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Coaxial Hyperplanes

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***Abstract*: For Euclidean geometry, the basis vectors are the source of inspiration for mutually perpendicular/orthogonal concepts that rules over the world in the form of tangent, normal and binormal or the rectangular co-ordinate system. So, the Pythagoras theorem is valid in 3 – dimensions. But when the 4^{th} and higher dimensional space does not obey Pythagoras and thus the higher ideas like 'norm' , 'ortho – normal basis' and more are required. So, the plane in a Euclidean geometry becomes a 'Hyperplane' in the *n* – dimensional space for *n* > 3. Pivoting in the linear programming or the construction of ortho-normal basis using Gram – Schmidt's process and the p^{th} norm are some of the methods that scales to the required heights. In the astronomical discussions, the planets in the solar system will come as close as possible and the distance required to go round and reach back is an idea or the thought source in which the shortest distance between two hyperplanes is presented.**

1. INTRODUCTION:

Hyper Plane:

In an *n* – dimensional linear space, how many hyper – planes would be there, of *n* – 1 – dimensional those form a family of symmetric subspaces?

Introduction: planes through origin only will be subspaces. But the collection of parallel planes cannot form the collection of subspaces while one and only one plane will have the origin on it. For instance,

$$a_1x_1 + a_2x_2 + ... + a_{n-1}x_{n-1} + a_nx_n = c \qquad (1.1)$$

will be a hyperplane in *n* – dimensional space $V_n(F)$ for *c* is any constant and when *c* = 0,

the set of vectors $x_1, x_2, ..., x_n$ that satisfy the condition (1.1) will form the subspace showing $a_1 \neq 0, a_2 \neq 0, a_{n-1} \neq 0, a_n = 0$ whenever $c \neq 0$. In the case of *c* = 0, if $a_i = 0 \forall 1 \le i \le n$, then $\{x_1, x_2, ..., x_{n-1}\}$ will be the basis of the subspace which is a hyperplane through the origin.

See that $\pi_j = \alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_n x_n = 0$, $\alpha_k, 1 \le k \le n, k \ne i$ (1.2)

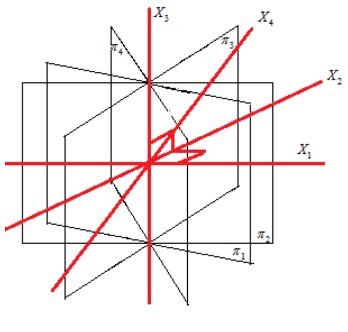
are the hyper planes that contain the X_i – axis as the axis of intersection or common axis or the axis of rotation for different

non zero values of α_k and $\left(\sum_{j=1}^{n} \alpha_j^{\,n}\right)^{1/n} = 1$ (1.3)

Definition: a family of hyperplanes through a common line in $V_n(F)$ is a co-axial family of hyperplanes. (1.4)

There are infinitely many co-axial families of hyperplanes through origin. A class of co-axial families will have co-ordinate axes as the axes of symmetry and other families will have lines through origin as axes of symmetry.

Definition: a non – homogeneous co-axial family of hyperplanes are the family that have the line of symmetry not through the origin. (1.5)



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There will be a unique plane through origin in the family of non – homogeneous co-axial family of hyperplanes. (1.6)

The X_i - axis can be represented by $X_1 = 0, X_2 = 0, \ldots, X_{i-1} = 0, X_{i+1} = 0, \ldots, X_n = 0$ (1.7)

2. ***Line of Intersection of Hyper Planes***:

Two hyper planes $\pi_r = \alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n = 0$ and $\pi_s = \beta_1 x_1 + \beta_2 x_2 + \ldots + \beta_n x_n = 0$ are clearly having the common line X_i and so are not coincident if and only if

$$\frac{\alpha_k}{\beta_k} \neq \frac{\alpha_m}{\beta_m} \quad (2.1)$$

for at least one distinct pair *k*, *m* among 1, 2, ...,*i* – 1, *i* +1, ...,*n*.

Any line not coincident with any coordinate axis will be of the axis of rotation for the family of the hyper planes

$$\left\{\pi_j = \alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_{i-1} x_{i-1} + \alpha_i x_i + \alpha_{i+1} x_{i+1} + \ldots + \alpha_n x_n = 0\right\} \quad (2.2)$$

If a line λ is not through origin in $V_n(F)$, then it can be parametrized as

$$x_1 = \alpha_1 t + \beta_1, x_2 = \alpha_2 t + \beta_2, \ldots, x_n = \alpha_n t + \beta_n \quad (2.3)$$

Two lines λ_1 defined by $x_1 = \alpha_1 t + \beta_1, x_2 = \alpha_2 t + \beta_2, \ldots, x_n = \alpha_n t + \beta_n$ and λ_2 defined by $x_1 = \gamma_1 t + \delta_1, x_2 = \gamma_2 t + \delta_2, \ldots, x_n = \gamma_n t + \delta_n$ that are not intersecting if $\frac{\alpha_k}{\gamma_k} \neq \frac{b_k}{\delta_k}$ for at least one *k* such that $1 \le k \le n$.

Definition: If λ_1 and λ_2 are the lines on two different members of a family of hyperplanes that do not intersect, then they are ***skew hyper lines*** on hyperplanes. (2.4)

Theorem: Two skew hyper lines in different co-axial families will be on a unique pair of parallel hyperplanes and so, the distance between the planes will be the shortest distance between the lines. (2.5)

Proof: there is a unique position in which the line λ_1 is in the plane

$\pi_1 = \alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_{i-1} x_{i-1} + \alpha_{i+1} x_{i+1} + \ldots + \alpha_n x_n = 0;$ and the line λ_2 is in

$\pi_2 = \beta_1 x_1 + \beta_2 x_2 + \ldots + \beta_{i-1} x_{i-1} + \beta_{i+1} x_{i+1} + \ldots + \beta_n x_n = c$, $c \neq 0$ such that $\frac{\alpha_k}{\beta_k} = \frac{\alpha_m}{\beta_m}$ for all $1 \le k, m \le n$

Since λ_1 and λ_2 are the skew hyper lines, at one instance, these lines will be as close as possible and the next moment, the distance between them will be more and keep increasing between any two points, each is on either the lines.

In other words, there is a unique pair of points $p_1 \in \lambda_1$ and $p_2 \in \lambda_2$ such that $\left\| p_2 - p_1 \right\|_n$ will be the shortest distance between the given lines.

It can be followed that the line of minimum distance between λ_1 and λ_2 is $\lambda_3 = \overleftrightarrow{p_1 p_2}$ such that λ_3 is normal to both λ_1 and λ_2.
So, the line λ_3 is through the feet of the normal that are the points of intersection of the lines namely $p_1 \in \lambda_1$ and $p_2 \in \lambda_1$
Incidentally $\lambda_1 \in \pi_1$ and $\lambda_2 \in \pi_2$ where π_1 is in one co – axial system and π_2 is in the other co – axial system such that π_1 & π_2 will be parallel.
The distance between the feet of the normal skew hyper line is the distance between the parallel planes π_1 and π_2.
The minimum distance possible between the planes π_1 & π_2 the length of the skew hyper line of shortest distance is

$$\frac{|c|}{\left(\sum_{j=1}^{n}\left(\alpha_j - \beta_j\right)^n\right)^{1/n}}, 1 \le i \le n \quad (2.6)$$

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Note: if both co – axial systems are not through origin, then the shortest distance between the skew hyper lines

$$\frac{\left\| d_1 - d_2 \right\|}{\left[\sum_{j=1}^{n} \left(\alpha_j - \beta_j \right)^n \right]^{1/n}} \qquad (2.7)$$

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