability of (1). Calculation of the state transition matrix for linear time-invariant system is a straight forward task. Unfortunately for linear time-varying systems, it is often difficult if not impossible to calculate the state transition matrix.

3. CALCULATION OF STATE TRANSITION MATRIX

Consider the general linear time –varying system, defined as

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ -c_1k_1\omega\cos\omega t & -c_2k_1\omega\cos\omega t & \cdots & -c_{n-1}k_1\omega\cos\omega t & -c_nk_1\omega\cos\omega t \end{bmatrix} x(t) \tag{10}$$

where $c_1, c_2, c_3, \ldots.., c_{n-1}, c_n$ are constants and $\omega\cos\omega t$ is time-varying factor.

The solution of the time-varying varying system (10) under the zero initial conditions is

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$x(t) = \phi(t,0)x(0)$

where $\phi(t,0)$ is the state transition matrix of (10). According to Cayley-Hamilton technique

$\phi(t,0) = e^{B}$ where

$$B = \int_0^t \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ -c_1k_1\omega\cos\omega\tau & -c_2k_1\omega\cos\omega\tau & \cdots & -c_{n-1}k_1\omega\cos\omega\tau & -c_nk_1\omega\cos\omega\tau \end{bmatrix} d\tau$$

$$= \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ -c_1k_1\sin\omega t & -c_2k_1\sin\omega t & \cdots & -c_{n-1}k_1\sin\omega t & -c_nk_1\sin\omega t \end{bmatrix}$$

Using Cayley-Hamilton technique, we say that

$$e^{B} = \alpha_1 I + \alpha_2 B + \alpha_3 B^2 + \alpha_4 B^3 + \alpha_5 B^4 + \cdots\cdots\cdots + \alpha_n B^{n-1}$$

and

$$e^{\lambda_i} = \alpha_1 + \alpha_2\lambda_i + \alpha_3\lambda_i^2 + \alpha_4\lambda_i^3 + \alpha_5\lambda_i^4 + \cdots\cdots\cdots + \alpha_n\lambda_i^{n-1} \text{ for } i = 1,2,3,4 \ldots\ldots\text{n}$$

Solving for $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \cdots\cdots \lambda_n$

$$\det(\lambda I - B) = \det \begin{bmatrix} \lambda & 0 & \cdots & 0 & 0 \\ 0 & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda & 0 \\ c_1k_1\sin\omega t & c_2k_1\sin\omega t & \cdots & c_{n-1}k_1\sin\omega t & \lambda + c_nk_1\sin\omega t \end{bmatrix}$$

$$\lambda^{n-1}\ \ \lambda + c_nk_1\sin\omega t\ \ = 0$$

$$\lambda_1 = 0, \lambda_2 = 0, \lambda_3 = 0, \cdots\cdots, \lambda_{n-1} \text{ and } \lambda_n = -c_nk_1\sin\omega t$$

Solving for $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \cdots\cdots, \alpha_n$

For $\lambda_1 = 0$, $\alpha_1 = 1$

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$$\frac{d}{d\lambda_i}(e^{\lambda_i}) = \frac{d}{d\lambda_i}(\alpha_1 + \alpha_2\lambda_i + \alpha_3\lambda_i^2 + \alpha_4\lambda_i^3 + \alpha_5\lambda_i^4 + \cdots\cdots + \alpha_n\lambda_i^{n-1})$$

$$e^{\lambda_i} = \alpha_2 + 2\alpha_3\lambda_i + 3\alpha_4\lambda_i^2 + 4\alpha_5\lambda^3 + \cdots\cdots + (n-1)\alpha_n\lambda_i^{n-2}$$

For $\lambda_2 = 0$, $\alpha_2 = 1 = 1/1!$

$$\frac{d^2}{d\lambda_i^2}(e^{\lambda_i}) = \frac{d^2}{d\lambda_i^2}(\alpha_1 + \alpha_2\lambda_i + \alpha_3\lambda_i^2 + \alpha_4\lambda_i^3 + \alpha_5\lambda_i^4 + \cdots\cdots + \alpha_n\lambda_i^{n-1})$$

$$e^{\lambda_i} = 2\alpha_3 + 6\alpha_4\lambda_i + 12\alpha_5\lambda_i^2 + \cdots\cdots + (n-1)(n-2)\alpha_n\lambda_i^{n-3}$$

For $\lambda_3 = 0$, $\alpha_3 = 1/2 = 1/2!$

Similarly

$$\alpha_4 = 1/6 = 1/3!, \alpha_5 = 1/24 = 1/4!$$
$$\alpha_6 = 1/120 = 1/5!,\ \alpha_7 = 1/720 = 1/6!, \cdots\cdots, \alpha_{n-1} = 1/(n-2)!$$

For $\lambda_n = -c_n k_1 \sin\omega t$

$$\alpha_n = \frac{-(+1) - 1/1!(-c_n k_1 \sin\omega t) - 1/2!(-c_n k_1 \sin\omega t)^2 - 1/3!(-c_n k_1 \sin\omega t)^3 - 1/4!(-c_n k_1 \sin\omega t)^4 - \cdots\cdots - 1/(n-2)!(-c_n k_1 \sin\omega t)^{n-2} + e^{-c_n k_1 \sin\omega t}}{(-c_n k_1 \sin\omega t)^{n-1}}$$

Let

$$A = -(+1) - 1/1!(-c_n k_1 \sin\omega t) - 1/2!(-c_n k_1 \sin\omega t)^2 - 1/3!(-c_n k_1 \sin\omega t)^3 - 1/4!(-c_n k_1 \sin\omega t)^4 - \cdots\cdots - 1/(n-2)!(-c_n k_1 \sin\omega t)^{n-2} + e^{-c_n k_1 \sin\omega t}$$

$$\alpha_n = \frac{A}{(-c_n k_1 \sin\omega t)^{n-1}}$$

$$\phi(t,0) = \alpha_1 I + \alpha_2 B + \alpha_3 B^2 + \alpha_4 B^3 + \alpha_5 B^4 + \cdots\cdots + \alpha_n B^{n-1}$$

$$=\begin{bmatrix}1 & 0 & \cdots & 0 & 0\\ 0 & 1 & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & 1 & 0\\ 0 & 0 & \cdots & 0 & 1\end{bmatrix}+1/1!\begin{bmatrix}0 & 0 & \cdots & 0 & 0\\ 0 & 0 & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & 0 & 0\\ -c_1k_1\sin wt & -c_2k_1\sin wt & \cdots & -c_{n-1}k_1\sin wt & -c_nk_1\sin wt\end{bmatrix}+$$

$$1/2!\begin{bmatrix}0 & 0 & \cdots & 0 & 0\\ 0 & 0 & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & 0 & 0\\ c_1c_nk_1^2\sin^2 wt & c_2c_nk_1^2\sin^2 wt & \cdots & c_{n-1}c_nk_1^2\sin^2 wt & c_n^2k_1^2\sin^2 wt\end{bmatrix}+$$

$$1/3!\begin{bmatrix}0 & 0 & \cdots & 0 & 0\\ 0 & 0 & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & 0 & 0\\ -c_1c_n^2k_1^3\sin^3 wt & -c_2c_n^2k_1^3\sin^3 wt & \cdots & -c_{n-1}c_n^2k_1^3\sin^3 wt & -c_n^2k_1^3\sin^3 wt\end{bmatrix}+$$

$$1/4!\begin{bmatrix}0 & 0 & \cdots & 0 & 0\\ 0 & 0 & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & 0 & 0\\ c_1c_n^3k_1^4\sin^4 wt & c_2c_n^3k_1^4\sin^4 wt & \cdots & c_{n-1}c_n^3k_1^4\sin^4 wt & c_n^4k_1^4\sin^4 wt\end{bmatrix}+\cdots\cdots+$$

$$A/(-c_nk_1\sin\omega t)^{n-1}\begin{bmatrix}0 & 0 & \cdots & 0 & 0\\ 0 & 0 & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & 0 & 0\\ -c_1c_n^{n-2}k_1^{n-1}\sin^{n-1} wt & -c_2c_n^{n-2}k_1^{n-1}\sin^{n-1} wt & \cdots & -c_{n-1}c_n^{n-2}k_1^{n-1}\sin^{n-1} wt & -c_n^{n-1}k_1^{n-1}\sin^{n-1} wt\end{bmatrix}$$

$$\phi(t,0)=\begin{bmatrix}1 & 0 & \cdots & 0 & 0\\ 0 & 1 & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & 1 & 0\\ n_1(t,0) & n_2(t,0) & \cdots & n_{n-1}(t,0) & n_n(t,0)\end{bmatrix}$$

where

$$n_1(t,0) = -c_1k_1\sin\omega t + \frac{1}{2!}c_1c_nk_1^2\sin^2\omega t - \frac{1}{3!}c_1c_n^2k_1^3\sin^3\omega t + \frac{1}{4!}c_1c_n^3k_1^4\sin^4\omega t + \cdots\cdots$$
$$+\frac{1}{n-2!}c_1c_n^{n-1}k_1^{n-2}\sin^{n-2}\omega t + \frac{A(-c_1c_n^{n-2}k_1^{n-1}\sin^{n-1}\omega t)}{(-c_nk_1\sin\omega t)^{n-1}}$$

$$n_2(t,0) = -c_2k_1\sin\omega t + \frac{1}{2!}c_2c_nk_1^2\sin^2\omega t - \frac{1}{3!}c_2c_n^2k_1^3\sin^3\omega t + \frac{1}{4!}c_2c_n^3k_1^4\sin^4\omega t$$
$$+\cdots\cdots+\frac{1}{n-2!}c_2c_n^{n-1}k_1^{n-2}\sin^{n-2}\omega t + \frac{A(-c_2c_n^{n-2}k_1^{n-1}\sin^{n-1}\omega t)}{(-c_nk_1\sin\omega t)^{n-1}}$$

$\vdots$

$\vdots$

$$n_{n-1}(t,0) = -c_3k_1\sin\omega t + \frac{1}{2!}c_3c_nk_1^2\sin^2\omega t - \frac{1}{3!}c_3c_n^2k_1^3\sin^3\omega t + \frac{1}{4!}c_3c_n^3k_1^4\sin^4\omega t$$
$$+\cdots\cdots+\frac{1}{n-2!}c_3c_n^{n-1}k_1^{n-2}\sin^{n-2}\omega t + \frac{A(-c_3c_n^{n-2}k_1^{n-1}\sin^{n-1}\omega t)}{(-c_nk_1\sin\omega t)^{n-1}}$$

$$n_n(t,0) = 1 - c_nk_1\sin\omega t + \frac{1}{2!}c_n^2k_1^2\sin^2\omega t - \frac{1}{3!}c_n^3k_1^3\sin^3\omega t + \frac{1}{4!}c_n^4k_1^4\sin^4\omega t$$
$$+\cdots\cdots+\frac{1}{n-2!}c_n^{n-2}k_1^{n-2}\sin^{n-2}\omega t + \frac{A(-c_n^{n-1}k_1^{n-1}\sin^{n-1}\omega t)}{(-c_nk_1\sin\omega t)^{n-1}}$$

Solving for $n_1(t,0)$, $n_2(t,0)$, ……,$n_{n\text{-}1}(t,0)$, and $n_n(t,0)$

$$n_1(t,0) = \frac{c_1}{c_n}[-1 + e^{-c_nk_1\sin\omega t}]$$

$$n_2(t,0) = \frac{c_2}{c_n}[-1 + e^{-c_nk_1\sin\omega t}]$$

Similarly, $n_{n\text{-}1}(t,0) = \dfrac{c_{n-1}}{c_n}[-1 + e^{-c_nk_1\sin\omega t}]$

And $\mathrm{n}_n(\mathrm{t},0) = e^{-c_nk_1\sin\omega t}$

The state transition matrix is given as

$$\phi(t,0)=\begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ \frac{c_1}{c_n}(e^{-c_n k_1 \sin\omega t}-1) & \frac{c_2}{c_n}(e^{-c_n k_1 \sin\omega t}-1) & \cdots & \frac{c_{n-1}}{c_n}(e^{-c_n k_1 \sin\omega t}-1) & e^{-c_n k_1 \sin\omega t} \end{bmatrix}$$

The solution of (10) is

$$x(t)=\begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ \frac{c_1}{c_n}(e^{-c_n k_1 \sin\omega t}-1) & \frac{c_2}{c_n}(e^{-c_n k_1 \sin\omega t}-1) & \cdots & \frac{c_{n-1}}{c_n}(e^{-c_n k_1 \sin\omega t}-1) & e^{-c_n k_1 \sin\omega t} \end{bmatrix}$$

4. EXAMPLES

Example1: Consider the system (1) with[7]

$$A(t)=\begin{bmatrix} -6t^2 & 3t^5 \\ 0 & -3t^2 \end{bmatrix}, \qquad t>0$$ and zero initial conditions.

Thus $\phi(t,0)=e^B$ where

$$B=\int_0^t A(\tau)d\tau=\begin{bmatrix} -2t^3 & \frac{t^6}{2} \\ 0 & -t^3 \end{bmatrix}$$

Using the Cayley-Hamilton technique where we assume that

$$e^B=\alpha_1+\alpha_2 B$$

$e^{\lambda_i}=\alpha_1+\alpha_2\lambda_i$ for $i=1$ and 2

Solving for λ_1 and λ_2

$\lambda_1=-2t^3$ and $\lambda_2=-t^3$

Solving for α_1 and α_2 yields

$$\alpha_1 = 2e^{-t^3} - e^{-2t^3} \quad , \quad \alpha_2 = \frac{e^{-t^3} - e^{-2t^3}}{t^3}$$

Then $\phi(t,0) = \begin{bmatrix} e^{-2t^3} & \frac{t^3}{2}(e^{-t^3} - e^{-2t^3}) \\ 0 & e^{-t^3} \end{bmatrix}$

Example 2: Consider the system equation (1) with[8]

$A(t) = \begin{bmatrix} 1 & 0 \\ 0 & 2t \end{bmatrix}, \quad t > 0$ and non-zero initial conditions.

Thus $\phi(t,t_0) = e^B$

where $B = \int_{t_0}^{t} A(\tau)d\tau = \begin{bmatrix} t - t_0 & 0 \\ 0 & t^2 - t_0^2 \end{bmatrix}$

Using the Cayley-Hamilton technique where we assume that

$e^B = \alpha_1 + \alpha_2 B$

$e^{\lambda i} = \alpha_1 + \alpha_2 \lambda_i$ for $i = 1$ and 2

$\lambda_1 = t - t_0$ and $\lambda_2 = t^2 - t_0^2$

Solving for α_1 and α_2 yields

$$\alpha_1 = e^{t-t_0} - \left(\frac{e^{t-t_0} - e^{t^2-t_0^2}}{1 - t + t_0}\right) \quad , \quad \alpha_2 = \frac{e^{t-t_0} - e^{t^2-t_0^2}}{t - t_0 - t^2 + t_0^2}$$

$$\phi(t,t_0) == \begin{bmatrix} e^{t-t_0} & 0 \\ 0 & e^{t^2-t_0^2} \end{bmatrix}$$

5. CONCLUSION

The paper presents a method to calculate the state transition matrix for linear, continuous time-varying systems utilizing Cayley -Hamilton theorem. Periodic systems can also be evaluated by this methodology. A powerful technique for studying the important properties of time -varying systems viz stability, controllability and observability has been presented. The application of the method has been demonstrated through two examples.

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REFERENCES

[1] Rugh,W.J. *, Linear system theory,* Second Edition, Prentice Hall, Information and System Sciences, New Jersey,1996.

[2] Wu,M.Y., " Solution of certain classes of linear time-varying systems" *International Journal of Control,* 31(1),11-20,1980.

[3] Kloet, P. V. and F.L.Neerhoff, "Diagonalization algorithms for linear time-varying dynamic systems", *International Journal of System Sciences*, 31(8),1053-1057,2000.

[4] Farkas, M., *Periodic Motions*, Appl. Math. Sci., Vol. 104, Springer- Verlag, Berlin and New York ,1994.

[5] Moler, C. and C.V.Loan, "Nineteen dubious ways to compute the exponential of a matrix" *SIAM Review*, 20(4), 801-833,1978.

[6] Chen, C., *Linear system theory and design*, Orlando, Florida: Holt: Rinehart and Winston Inc.,1984.

[7] Wu,M.Y., "An new method of computing the state transition matrix of linear time-varying systems" *Proceedings of the IEEE International Symposium on Circuits and Systems,* San Francisco, 269-272,1974.

[8] Wu,M.Y. , " Some new results in linear time-varying systems", *IEEE Transaction on Automatic Control,*.20(1) , 159-161,1975.

[9] Horn,R.A. and C.R. Johnson, *Topics in matrix analysis*, New York : Cambridge University Press ,1991.

[10] Wu,M.Y.and A.Sherif, " On the commutative class of linear time-varying systems" *International Journal of Control*, 23(3), 433-444, 1976.

[11] Kloet, P. V. and F.L.Neerhoff, "Normalized dynamic eigen values for scalar time-varying systems", Proc. NDES 2002, Izmir, Turkey, 21-23 June .2002.

[12] Zhu, A. and C.D.Johnson, "New results in the reduction of linear time- varying dynamical systems", *SIAM Journal of Control and Optimization,* 27(3), 476-494 ,1989.