Comparing Numerical Solution to a Second Order Boundary Value Problem by Variational Techniques

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Abstract— **This Research paper presents the techniques to determine the solution for a second-order Boundary Value Problem (BVP) that is frequently witnessed in engineering phenomena, by multiple variational methods. Further, the solutions obtained by these methods are compared with the exact solution of the second-order differential equation. The methods that are illustrated here are Subdomain Method, Collocation Method, Least Square Method, Petrov Galerkin Method, Galerkin Method, Rayleigh-Ritz Method and the Finite Difference Method. The solutions obtained by numerical methods seem to be fairly in good agreement with the exact solutions. Further, the comparison and average percentage error between solutions obtained by Exact Method and above-stated Numerical Methods have been carried out in MS-Excel. The interval or spacing of 0.1 is considered as the step size within the specified domain of 0 to 1. It is observed and confirmed that to achieve higher accuracy, the intervals or step size must be reduced. This analysis reveals that the Finite Difference Method (FDM) gives the most accurate solution among the other Numerical Methods considered in this study. And therefore, most of the Finite Element Analysis (FEA) and Computational Fluid Dynamics (CFD) software includes FDM. The accuracy level attained by various methods from high to low is Finite Difference Method, Rayleigh-Ritz Method, Galerkin Method, Petrov Galerkin Method, Collocation Method, Subdomain Method, Least Square Method. Galerkin Method and Rayleigh-Ritz Method give identical solutions. The study also concludes that the Finite Difference Method is high in accuracy and Least Square Method appears to be the poorest in achieving the accuracy.**

Keywords— ***Boundary Value Problem, Approximate Method, Weighted Residue Methods.***

I. INTRODUCTION

It is said that the backbone of any physical system is its differential equation. Ordinary differential or partial differential equation is used to model the physical system [1]. The ultimate aim of numerical analysis is to find the approximate solution to the differential equation when the exact solution is difficult to obtain. Both finite difference method and variational methods of approximation attempts are performed to find the approximate solution in the form of a linear combination of suitable approximation function and undetermined coefficients [2]. The given differential equation is $\frac{d^2y}{dx^2} + y + 1 = 0$, comparing with $A\frac{d^2y}{dx^2} + B\frac{d^2y}{dxdu} + C\frac{d^2y}{du^2} + D\frac{dy}{dx} + E\frac{dy}{du} + Fy + G = 0$. Here, $A = 1$, $B = C = 0$. So, $B^2 - 4AC = 0$. Hence, the given differential equation is parabolic [3]. This research compares the accuracy of various weighted residue methods and finite difference methods with the exact method by comparing their solutions to the second ordered differential boundary value problem.

II. SOLVING SECOND ORDER DIFFERENTIAL BOUNDARY VALUE PROBLEM BY VARIOUS NUMERICAL METHODS

Given Differential Equation is: $\frac{d^2y}{dx^2} + y + 1 = 0$; Domain: $0 \le x \le 1$

Boundary Conditions: i. $Y(0) = 0$; ii. $Y(1) = 0$.

By Exact method:

By Exact Method, the solution is given by:

$$\text{Yexact} = C1\cos(x) + C2\sin(x) - 1 \quad (1)$$

Applying boundary condition i to equation 1:

$0 = C1 - 1$

$C1 = 1$

Applying boundary condition ii to equation 1:

$0 = (1)\cos(1) + C2\sin(1) - 1$

$C2 = 0.5463$

Putting values of C1 and C2 in equation 1:

$$\text{Yexact} = \cos(x) + (0.5463)\sin(x) - 1 \quad (2)$$

By Variational methods of approximations:

Given Differential Equation is:

$$\frac{d^2y}{dx^2} + y + 1 = 0. \quad (3)$$

Let $Y \neq \text{Yexact} = \text{Yapproximate}$.

Let C1, C2, C3 and C4 be the constants.

Let us assume a polynomial:

$$\text{Yapproximate} = C1 + C2x + C3x^2 + C4x^3 \quad (4)$$

Applying boundary condition i to equation 4:

$C1 = 0$

Applying boundary condition ii to equation 4:

$0 = 0 + C2 + C3 + C4$

$C2 = -C3 - C4$

Putting values of C1 and C2 in equation 4:

$$\text{Yapproximate} = (-C3 - C4)x + C3x^2 + C4x^3 \quad (5)$$

$$\text{Yapproximate} = -C3x - C4x + C3x^2 + C4x^3 \quad (6)$$

$$\text{Yapproximate} = C3(x^2 - x) + C4(x^3 - x) \quad (7)$$

Differentiating equation 7 with respect to x:

$$\frac{d\text{Yapproximate}}{dx} = C3(2x - 1) + C4(3x^2 - 1) \quad (8)$$

Substituting $-C3 = A$ and $-C4 = B$ in equation 5:

$$\text{Yapproximate} = (A + B)x - Ax^2 - Bx^3 \quad (9)$$

$$\text{Yapproximate} = A(x - x^2) + B(x - x^3) \quad (10)$$

Differentiating equation 9 with respect to x:

$$\frac{d\text{Yapproximate}}{dx} = (A + B) - 2Ax - 3Bx^2 \quad (11)$$

Differentiating equation 11 with respect to x:

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$\frac{d^2Yapproximate}{dx^2} = -2A - 6Bx$ (12)

Substituting equations 9 and 12 in equation 3:

$-2A - 6Bx + (A + B)x - Ax^2 - Bx^3 + 1 = R$

Where R = Residue.

$(-2A + 1) + (A - 5B)x - Ax^2 - Bx^3 = R$ (13)

$A(x - x^2 - 2) + B(-x^3 - 5x) + 1 = R$ (14)

Now, we have Weighted Residue Statement for the given differential equation is:

$\int Wi\ R\ dx = 0\ ;\ 0 \le x \le 1$ (15)

A. Subdomain method

As the given domain is $0 \le x \le 1$. In the subdomain method, we divide the domain into two parts and we will solve each part individually.

In the subdomain, we take the weight function as a unity.

i.e. $w1 = w2 = 1$

Solving for the first part of domain $= 0 \le x \le \frac{1}{2}$; i.e. i = 1

$\int_0^{\frac{1}{2}} (w1)R\ dx = 0$

$\int_0^{\frac{1}{2}} (1)\left((-2A + 1) + (A - 5B)x - Ax^2 - Bx^3\right)dx = 0$

$\frac{-11}{12}A - \frac{41}{64}B = \frac{-1}{2}$ (16)

Solving for second part of domain $= \frac{1}{2} \le x \le 1$; i.e. i = 2

$\int_{\frac{1}{2}}^1 (w2)Rdx = 0$

$\int_{\frac{1}{2}}^1 (1)\left((-2A + 1) + (A - 5B)x - Ax^2 - Bx^3\right)dx = 0$

$\frac{-11}{12}A - \frac{135}{64}B = \frac{-1}{2}$ (17)

Solving equations 16 and 17 simultaneously:

$A = \frac{6}{11}$ and $B = 0.$

Substituting the values of A and B in equation 9:

$Ysubdomain\ method = \frac{-6}{11}x^2 + \frac{6}{11}x$ (18)

B. Collocation method

In the collocation method, we take residue at gauss point 0. i.e. $R(x) = 0$.

A domain is $0 \le x \le 1$. We will take two gauss points $x = \left(\frac{1}{3}\right)$ and $x = \left(\frac{2}{3}\right)$, where the value of residue is 0.

Solving both cases as given below:

$R\left(x = \frac{1}{3}\right) = 0$

Putting $x = \frac{1}{3}$ in equation 14:

$\frac{-16}{9}A - \frac{46}{27}B = -1$ (19)

$R\left(x = \frac{2}{3}\right) = 0$

Putting $\left(x = \frac{2}{3}\right)$ in equation 14:

$\frac{-16}{9}A - \frac{98}{27}B = -1$ (20)

Solving equations 19 and 20 simultaneously:

$A = \frac{9}{16}$ and $B = 0.$

Substituting the values of A and B in equation 9:

$Ycollocation\ method = \frac{9}{16}x - \frac{9}{16}x^2$ (21)

C. Least square method

In the least square method, differentiate the equation of residue i.e. equation (13) with respect to A and B to get weight function w2 and w1 respectively.

$w1 = \frac{\partial R}{\partial A} = x - x^2 - 2$

$w2 = \frac{\partial R}{\partial B} = -x^3 - 5x$

Solving both cases as given below:

$\int Wi\ R\ dx = 0\ ;\ 0 \le x \le 1$; i.e. i = 1

$\int_0^1 (w1)Rdx = 0$

$\int_0^1 (x - x^2 - 2)[(-2A + 1) + (A - 5B)x - Ax^2 - Bx^3]dx = 0$

$\frac{101}{30}A - \frac{119}{20}B = \frac{11}{6}$ (22)

$\int Wi\ R\ dx = 0\ ;\ 0 \le x \le 1$; i.e. i = 2

$\int_0^1 (w2)Rdx = 0$

$\int_0^1 (-5x - x^3)[(-2A + 1) + (A - 5B)x - Ax^2 - Bx^3]dx = 0$

$\frac{101}{20}A - \frac{220}{21}B = \frac{11}{4}$ (23)

Solving equations 22 and 23 simultaneously:

$A = \frac{55}{101}$ and $B = 0.$

Substituting the values of A and B in equation 9:

$Y\ least\ square\ method = \frac{55}{101}x - \frac{55}{101}x^2$ (24)

D. Petrov Galerkin method

In the Petrov Galerkin Method we take weight function (w1 and w2) in combination such as (1 and x) or (x and x^2) or (1 and x^2) or (1 and x^3) or (x^3 and x^9) respectively.

Here, we will solve by using x and x^2 as a weight function.

$w1 = x$

$w2 = x^2$

Solving both cases as given below:

$\int Wi\ R\ dx = 0\ ;\ 0 \le x \le 1$; i.e. i = 1

$\int_0^1 (w1)R\ dx = 0$

$\int_0^1 (x)[(-2A + 1) + (A - 5B)x - Ax^2 - Bx^3]dx = 0$

$\frac{-11}{12}A - \frac{28}{15}B = \frac{-1}{2}$ (25)

$\int Wi\ R\ dx = 0\ ;\ 0 \le x \le 1$; i.e. i = 2

$\int_0^1 (w2)R\ dx = 0$

$\int_0^1 (x^2)\ [(-2A + 1) + (A - 5B)x - Ax^2 - Bx^3]dx = 0$

$\frac{-37}{60}A - \frac{17}{12}B = \frac{-1}{3}$ (26)

Solving equations 25 and 26 simultaneously:

$A = \frac{310}{531}$ and $B = \frac{-10}{531}$

Substituting the values of A and B in equation 9:

$Ypetrov\ galerkin\ method = \frac{100}{177}x - \frac{310}{531}x^2 + \frac{10}{531}x^3$ (27)

E. Galerkin method

In Galerkin Method, differentiate equation of $Y_{approximate}$ i.e. equation (10) with respect to A and B to get weight function w1 and w2 respectively.

$w1 = \frac{\partial yapproximate}{\partial A} = (x - x^2)$

$w2 = \frac{\partial yapproximate}{\partial B} = (x - x^3)$

Solving both cases as given below:

$\int Wi\ R\ dx = 0\ ;\ 0 \le x \le 1$; i.e. i = 1

$\int_0^1 (w1)\ R\ dx = 0$

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$\int_0^1 (x - x^2)[(-2A + 1) + (A - 5B)x - Ax^2 - Bx^3]dx = 0$

$\frac{-3}{10}A - \frac{9}{20}B = \frac{-1}{6}$ (28)

$\int$ Wi R dx = 0 ; $0 \le x \le 1$; i.e. i = 2

$\int_0^1 (w2)R\,dx = 0$

$\int_0^1 (x - x^3)[(-2A + 1) + (A - 5B)x - Ax^2 - Bx^3]dx = 0$

$\frac{-9}{20}A - \frac{76}{105}B = \frac{-1}{4}$ (29)

Solving equations 28 and 29 simultaneously:

$A = \frac{5}{9}$ and $B = 0$.

Substituting the values of A and B in equation 9:

Ygalerkin method $= \frac{5}{9}x - \frac{5}{9}x^2$ (30)

F. Rayleigh-Ritz method

In Rayleigh-Ritz Method we take Residue (R) as a given differential equation.

So, $R = \frac{d^2y}{dx^2} + y + 1$.

In Rayleigh-Ritz Method, we take the coefficient of C3 and C4 from equation 7 as weight function W1 and W2 respectively.

$W1 = \frac{\partial yapproximate}{\partial(C3)} = (x^2 - x)$

$W2 = \frac{\partial yapproximate}{\partial(C4)} = (x^3 - x)$

$\frac{d(w1)}{dx} = (2x - 1)$

$\frac{d(w2)}{dx} = (3x^2 - 1)$

Solving both cases as given below:

$\int$ Wi R dx = 0 ; $0 \le x \le 1$; i.e. i = 1

$\int_0^1 (wi)\left[\frac{d^2y}{dx^2} + y + 1\right]dx = 0$

$\int_0^1 (wi)\frac{d^2y}{dx^2}dx + \int_0^1 (wi)\,ydx + \int_0^1 (wi)\,dx = 0$

$\int_0^1 (wi)\frac{d^2y}{dx^2}dx = -\int_0^1 (wi)\,ydx - \int_0^1 (wi)\,dx$

Applying integration by parts to term $(\int_0^1 (wi)\frac{d^2y}{dx^2}dx)$, considering $u = wi$ and $v = \frac{d^2y}{dx^2}$, where the formula is $\int u\,v\,dx = u\int v\,dx - \int\left[\frac{du}{dx}\int v\,dx\right]dx$.

$$[wi\frac{dy}{dx}]_0^1 - \int_0^1\left(\frac{dwi}{dx}\times\frac{dy}{dx}\right)dx = -\int_0^1 (wi)\,y\,dx - \int_0^1 (wi)\,dx$$

$$[wi\frac{dy}{dx}]_0^1 = \int_0^1\left(\frac{dwi}{dx}\times\frac{dy}{dx}\right)dx - \int_0^1 (wi)\,ydx - \int_0^1 (wi)\,dx$$

Putting i = 1:

$$[w1\frac{dy}{dx}]_0^1 = \int_0^1\left(\frac{dw1}{dx}\times\frac{dy}{dx}\right)dx - \int_0^1 (w1)ydx - \int_0^1 (w1)dx$$

$[(x^2 - x)\frac{dy}{dx}]_0^1 = \int_0^1 (2x-1)\big(C3(2x-1) + C4(3x^2 - 1)\big)dx - \int_0^1 (x^2 - x)ydx - \int_0^1 (x^2 - x)dx$

$\frac{3}{10}C3 + \frac{9}{20}C4 = \frac{-1}{6}$ (31)

$\int$ Wi R dx = 0 ; $0 \le x \le 1$; i.e. i = 2

$\int_0^1 (wi)\left[\frac{d^2y}{dx^2} + y + 1\right]dx = 0$

$\int_0^1 (wi)\frac{d^2y}{dx^2}dx + \int_0^1 (wi)\,ydx + \int_0^1 (wi)\,dx = 0$

$\int_0^1 (wi)\frac{d^2y}{dx^2}dx = -\int_0^1 (wi)\,ydx - \int_0^1 (wi)\,dx$

Applying integration by parts to term $(\int_0^1 (wi)\frac{d^2y}{dx^2}dx)$, considering $u = wi$ and $v = \frac{d^2y}{dx^2}$, where the formula is $\int u\,v\,dx = u\int v\,dx - \int\left[\frac{du}{dx}\int v\,dx\right]dx$.

$$[wi\frac{dy}{dx}]_0^1 - \int_0^1\left(\frac{dwi}{dx}\times\frac{dy}{dx}\right)dx = -\int_0^1 (wi)\,y\,dx - \int_0^1 (wi)\,dx$$

$$[wi\frac{dy}{dx}]_0^1 = \int_0^1\left(\frac{dwi}{dx}\times\frac{dy}{dx}\right)dx - \int_0^1 (wi)\,ydx - \int_0^1 (wi)\,dx$$

Putting i = 2:

$$[w2\frac{dy}{dx}]_0^1 = \int_0^1\left(\frac{dw2}{dx}\times\frac{dy}{dx}\right)dx - \int_0^1 (w2)ydx - \int_0^1 (w2)dx$$

$[(x^3 - x)\frac{dy}{dx}]_0^1 = \int_0^1 (3x^2 - 1)(C3(2x - 1) + C4(3x^2 - 1))dx - \int_0^1 (x^3 - x)(C3(x^2 - x) + C4(x^3 - x)dx - \int_0^1 (x^3 - x)\,dx$

$\frac{9}{20}C3 + \frac{76}{105}C4 = \frac{-1}{4}$ (32)

Solving equations 31 and 32 simultaneously:

$C3 = \frac{-5}{9}$ and $C4 = 0$.

Substituting the values of C_3 and C_4 in equation 7:

Y rayleigh ritz method $= -\frac{5}{9}(x^2 - x)$ (33)

G. Finite difference method (FDM)

In the Finite Difference Method, we have to define the nodes at a point where the value is to be determined. As Domain is $0 \le x \le 1$, we will take a step size of 0.1 i.e. h = 0.1. As h = 0.1 ; Therefore $h^2 = \frac{1}{100}$. Let Y0, Y1, Y2, Y3, Y4, Y5, Y6, Y7, Y8, Y9, Y10 be the values of Y at a particular node when value of x is 0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1 respectively.

According to given boundary conditions i and ii:

Y0 = Y(0) = 0, Y10 = Y(1) = 0.

For the value of an unknown node we have a formula:

$\frac{[y_{i+1}]+[y_{i-1}]-2y_i}{h^2} + Y_i + 1 = 0.$

For the value of i equals to 1, 2, 3, 4, 5, 6, 7, 8, 9 equations are:

$199Y1 - 100Y2 = 1$ (34)

$100Y1 - 199Y2 + 100Y3 = -1$ (35)

$100Y2 - 199Y3 + 100Y4 = -1$ (36)

$100Y3 - 199Y4 + 100Y5 = -1$ (37)

$100Y4 - 199Y5 + 100Y6 = -1$ (38)

$100Y5 - 199Y6 + 100Y7 = -1$ (39)

$100Y6 - 199Y7 + 100Y8 = -1$ (40)

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$100Y7 - 199Y8 + 100Y9 = -1$ (41)
$100Y8 - 199Y9 = -1$ (42)

III. SOLUTION OF SECOND-ORDER DIFFERENTIAL BOUNDARY VALUE PROBLEM BY VARIOUS NUMERICAL METHODS

Substituting the values of x in equations 2, 18, 21, 24, 27, 30, 33 and solving equations 34, 35, 36, 37, 38, 39, 40, 41, 42 simultaneously we get the results shown in table 1. The solutions of second order boundary value problem by various numerical methods at 0.0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0 is mentioner under table 1.

The graphical representation of solutions of second-order differential boundary value problem by various numerical methods has been shown in figure 1 in scatter with smooth line form. From figure 1, it can be concluded that the considered differential equation is parabolic. The parabolic nature of the considered second-order differential equation has been explained in the introduction portion of this research article. But, as the solutions by various numerical methods are in good agreement with each other the curves in figure 1 is overlapping on each other and the vision of graphs of each numerical method is not much clear. To overcome this issue in figure 2 bar graph representation is shown.

Table 1. Solution of second-order differential BVP by various numerical methods

Value of x	Value of y							
	Exact	Subdomain	Collocation	Least Square	Petrov Galerkin	Galerkin	FDM	Rayleigh Ritz
0.0	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.1	0.04954	0.04909	0.05063	0.04901	0.05068	0.05000	0.04959	0.05000
0.2	0.08860	0.08727	0.09000	0.08713	0.08979	0.08889	0.08868	0.08889
0.3	0.11678	0.11455	0.11813	0.11436	0.11746	0.11667	0.11689	0.11667
0.4	0.13380	0.13091	0.13500	0.13069	0.13379	0.13333	0.13393	0.13333
0.5	0.13949	0.13636	0.14063	0.13614	0.13889	0.13889	0.13962	0.13889
0.6	0.13380	0.13091	0.13500	0.13069	0.13288	0.13333	0.13393	0.13333
0.7	0.11678	0.11455	0.11813	0.11436	0.11588	0.11667	0.11689	0.11667
0.8	0.08860	0.08727	0.09000	0.08713	0.08798	0.08889	0.08868	0.08889
0.9	0.04954	0.04909	0.05063	0.04901	0.04932	0.05000	0.04959	0.05000
1.0	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

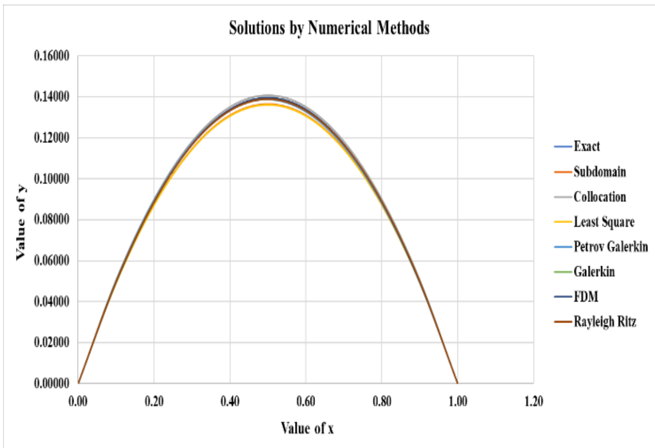

Fig. 1 Scatter with smooth lines graph representation of the solution of second-order differential boundary value problem by various numerical methods

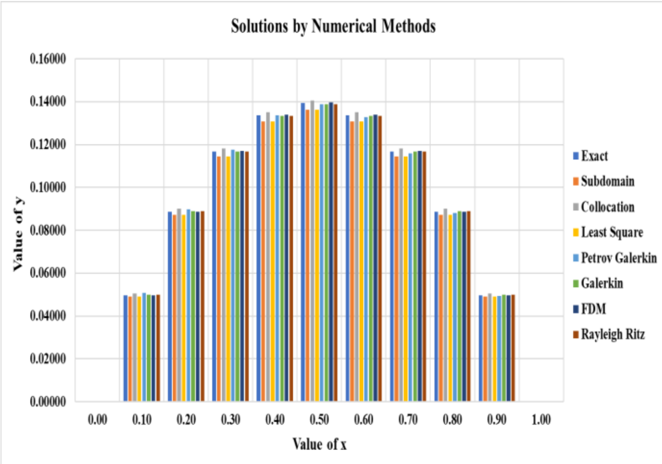

Fig. 2 Bar graph representation of the solution of second-order differential boundary value problem by various numerical methods

IV. ERROR ANALYSIS FOR THE SOLUTIONS OF VARIOUS NUMERICAL METHODS WITH RESPECT TO EXACT METHOD

For error analysis, the solutions by the exact method are considered to be real without any bias i.e. with zero error. For calculating the error in the solutions of numerical methods, the solution difference of the exact method and numerical method is estimated. If in some cases difference comes negative then modulus is applied. Table 2 indicates

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the error in the solution of various numerical methods with respect to exact method solutions. Figure 3 indicates a graphical representation of error analysis for the solutions of various numerical methods with respect to the exact method.

The assumptions considered during estimating error analysis is as follows:

1. Consider Exact Method has zero error.
2. Error in values of $Y_{approximate}$ is estimated by formula:

Error in values of $Y_{approximate} = |(Y_{exact\ method} - Y_{numerical\ method})|$

Table 2. Error analysis for the solutions of various numerical methods with respect to exact method

	Difference in value of y							
x	**Exact**	**Subdomain**	**Collocation**	**Least Square**	**Petrov Galerkin**	**Galerkin**	**FDM**	**Rayleigh Ritz**
0.0	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.1	0.00000	0.00045	0.00109	0.00053	0.00114	0.00046	0.00005	0.00046
0.2	0.00000	0.00133	0.00140	0.00147	0.00119	0.00029	0.00008	0.00029
0.3	0.00000	0.00223	0.00135	0.00242	0.00068	0.00011	0.00011	0.00011
0.4	0.00000	0.00289	0.00120	0.00311	0.00001	0.00047	0.00013	0.00047
0.5	0.00000	0.00313	0.00114	0.00335	0.00060	0.00060	0.00013	0.00060
0.6	0.00000	0.00289	0.00120	0.00311	0.00092	0.00047	0.00013	0.00047
0.7	0.00000	0.00223	0.00135	0.00242	0.00090	0.00011	0.00011	0.00011
0.8	0.00000	0.00133	0.00140	0.00147	0.00062	0.00029	0.00008	0.00029
0.9	0.00000	0.00045	0.00109	0.00053	0.00022	0.00046	0.00005	0.00046
1.0	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

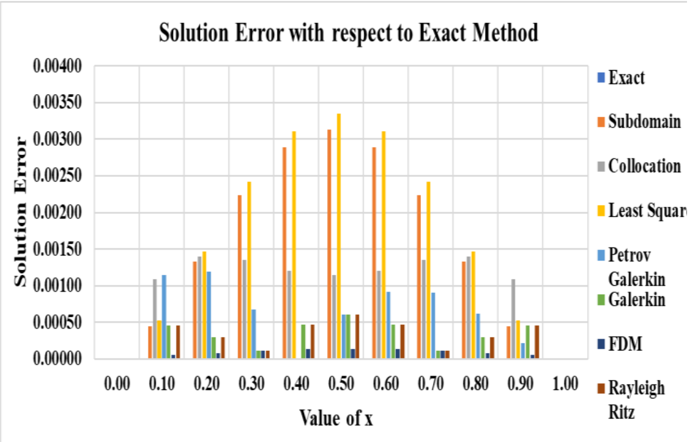

Fig. 3 Graphical representation of error analysis for the solutions of various numerical methods with respect to the exact method

V. PERCENTAGE ERROR ANALYSIS FOR THE SOLUTIONS OF VARIOUS NUMERICAL METHODS WITH RESPECT TO EXACT METHOD

In this section, the percentage error for solutions of numerical methods with respect to the exact method at eleven points within the limit of 0 to 1 is estimated. Along with this, the average percentage error in the solution of each numerical method is also calculated. It has been assumed that the solution obtained by the exact method is a true solution with zero percentage of error. Table 3 indicates percentage error analysis for solutions of numerical methods with respect to the exact method. The average percentage error with respect to exact method in solutions of subdomain method, collocation method, least square method, petrov galerkin method, galerkin method, finite difference method, rayleigh ritz method is 1.38199 %, 1.13488 %, 1.51389 %, 0.66057 %, 0.34850 %, 0.07803 %, 0.34850 % respectively.

Table 3 Percentage error analysis for solutions of numerical methods with respect to the exact method

	Percentage error in value of y with respect to exact method (%)							
x	**Exact**	**Subdomain**	**Collocation**	**Least Square**	**Petrov Galerkin**	**Galerkin**	**FDM**	**Rayleigh Ritz**
0.0	0	0	0	0	0	0	0	0
0.1	0	0.90835	2.20024	1.06984	2.30117	0.92854	0.10092	0.92854
0.2	0	1.50112	1.58013	1.65914	1.34311	0.32731	0.09029	0.32731
0.3	0	1.90957	1.15601	2.07227	0.58229	0.09491	0.09419	0.09491
0.4	0	2.15994	0.89686	2.32436	0.00747	0.35127	0.09715	0.35127
0.5	0	2.24388	0.81726	2.40160	0.43013	0.43013	0.09319	0.43013
0.6	0	2.15994	0.89686	2.32436	0.68759	0.35127	0.09715	0.35127
0.7	0	1.90957	1.15601	2.07227	0.77067	0.09419	0.09419	0.09419
0.8	0	1.50112	1.58010	1.65914	0.69977	0.32731	0.09029	0.32731
0.9	0	0.90835	2.20024	1.06984	0.44408	0.92854	0.10092	0.92854
1.0	0	0	0	0	0	0	0	0
Average Percentage Error (%)	0	1.38199	1.13488	1.51389	0.66057	0.34850	0.07803	0.34850

Figure 4 and figure 5 indicate scatter with smooth lines and bar graph representation of percentage solution error by numerical methods with respect to exact method respectively.

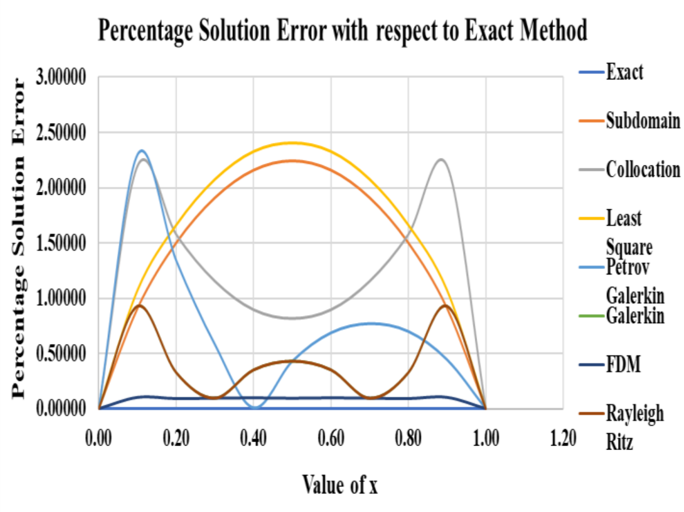

Fig. 4 Scatter with smooth lines graph representation of percentage solution error by numerical methods with respect to the exact method

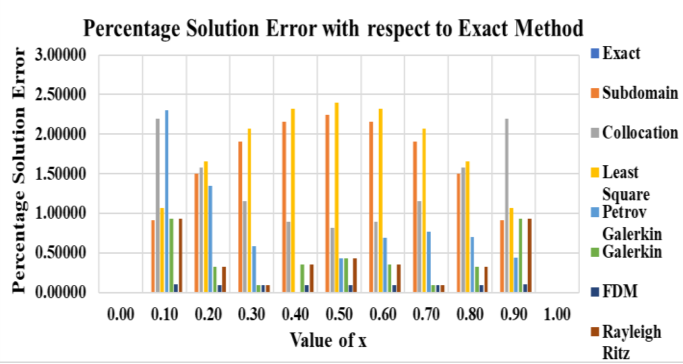

Fig. 5 Bar graph representation of percentage solution error by numerical methods with respect to the exact method

VI. RESULT AND DISCUSSION

The chosen second order differential equation has been solved by Subdomain Method, Collocation Method, Least Square Method, Petrov Galerkin Method, Galerkin Method, Rayleigh-Ritz Method and Finite Difference Method. The solutions obtained by these numerical methods are compared with the exact solutions. The average percentage error has been estimated between solutions of Numerical Methods and Exact Methods. Figure 6 indicates a bar graph of average percentage error by numerical methods with respect to the exact method and table 4 indicates the ranking of various numerical methods on the basis of average percentage error with the exact method.

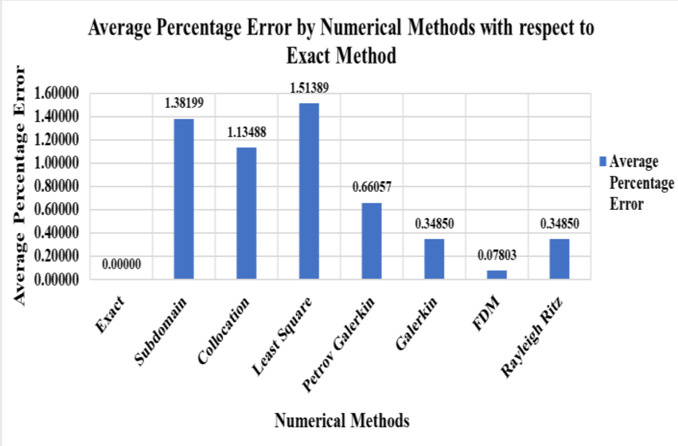

Fig. 6 Average percentage error by numerical methods with respect to the exact method

Table 4. Ranking of various numerical methods on the basis of average percentage error with the exact method.

Numerical Methods	Average Percentage Error (%)	Rank
Finite Difference Method	0.07803	1
Rayleigh Ritz Method	0.34850	2
Galerkin Method	0.34850	2
Petrov Galerkin Method	0.66057	3
Collocation Method	1.13488	4
Subdomain Method	1.38199	5
Least Square Method	1.51389	6

From table 4, the numerical methods on the basis of minimum average percentage error with respect to exact solution and accuracy in their solution is listed as below:
1. Finite Difference Method (FDM)
2. Rayleigh-Ritz Method and Galerkin Method
3. Petrov Galerkin Method
4. Collocation Method
5. Subdomain Method
6. Least Square Method

VII. CONCLUSION

1. In this research work, an attempt has been made to solve second-order differential equations by various methods of approximations and the Finite Difference Method. It has been found that from table 4, the Finite Difference Method gives closest solutions to exact solutions. Hence, the Finite Difference Method can be concluded as the most accurate method for solving a differential equation.
2. From Table 1, It can be concluded that the Galerkin and Rayleigh-Ritz Method gives identical solutions. Hence, Galerkin and Rayleigh-Ritz methods are almost in agreement with each other.
3. From Table 1, it is observed that at x = 0.5 (i.e. centre point of the domain) we get the peak value for all numerical methods as the equation is parabolic.
4. Further from Table 1, at points (0.4 and 0.6), (0.3 and 0.7), (0.2 and 0.8) and (0.1 and 0.9), the identical solution

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obtained in each respective method, but only in the Petrov Galerkin Method, this trend can't be observed.

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