Coloring Edge Connectivity of Fuzzy Graph

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***Abstract*— Fuzzy graphs were introduced by Rosenfeld in 1975. Fuzzy graph theory has numerous applications in modern science and technology, especially in the fields of information theory, neural networks, expert systems, cluster analysis, medical diagnosis, control theory etc. In this paper, we deal with coloring of the edge connectivity of a fuzzy graph to obtain the better result in diagnosis of cancer cells.**

***Keywords*— Fuzzy graphs; fuzzy bonds; Fuzzy k-coloring; Fuzzy edge connectivity.**

I. INTRODUCTION

The fuzzy graph approach is more powerful in cluster analysis than the usual graph-theoretic approach due to its ability to handle the strengths of edges effectively. As in graphs, connectivity concepts play a key role in applications related with fuzzy graphs [1]. Fuzzy graphs were introduced by Rosenfeld [2] and Yeh and Bang [3] independently in 1975. Rosenfeld in his paper "Fuzzy Graphs" presented the basic structural and connectivity concepts while Yeh and Bang introduced different connectivity parameters of a fuzzy graph and discussed their applications in the paper titled "Fuzzy relations, Fuzzy graphs and their applications to clustering analysis" [3]. In [4] the authors have defined the concepts of strong arcs and strong paths. Also, in modern fuzzy graph theory, we have the notions of strong as well as strongest path [4] between any pair of vertices and a fuzzy edge cut can be viewed as a set of strong edges whose removal from G reduces the strength of connectedness between some pair of vertices of G, at least one of them differing from the end vertices of edges in the cut. The edges which are not strong need not be considered because the flow through such edges can be redirected through a different path having more strength. In the same manner we have to coloring the edge connectivity to diagnosis the cancer cells for better result.

II. FUZZY GRAPH

Definition 2.1. *Let V be a non-empty set, A fuzzy graph is a pair of functions* $G:(\sigma,\mu)$ *where* σ *is a fuzzy subset of V and* μ *is a symmetric fuzzy relation on* σ*. i.e.,* $\sigma: V \to [0,1]$ *and* $\mu: V\times V \to [0,1]$ *such that* $\sigma(u)\wedge\sigma(v) \geq \mu(u,v)$ *for all u, v in V.*

Definition 2.2. *The fuzzy graph* $H:(\tau,\nu)$ *is called a partial fuzzy sub graph of* $G:(\sigma,\mu)$ *if* $\tau\leq\sigma$ *and* $\nu\leq\mu$*. In particular, we call* $H:(\tau,\nu)$ *a fuzzy sub graph of* $G:(\sigma,\mu)$ *if* $\tau(u)=\sigma(u)\forall u\in\tau^*$ *and* $V(u,v)=\mu(u,v)\forall(u,v)\in\nu^*$.

Definition 2.3. *A fuzzy graph* $G:(\sigma,\mu)$ *is strong if* $\sigma(u)\wedge\sigma(v)=\mu(u,v)\forall u,v\in\mu^*$ *and is complete if* $\sigma(u)\wedge\sigma(v)=\mu(u,v)\forall u,v\in\sigma^*$.

Note that every complete fuzzy graph is strong but not conversely.
Also if $G:(\sigma,\mu)$ is a complete fuzzy graph than $G^*:(\sigma^*,\mu^*)$ is a complete graph.

Definition 2.4. *A path P in a fuzzy graph* $G:(\sigma,\mu)$ *is a sequence of distinct vertices* $u_0,u_1,u_2,\ldots,u_n$ *such that* $\mu(u_{i-1},u_i)>0,\ 1\leq i\leq n.$

Here $n\geq 1$ is called the length of the path P. A single vertex u may also be considered as a path. In this case the path is of length 0. The consecutive pairs (u_{i-1},u_i) are called edges of the path. We call P a cycle if $u_0=u_n$ and $n\geq 3$.

Definition 2.5. *The strength of a path P is defined as* $\bigwedge_{i=1}^{n}\mu(u_{i-1},u_i)$*. In other words, the strength of a path is defined to be the degree of membership of a weakest edge of the path. If the path has length 0, it is convenient to define its strength to be* $\sigma(u_0)$.

Definition 2.6. *Let* $G:(\sigma,\mu)$ *be a fuzzy graph. The strong degree of a vertex* $v\in\sigma^*$ *is defined as the sum of membership values of all strong edges incident at* v*. It is denoted by* $D_s(v)$*. Also if* $n_s(v)$ *denote the set of all strong neighbours of* v*, then* $D_s(v)=\sum_{u\in n_s(v)}\mu(u,v)$.

Example-2.7.
$\sigma(u_1)=0.7,\sigma(u_2)=1,\sigma(u_3)=0.6,$
$\sigma(u_4)=0.8,\sigma(u_5)=0.9$ *and*
$\mu(u_1,u_2)=0.3,\mu(u_1,u_3)=0.4,\mu(u_2,u_3)=0.2,$
$\mu(u_2,u_4)=0.7,\mu(u_2,u_5)=0.8,$
$\mu(u_3,u_5)=0.5,\mu(u_4,u_5)=0.6.$

Since the values satisfy $\mu(u,v) \le \sigma(u) \wedge \sigma(v)$ which is a fuzzy graph a strongest path joining u_2 *and* u_5 is the path

$P: u_2, u_4, u_5$ with $\mu^{\infty}(u_2,u_5) = 0.6$

$\mu^{\infty}(u_1,u_2) = \mu^{\infty}(u_1,u_4) = \mu^{\infty}(u_1,u_5) = \mu^{\infty}(u_1,u_3) = 0.3$

$\mu^{\infty}(u_2,u_3) = 0.2, \mu^{\infty}(u_2,u_4) = 0.7, \mu^{\infty}(u_2,u_5) = 0.6$

$\mu^{\infty}(u_3,u_5) = 0.5 \text{ and } \mu^{\infty}(u_4,u_5) = 0.6.$

III. FUZZY EDGE CONNECTIVITY

In [3], the notion of edge connectivity of a fuzzy graph if defined as given below. As mentioned in the introduction this definition is more close to a graph rather than a fuzzy graph since, in a fuzzy graph the concept of strength of connectedness plays a crucial role.

Definition 3.1. *Let G be a fuzzy graph and* $\{V_1, V_2\}$ *be a partition of its vertex set. The set of edges joining vertices of* V_1 *and vertices of* V_2 *is called a cut-set of G, denoted by* (V_1, V_2) *relative to the partition* $\{V_1, V_2\}$. *The weight of the cut-set* (V_1, V_2) *is defined as* $\sum_{u \in V_1, v \in V_2} \mu(u,v)$.

Definition 3.2. *Let G be a fuzzy graph. The edge connectivity of G denoted by* $\lambda(G)$ *is defined to be the minimum weight of cut-sets of G.*

Definition 3.3. *A one fuzzy edge connectivity is called a fuzzy bond.*

Remark 3.4. Note that fuzzy bonds are special type of fuzzy bridges. Note all fuzzy bridges are fuzzy bonds.

Example 3.5.

Let $G:(\sigma,\mu)$ be with $\sigma^* = \{a,b,c,d,e\}$ with

$\mu(a,b) = 0.4, \mu(a,c) = 0.2, \mu(b,c) = 0.3,$

$\mu(c,d) = 0.5, \mu(d,a) = 0.7, \mu(a,e) = 0.6,$.

$\mu(b,e) = 0.5$

There are four fuzzy bonds in this fuzzy graph namely edges (a, d), (a, e), (d, c) and (e, b).

IV. COLORING FUZZY EDGE CONNECTIVITY

In a crisp graph G = (V, E), a coloring function C assigns an integer value $c(i)$ to each vertex $i \in V$ in such a way that the extremes of any edge $(i, j) \in E$ cannot share the same color,

i.e., $c(i) \neq c(j)$. In a fuzzy graph $G = (V, \mu)$, its chromatic number is the fuzzy number $\chi(G) = \{(x, v(x)) / x \in X\}$ where

$X = \{1, \ldots, |v|\}, v(x) = \sup\{\alpha \in I / x \in A_\alpha\} \forall x \in X$ and

$A_\alpha = \{1, \ldots, \chi_\alpha\} \forall \alpha \in I.$

Definition 4.1. *A family* $\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_k\}$ *of fuzzy sets on X is called a k-fuzzy coloring of* $G = (X, \sigma, \mu)$ *if*

(i) $\vee\Gamma = \sigma,$

(ii) $\gamma_i \wedge \gamma_j = 0,$

(iii) for every strong edge xy *of G,* $\min\{\gamma_i(x), \gamma_i(y)\} = 0 \ (1 \le i \le k).$

The least value of k for which G has a k-fuzzy coloring, denoted by $\chi_F(G)$, is called the fuzzy chromatic number of G.

Definition 4.2. *A family* $\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_k\}$ *of fuzzy sets on* $X \cup E$ *is called a k-fuzzy total coloring of* $G = (V, \sigma, \mu)$ *if*

(i) $\max_i\{\gamma_i(x)\} = \sigma(x)$ *for all* $x \in X$ *and* $\max_i\{\gamma_i(xy)\} = \mu(xy)$ *for all edges* $xy \in E$.

(ii) $\gamma_i \wedge \gamma_j = 0,$

(iii) For every adjacent vertices x, y of $\min\{\gamma_i(y_j y_k) / y_j y_k\} = 0,$ *(are set of incident edges from the vertex* x_j*)* $j = 1, 2, \ldots, |x|$.

Definition 4.3. *Let G be a fuzzy graph. The coloring edge connectivity of G denoted by* $\lambda_F(G)$ *is defined to be the minimum weight of cut-sets of G. Given a fuzzy graph* $G = (V, \sigma_F, \mu_F)$ *its edge chromatic number in fuzzy number* $\lambda_F(G) = \{x_\alpha, \alpha\}$ *where* x_α *is the edge connectivity chromatic number of* G_α *and ∝ values are the different membership value of vertex and edge of graph G.*

IV. ILLUSTRATION: CANCER DETECTION PROBLEM

In our human body, based on the location of the cells in the low magnification image of a tissue sample, surgically removed from a patient, it is possible to construct a graph with G with vertices as cells, called cell graph [5]. By analyzing the physical features of the cells; for example color and size, we can assign a membership value to the vertices of G. This value will range over (0, 1] depending of the nature of the cell; that is healthy, inflammatory or cancerous. Also, edges of G can assign a membership value based on the distances between the cells. Thus the cell graph can be converted to a fuzzy graph in this manner.

Applying the above clustering procedure to such a fuzzy graph, the cancerous cell clusters can be detected at the cellular level in principle. This process, classifies cell clusters in a tissue into different phases of cancer, depending of the distribution, density and fuzzy connectivity of the cell clusters within the tissue using coloring the edge connectivity.

Assume that the vertices with weights more than 0.6 represent cancerous cells, edges with weights between 0.3 and 0.5 inflammatory cells and between 0 and 0.4 healthy cells. Let the vertex set of G be $\{v_1, v_2, v_3, v_4, v_5\}$

Let $G:(\sigma, \mu)$ be an fuzzy graph with

$$\sigma^* = \{v_1, v_2, v_3, v_4, v_5\} \, with$$

and

$$\sigma(v_1) = \sigma(v_2) = \sigma(v_3) = \sigma(v_4) = \sigma(v_5) = 0.8$$

$$\mu(a,b) = \mu(a,c) = 0.4,$$

$$\mu(b,d) = \mu(d,e) = \mu(c,e) = 0.6$$

$$\mu(b,c) = 0.3, \ \mu(c,d) = 0.5$$

using coloring edge connectivity

$$\sigma(x_i) = 0.8 \ for \, i = 1, \ldots, 5$$

$$\mu(x_i, y_j) = \begin{cases} 0.4 \ for \, ij = 12, 13 \\ 0.6 \ for \, ij = 24, 45, 35 \\ 0.3 \ for \, ij = 23 \\ 0.4 \ for \, ij = 34 \end{cases}$$

Level	Maximal edge connectivity	Clusters
(0, 0.6]	{1, 2, 3}	C_1={1, 2, 3}
[0.6, 1]	{2}, {4}, {5}	C_2={2}, C_3={4}, C_4={5}

After applying the coloring edge connectivity of fuzzy graph G is $\lambda_F(G)$=0.6. Using the Yeh and Bang procedure in coloring we obtain the level as given below

From the above clusters corresponding to $\lambda_F(G)$=0.6, it is observed that {4, 5} is a cell cluster which is affected seriously by cancer whereas its neighbouring area containing the cells {2} and {3} can be found inflammatory. Note that the cell {1} is healthy.

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