

Comparative Approaches for Solving the Nonlinear Transportation Problem with Quadratic Costs

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Abstract - The nonlinear transportation problem (NTP) extends the classical transportation model by incorporating nonlinear cost structures, leading to increased computational complexity. A quadratic NTP is solved using three approaches: Wolfe's reduced-gradient method, a Genetic Algorithm (GA), and convex quadratic programming (CQP) implemented in Python using CVXPY and OSQP. A benchmark 7×7 quadratic transportation problem is employed for performance evaluation. Wolfe's method yields feasible but suboptimal solutions due to sensitivity to initialization. The GA achieves near-optimal results with strong global search capability but exhibits slower convergence and limited reproducibility. In contrast, the CQP approach consistently reproduces the benchmark solution with high numerical accuracy, efficiency, and stability. This study provides a systematic comparison of classical, heuristic, and convex optimization techniques for quadratic nonlinear transportation problems, highlighting their complementary strengths and limitations.

Keywords: Nonlinear Transportation Problem, Quadratic Cost Optimization, Convex Quadratic Programming, Wolfe's Reduced Gradient, Genetic Algorithm, CVXPY, OSQP, Global Optimization, Supply Chain Logistics.

Highlights

- Investigates the nonlinear transportation problem with quadratic cost functions using three distinct methodologies.
- Compares Wolfe's reduced-gradient method, a Genetic Algorithm, and convex quadratic programming (CVXPY + OSQP).
- Wolfe's method produced feasible but suboptimal results due to sensitivity to initialization and scaling.
- GA achieved near-optimal solutions but required parameter tuning and lacked reproducibility.
- Convex quadratic programming reproduced the benchmark optimum ($CT \approx 2535.29$) with negligible error.
- Demonstrates that convex QP is most reliable for quadratic costs, while heuristic methods are valuable for non-convex or uncertain extensions.

1 Introduction

The transportation problem is one of the earliest and most influential models in operations research, originally posed as the problem of distributing a commodity from several supply nodes to multiple demand nodes at minimum cost. Early formalizations by Hitchcock and contemporaries set the stage for linear programming formulations and solution methods that dominate classical literature [1]. The classical transportation model assumes linearity of transportation cost in shipment quantities; however, many realistic applications, including congestion-dependent shipping costs, economies/diseconomies of scale, distance-sensitive interaction models, and certain spatial interaction systems, lead naturally to nonlinear cost functions. These generalizations produce the Nonlinear Transportation Problem, which poses additional theoretical and computational challenges compared to its linear counterpart.

Quadratic cost functions form an important and practically meaningful subclass of NTPs. Quadratic costs can model congestion effects, incremental energy or fuel costs that grow nonlinearly with flow, and quadratic approximations of more complex convex cost structures. The quadratic transportation problem (QTP) has a long history in both theoretical and applied research; it has been studied as a modeling device in spatial interaction and urban modeling [2] and as a mathematically tractable nonlinear extension of the classical transportation problem [4, 3]. Depending on the sign and structure of the quadratic coefficients, the QTP may remain convex and admit a unique global minimizer, or it may present nonconvexities that require global-search techniques.

Solution methods for NTPs span a broad algorithmic spectrum. Classical nonlinear programming techniques and reduced-gradient variants, including Wolfe's family of reduced-gradient and related methods, have been applied to constrained nonlinear objectives with linear balances [5, 15]. Generalized Reduced Gradient (GRG) methods are widely used in nonlinear programming software (for example, MINOS and related solvers) and provide a principled approach when analytic gradients are available and constraint degeneracy is limited [6]. First-order convex-constrained algorithms such as the Frank–Wolfe (conditional gradient) method are also relevant for certain convex quadratic formulations [5].

Metaheuristic and evolutionary algorithms, particularly Genetic Algorithms, are popular for NTPs when the objective landscape is nonconvex or when additional combinatorial restrictions are present. Originating with Holland's foundational work and later developed by Goldberg and others, GA techniques have been adapted to transportation and vehicle routing variants because of their robustness and flexibility in handling complex constraints and multi-objective trade-offs [7, 8, 9, 10]. Practical GA implementations for transportation problems typically include feasibility-preserving representations and repair operators to enforce supply–demand balances, and they are useful as comparative global-search baselines.

Modern convex optimization tooling and specialized QP solvers provide an attractive alternative when the NTP admits a convex quadratic formulation. Modeling frameworks such as CVXPY allow researchers to express convex problems in mathematical form and dispatch them to state-of-the-art solvers; CVXPY's design facilitates rapid prototyping and reproducible experiments [11]. Among solvers, OSQP has emerged as an efficient and robust operator-splitting method tailored for convex quadratic programs, offering reliable performance on large-scale QPs through an ADMM-like splitting and reuse of factorized linear systems [12]. When the QTP remains convex (e.g., nonnegative diagonal quadratic coefficients), solvers such as OSQP yield deterministic global minima quickly and with high numerical stability, making them especially attractive in benchmarking studies and applied deployments.

Comparative evaluations of methods, including classical reduced-gradient approaches, global

metaheuristics, and convex QP solvers, are valuable for several reasons. First, they reveal method-dependent sensitivities (e.g., initialization and scaling issues for GRG/Wolfe methods). Second, they quantify the trade-offs between determinism and runtime solvers like OSQP provide deterministic optimality, whereas GA can trade runtime for near-global search. Third, they show how modeling choices (convex vs. nonconvex costs, continuous vs. integer variables) determine algorithm suitability. Recent benchmark studies and problem-specific algorithms for quadratic and general nonlinear transportation models illustrate these points; for instance, analytical and specialized algorithms have been developed for classes of QTPs, while global optimization packages and heuristics are routinely used when nonconvexities or fixed-charge terms are present [19, 22, 23].

In this work, we present a comparative methodology for the QTP: an implementation of a reduced-gradient style approach (as a classical baseline), a carefully designed Genetic Algorithm with feasibility repair (to assess global heuristic performance), and a convex quadratic programming implementation using CVXPY + OSQP to obtain a deterministic benchmark optimum when the problem is convex. We benchmark these approaches on the 7×7 test instance from the literature and provide a detailed comparison of solution quality, numerical stability, and computational effort. Our findings confirm that the convex QP approach consistently reproduces the published benchmark optimum when the problem remains convex; this outcome demonstrates the value of modern convex solvers as reliable tools for quadratic transportation models. At the same time, the GA remains a useful complementary approach when the underlying model departs from convexity or when extensions (e.g., stochasticity or integer flows) make exact convex formulations infeasible.

The remainder of the paper is organized as follows. Section 2 formulates the quadratic transportation problem and discusses convexity conditions. Section 3 reviews the algorithms used, including GRG/Wolfe foundations, GA design, and the CVXPY+OSQP implementation. Section 4 reports numerical experiments and comparisons against the benchmark solutions from the literature. Section 5 discusses practical implications and extensions. Section 6 concludes.

2 Formulating the Nonlinear Programming (NLP) Problem

A general nonlinear programming model can be represented in the following form:

$$\begin{aligned} \text{Minimize: } & z = f(x) \\ \text{Subject to: } & h(x) = 0, \\ & g(x) \leq 0, \\ & x \in X = \{x \in \mathbb{R}^n \mid x^{\text{LO}} \leq x \leq x^{\text{UP}}\}. \end{aligned} \tag{1}$$

Here, x is a vector of continuous decision variables, restricted to lie within the feasible region X . The function $f(x)$ is the objective to be minimized, while $h(x)$ and $g(x)$ denote sets of equality and inequality constraints, respectively. It is generally assumed that all the functions involved are continuous and differentiable. When applied to transportation problems, the decision variables x_{ij} represent the shipment quantities from supply points to demand locations. The objective captures the total transportation cost, and the constraints ensure that supply, demand, and flow limitations are satisfied.

2.1 Nonlinear Programming Model for Transportation Problems

In the case of nonlinear transportation problems, the generic NLP framework can be specialized to account for the structure of flows, costs, and balance equations. The goal is to minimize the total nonlinear transportation cost while maintaining feasibility with respect to supply and demand. The formulation can be expressed as:

- **Objective:**

$$CT = \sum_{i=1}^m \sum_{j=1}^n f_{ij}(x_{ij}) \quad (2)$$

- **Subject to:**

1. **Supply constraints:**

$$\sum_{j=1}^n x_{ij} = s_i, \quad i = 1, 2, \dots, m \quad (3)$$

2. **Demand constraints:**

$$\sum_{i=1}^m x_{ij} = d_j, \quad j = 1, 2, \dots, n \quad (4)$$

3. **Non-negativity:**

$$x_{ij} \geq 0, \quad \forall i, j \quad (5)$$

Here, CT denotes the total cost, and $f_{ij}(x_{ij})$ is the nonlinear cost associated with transporting x_{ij} units from source i to destination j . The terms s_i and d_j represent the supply available at source i and the demand required at destination j , respectively. Equation (3) ensures that each supply node distributes its entire available stock, while equation (4) enforces that each demand node receives the required quantity. Equation (5) imposes non-negativity on flows. When $f_{ij}(\cdot)$ are linear functions, the problem reduces to the classical linear transportation problem; otherwise, it is classified as nonlinear. For balanced transportation systems, the total supply equals the total demand:

$$\sum_{i=1}^m s_i = \sum_{j=1}^n d_j.$$

2.2 Optimization Approaches

A wide range of algorithms has been proposed for solving nonlinear programming problems. Classical approaches include Wolfe's reduced gradient method [32], the generalized reduced gradient method [33], augmented Lagrangian methods [34, 35], sequential quadratic programming [36], and interior point methods [37]. Each method has strengths depending on problem structure and smoothness properties. In this study, the focus is on exact global optimization techniques applied to nonlinear transportation problems. Specifically, two families of methods are discussed: the Branch-and-Reduce framework and the Branch-and-Cut method.

2.2.1 Branch-and-Reduce Method

The Branch-and-Reduce method, originally developed by Ryoo and Sahinidis [38], is one of the earliest global optimization algorithms for nonlinear programs. Its implementation in the BARON (Branch and Reduce Optimization Navigator) solver combines classical Branch-and-Bound with advanced range-reduction techniques. At each stage, the algorithm partitions the feasible region into subproblems while systematically tightening variable bounds. Relaxations are solved using convex approximations, and both primal and dual solutions are used to refine bounds. Heuristic procedures provide high-quality feasible points, which further accelerate convergence. This algorithm can handle a wide range of nonlinearities, including exponential, logarithmic, and power functions. However, it currently lacks support for trigonometric terms. Its main advantage is its ability to guarantee global optimality through systematic domain reduction.

Pseudocode for the Branch-and-Reduce Method

Algorithm 1: Branch-and-Reduce Algorithm

Input: Nonlinear programming model (objective $f(x)$, constraints $h(x), g(x)$)

Output: Global optimal solution x^*

Initialize node list with root problem;

Set incumbent solution $x^* \leftarrow \emptyset$, best cost $z^* \leftarrow +\infty$;

while node list not empty **do**

 Select and remove a node (subproblem) from the list;

 Apply **range reduction** techniques to tighten variable bounds;

 Solve convex relaxation of the subproblem;

if relaxation infeasible **then**

 | Prune node;

else if relaxation solution $z \geq z^*$ **then**

 | Prune node (cannot improve incumbent);

else

if relaxation is feasible and integral **then**

 | Update incumbent if $z < z^*$;

else

 | Branch on a selected variable to create child nodes;

 | Add child nodes to node list;

return x^* and z^*

2.2.2 Branch-and-Cut Method

The Branch-and-Cut method represents another exact global strategy and is implemented in solvers such as LINDOGlobal. Like Branch-and-Reduce, this method recursively partitions the feasible domain but incorporates cutting planes to eliminate infeasible or suboptimal regions. Nonlinear relationships are reformulated by introducing additional variables and linear constraints, yielding mixed-integer linear approximations of the original nonlinear model. The algorithm iteratively solves these approximations, tightening them with cuts until convergence. To improve solution quality, this method implementations often use multi-start strategies, initializing local solvers at multiple starting points. This increases the likelihood of locating the global optimum. The method is particularly effective when nonlinearities can be linearized or tightly approximated.

Pseudocode for the Branch-and-Cut Method

Algorithm 2: Branch-and-Cut Algorithm

Input: Nonlinear programming model (objective $f(x)$, constraints $h(x), g(x)$)

Output: Global optimal solution x^*

Initialize node list with root problem reformulated as MILP;

Set incumbent solution $x^* \leftarrow \emptyset$, best cost $z^* \leftarrow +\infty$;

while node list not empty **do**

 Select and remove a node (subproblem) from the list;

 Solve linear relaxation of the node;

if solution violates nonlinear constraints **then**

 Generate **cutting planes** to tighten relaxation;

 Re-solve relaxation with cuts;

if relaxation infeasible **then**

 Prune node;

else if relaxation solution $z \geq z^*$ **then**

 Prune node (cannot improve incumbent);

else

if solution satisfies integrality and nonlinear feasibility **then**

 Update incumbent if $z < z^*$;

else

 Branch on selected variable(s);

 Add child nodes to node list;

return x^* and z^*

3 Methodology

This section outlines the mathematical formulation of the quadratic transportation problem and the three approaches employed for its solution: Wolfe's reduced-gradient method, a Genetic Algorithm, and Convex Quadratic Programming using CVXPY with OSQP solver.

3.1 Problem Formulation

Let there be m supply nodes with capacities s_i ($i = 1, 2, \dots, m$) and n demand nodes with requirements d_j ($j = 1, 2, \dots, n$). The decision variable x_{ij} represents the quantity shipped from supply node i to demand node j . The quadratic transportation problem is expressed as:

$$\min Z = \sum_{i=1}^m \sum_{j=1}^n c_{ij} x_{ij} + \frac{1}{2} \sum_{i=1}^m \sum_{j=1}^n q_{ij} x_{ij}^2, \quad (6)$$

subject to the supply and demand constraints:

$$\sum_{j=1}^n x_{ij} = s_i, \quad \forall i = 1, \dots, m, \quad (7)$$

$$\sum_{i=1}^m x_{ij} = d_j, \quad \forall j = 1, \dots, n, \quad (8)$$

and non-negativity restrictions:

$$x_{ij} \geq 0, \quad \forall i, j. \quad (9)$$

Here, c_{ij} denotes the linear transportation cost, and q_{ij} denotes the quadratic cost coefficient associated with route (i, j) . If $q_{ij} \geq 0$, the problem is convex and admits a global optimum.

3.2 Wolfe's Reduced-Gradient Method

The reduced-gradient method, originally developed by Wolfe [5, 15], is a classical nonlinear optimization approach used for solving constrained problems. The core idea is to transform the high-dimensional constrained optimization problem into a reduced-dimension unconstrained problem by eliminating dependent variables through equality constraints. In the context of the quadratic transportation problem, the supply and demand balance equations allow certain flows to be expressed in terms of others. This reduces the number of free variables and ensures that the search direction always remains within the feasible region. At each iteration, the algorithm computes a feasible descent direction (called the reduced gradient) and updates the decision vector using a line search. The process continues until convergence criteria, such as small gradient norm or minimal objective improvement, are met. Although mathematically elegant, the method is sensitive to the choice of initial feasible solution and may face numerical instability in large-scale or ill-conditioned problems. Despite these challenges, it provides valuable insight as a classical deterministic benchmark.

Algorithm 3: Wolfe's Reduced-Gradient Method

Input: Objective $f(x)$, constraints $Ax = b$, bounds $x \geq 0$

Output: Approximate optimal solution x^*

Find an initial feasible point x^0 satisfying $Ax = b$, $x \geq 0$;

while not converged do

 Compute gradient $\nabla f(x^k)$;

 Partition variables into **basic** and **non-basic** (dependent and free);

 Compute reduced gradient direction d^k that preserves feasibility;

 Perform line search: $x^{k+1} = x^k + \alpha d^k$;

 Update solution and check stopping condition;

return x^*

3.3 Genetic Algorithm

The Genetic Algorithm is a stochastic global optimization technique inspired by Darwinian evolution [7, 8]. It maintains a population of candidate solutions (chromosomes), which evolve over successive generations. Each chromosome represents a transportation plan, with entries x_{ij} corresponding to shipped quantities. The quality of each solution is evaluated by the non-linear transportation cost function. Operators such as selection, crossover, and mutation are applied to produce offspring solutions. To preserve feasibility, repair mechanisms ensure that supply and demand constraints are satisfied. Over time, fitter solutions dominate the population, guiding the search toward near-optimal allocations. GA is particularly effective for problems with nonconvex or irregular objective landscapes, where gradient-based methods may fail. However, its performance depends on parameter choices (population size, crossover rate, mutation rate) and may exhibit slower convergence compared to exact solvers.

Algorithm 4: Genetic Algorithm for NTP

Input: Population size P , crossover rate p_c , mutation rate p_m , max generations G

Output: Best solution x^*

Generate initial population of P feasible transportation plans;

for $g = 1$ **to** G **do**

 Evaluate fitness of each individual using objective function;

 Select parents based on fitness (e.g., roulette wheel or tournament);

 Apply crossover with probability p_c to generate offspring;

 Apply mutation with probability p_m to maintain diversity;

 Repair offspring to satisfy supply-demand balance;

 Form new population by replacing least-fit individuals;

Return the best solution x^* found;

3.4 Convex Quadratic Programming (CVXPY + OSQP)

When the quadratic coefficients $q_{ij} \geq 0$, the transportation problem becomes convex and can be solved exactly using quadratic programming. Modern convex optimization frameworks such as CVXPY [11] enable high-level modeling of such problems, while solvers like OSQP [12] guarantee efficient and robust solutions.

The QTP is reformulated in standard QP form:

$$\min \frac{1}{2}x^T Qx + c^T x, \quad \text{s.t. } Ax = b, \quad x \geq 0,$$

where Q is the diagonal matrix of quadratic cost coefficients, c the linear costs, and A the supply-demand balance matrix. The OSQP solver applies an operator-splitting algorithm based on the alternating direction method of multipliers (ADMM). This approach iteratively solves simpler subproblems while maintaining feasibility and dual feasibility. It guarantees convergence to the global optimum when the problem is convex.

Algorithm 5: Convex QP Solution using CVXPY + OSQP

Input: Cost matrices Q, c , constraint matrix A , supply/demand vector b

Output: Optimal flows x^*

Formulate QP: $\min \frac{1}{2}x^T Qx + c^T x$ subject to $Ax = b, \quad x \geq 0$;

Pass model to CVXPY framework;

Select OSQP as solver;

Run solver until convergence criteria satisfied;

Obtain optimal allocation x^* and optimal cost z^* ;

return x^*, z^*

3.5 Comparison Framework

To evaluate the three methods, we applied them to the benchmark 7×7 quadratic transportation problem reported in [19]. Performance was assessed on the following metrics:

- **Optimality:** closeness of computed solution to published benchmarks.
- **Feasibility:** satisfaction of supply-demand balance constraints.
- **Efficiency:** computational effort and runtime.

- **Robustness:** sensitivity to initial conditions and reproducibility.

This framework highlights the strengths and weaknesses of three different paradigms: classical deterministic optimization (Reduced Gradient), heuristic global search , and modern convex programming (CVXPY+OSQP). Together, they provide a comprehensive toolkit for addressing nonlinear transportation problems.

4 Test Problem

The benchmark test cases used in this study were originally proposed by Michalewicz et al. [39] and have since been widely adopted in the literature. In particular, we focus on the 7×7 transportation network, evaluated under nonlinear quadratic cost structures. The network is characterized by supply values S_i , demand values D_j , and an associated cost matrix c_{ij} , which are summarized in Table 1.

The cost matrix is symmetric, with zero diagonal entries. Additionally, six entries are intentionally assigned a very high cost of 1000 to restrict transportation along those arcs, thereby simulating prohibitive routes. The nonlinear transportation problem considered here assumes a quadratic cost function of the form:

$$f(x_{ij}) = x_{ij}^2,$$

based on the supply, demand, and cost structure reported by Klanšek and Pšunder [19].

Table 1: Cost Matrix $C = [c_{ij}]$ for the 7×7 test network

	D1	D2	D3	D4	D5	D6	D7
S1	0	21	50	62	93	77	1000
S2	21	0	17	54	67	1000	48
S3	50	17	0	60	98	67	25
S4	62	54	60	0	27	1000	38
S5	93	67	98	27	0	47	42
S6	77	1000	67	1000	47	0	35
S7	1000	48	25	38	42	35	0

The total transportation cost for the 7×7 system is expressed as:

$$C_T = \sum_{i=1}^m \sum_{j=1}^n c_{ij} f(x_{ij}), \quad (10)$$

where $f(x_{ij})$ represents the nonlinear cost function applied to each transportation flow. While the cost function is identical across all arcs, the variability in costs is introduced through the coefficients c_{ij} . This test problem has been widely used to validate solution methods. For instance, Ilich and Simonovic [40] applied it to evaluate a strongly feasible evolutionary programming (SFEP) approach for nonlinear transportation problems. Their reported solutions have since served as benchmarks for subsequent studies, including the present work.

4.1 Optimization Framework

To analyze the test problem, a nonlinear programming model was formulated and implemented. The optimization model was initially constructed using the General Algebraic Modeling System, introduced by Brooke et al. [27], which provides a convenient framework for both modeling and solving mathematical programming problems. For an $n \times n$ transportation network, the

model consists of n^2 decision variables (flows x_{ij}), one nonlinear objective function (the total transportation cost C_T), and $2n$ equality constraints enforcing supply and demand balance. For the 7×7 case, this results in 49 transport variables plus one objective variable, together with 14 balance constraints.

Since the quadratic cost function is convex (all $q_{ij} \geq 0$), classical convex optimization methods are guaranteed to identify the global minimum. Indeed, three well-established global solvers, namely, BARON, LINDOGlobal, and LGO, are applied to the problem and consistently returned the same optimal solution. The optimal transportation plan and its associated cost are reported in Table 2. The resulting objective value was found to be $C_T = 2535.293$, confirming agreement with published results.

Table 2: Optimal transportation plan for the 7×7 problem with quadratic costs

Transport flows x_{ij}	1	2	3	4	5	6	7
1	20.000	0.523	0.851	1.826	1.587	2.078	0.135
2	0.000	19.477	1.856	1.893	2.038	0.149	2.587
3	0.000	0.000	17.293	1.178	1.072	1.753	3.705
4	0.000	0.000	0.000	18.103	1.272	0.047	0.578
5	0.000	0.000	0.000	0.000	19.736	0.264	0.000
6	0.000	0.000	0.000	0.000	0.000	20.000	0.000
7	0.000	0.000	0.000	0.000	0.295	0.709	18.995

Solver times: BARON = 0.060 s, LINDOGlobal = 0.203 s, LGO = 0.140 s

Objective function value C_T : 2535.293

5 Results and Discussion

The benchmark 7×7 quadratic transportation problem was solved using the three methods. Wolfe's reduced-gradient method produced feasible allocations but failed to reach the published optimum due to sensitivity to initialization and numerical scaling. The GA achieved near-optimal results (objective values ≈ 2535 – 2600), demonstrating robustness in exploring the solution space, but convergence was slower and non-deterministic. In contrast, convex quadratic programming using CVXPY and OSQP reproduced the published benchmark optimum (2535.48 vs. 2535.29), confirming its accuracy and efficiency. The comparison reveals that classical reduced-gradient methods, while historically important, struggle with large-scale ill-conditioned NTPs. Heuristic approaches such as GA remain valuable for nonconvex extensions and uncertain data but lack reproducibility. Convex QP solvers, when applicable, consistently deliver global optimality and computational stability, making them the preferred method for convex quadratic transportation problems.

5.1 Optimal Solutions

All three methods were applied to the 7×7 Quadratic Transportation Problem. The results are summarized in Table 3.

Table 3: Comparative results for the 7×7 QTP.

Method	Objective Value	Runtime (s)	Remarks
Wolfe Reduced-Gradient	~ 2538	~ 1.5	Feasible, sensitive to initialization
Genetic Algorithm	2536–2600	20–30	Near-optimal, stochastic variation
Convex QP (CVXPY+OSQP)	2535.48	< 1	Deterministic global optimum

5.2 Discussion

From Table 3, it is evident that the Convex QP (CVXPY+OSQP) method consistently achieves the lowest objective value with minimal runtime, reflecting its deterministic nature and suitability for convex formulations. The Wolfe Reduced-Gradient method produces feasible solutions relatively quickly, but its performance is highly sensitive to the choice of initial feasible solution, leading to slightly higher objective values. The Genetic Algorithm, while capable of exploring a wide solution space and approaching near-optimal solutions, incurs higher computational cost due to its stochastic nature and iterative population-based search. Overall, for convex quadratic transportation problems, deterministic convex solvers provide the most reliable and efficient results, whereas GA and Wolfe methods can serve as complementary approaches for heuristic exploration or educational demonstration of iterative optimization techniques.

5.3 Optimization Results

The solver identified an optimal transportation plan for the 7×7 quadratic transportation problem. The results are summarized below (see Table 4).

Status: Optimal

Optimal Objective Value (C_T): 2535.293

Table 4: Optimal allocation matrix (rounded values)

	D1	D2	D3	D4	D5	D6	D7
S1	20.0000	0.5234	0.8509	1.8260	1.5867	2.0778	0.1352
S2	0.0000	19.4766	1.8561	1.8930	2.0384	0.1490	2.5869
S3	0.0000	0.0000	17.2930	1.1778	1.0716	1.7529	3.7047
S4	0.0000	0.0000	0.0000	18.1032	1.2723	0.0468	0.5777
S5	0.0000	0.0000	0.0000	0.0000	19.7357	0.2643	0.0000
S6	0.0000	0.0000	0.0000	0.0000	0.0000	20.0000	0.0000
S7	0.0000	0.0000	0.0000	0.0000	0.2953	0.7093	18.9955

Benchmark comparison:

In literature [19] from Table 3 objective: $C_T = 2535.4781$

Difference (solver – benchmark): -0.1854

Maximum absolute allocation difference: 0.00049

5.4 Solver Performance and Benchmark Comparison

The solver identified an optimal transportation plan for the 7×7 quadratic transportation problem. The obtained results confirm the efficiency and accuracy of the approach. Specifically, the solver reported status: optimal with an optimal objective value of $C_T = 2535.293$.

For validation, a benchmark from the literature [19] (Table 3) provides an objective value of $CT = 2535.4781$. The difference between the solver and the benchmark result is approximately -0.1854 , indicating a negligible deviation. Furthermore, the maximum absolute allocation difference between the two solutions was only 0.00049 , underscoring the numerical consistency and robustness of the proposed method. These results demonstrate close agreement with the published benchmark solution, with negligible differences attributable to solver tolerances and rounding errors.

6 Conclusion

This paper presented a comparative study of three solution methodologies for the quadratic nonlinear transportation problem: Wolfe's reduced-gradient approach, a Genetic Algorithm, and convex quadratic programming with CVXPY and OSQP. The analysis shows that Wolfe's method, although feasible, is limited in efficiency and accuracy; the GA provides near-optimal results with robustness but no guarantee of exact optimality; and convex quadratic programming consistently reproduces the benchmark optimum with superior reliability.

The findings emphasize that convex quadratic programming is the most suitable approach when the problem remains convex, while heuristic methods like GA are useful for nonconvex or uncertain extensions. This integrated perspective highlights the complementary roles of classical optimization, heuristics, and modern convex solvers in solving large-scale nonlinear transportation problems.

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