β*- Closed And β*Open Maps In Topological Spaces

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Abstract

The aim of this paper is to introduce the notions of β*-closed maps, β*-open maps and semi-β*closed maps .Their relationships with other closed maps are investigated..It is found that the concept of β*closed maps-are stronger than the concept of gspclosed maps.However it is weaker than closed maps. It is shown that the composition of β*closed maps need not be β*closed .The applications of these maps in some topological spaces are also studied.Also ultra β*regular space and ultra β*-normal spaces are introduced.

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1. INTRODUCTION

T.Noiri,H.Maki and J.Umehara [9] introduced the concept of gp-closed and pre gp-closed map using gp-closed sets.Lellis Thivagar [6] introduced the concept of quasi α-open and strongly α-open map mappings using α-open sets.Here we have introduced the concept of β*-closed maps and semi β*closed maps using β*closed sets.Their respective open maps are also introduced

2.PRELIMINARIES

Throughout this paper spaces(X ,τ)and (Y,σ) mean topological spaces and f: X→Y represents a single valued map .The following definitions and Theorems are useful in the sequel.

Definitions 2.1

A subset A of a topological space (X, τ) is called

(i) A semi-open set [7] if A⊂cl(int(A)) and a semi-closed set if int(cl(A))⊂A,

(ii) An α-open set[6] if A⊂int(cl(int(A))) and an α-closed set if cl(int(cl(A)))⊂A,

(iii) A semipre open set [2] (= β-open set [1]) if A⊂cl(int(cl(A))) and a semi-pre closed set(=β closed) if int(cl(int(A)))⊂A and

The intersection of all semi-closed subsets of (X, τ) containing A is called the semi-closure of A and is denoted by scl (A). Also the intersection of all α closed (resp. semi-pre closed) subsets of (X, τ) containing A is called the α closure (resp. semi-pre closure) of A and is denoted by αcl(A) (resp. spcl(A)).

Definition-2.2

A subset A of a topological space (X, τ) is called

(i) A generalized closed set (briefly g-closed)[8] if cl(A) ⊂ U whenever A ⊂U and U is open in (X, τ).

(ii) A generalized semi-pre closed set (briefly gsp-closed) [5] if spcl(A) ⊂ U whenever A ⊂ U and U is open in (X, τ).

(iii) An ω-closed set [10] if cl(A) ⊂ U whenever A ⊂ U and U is semi-open in (X, τ)

(iv) An η̂* closed set [3] if spcl(A)⊂U whenever A⊂U and U is ω-open in (X, τ)

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(v) A β*-closed set [4] if spcl(A) ⊂int(U) whenever A⊂U and U is ω-open

The complement of g-closed(resp-gsp-closed, ω-closed, η̂*closed,β*closed) set is said to be g open (resp. gsp-open, ω-open, η̂*-open,β*open).

Definition-1.1.4: A topological space(X, τ) is

(i) A T_ω – space [10] if every ω-closed subset of (X, τ) is closed (X, τ)

Definition 2.3

A map f: X→Y is said to be

(i) g closed [8] (resp. g-open) if f(V) is g-closed (resp. g-open) in (Y, σ) for every closed (resp.open) set V (X, τ)

(ii) ω-closed [10] (resp.ω-open) if f(V) is ω-closed in (Y, σ) for every set V of (X, τ)

(iii) gsp-closed (resp.gsp-open)[5] if f(V) is gsp-closed in (Y, σ) for every closed set V of (X, τ)

(iv) η̂*-closed(resp.η̂*-open) [3] if f(V) is η̂*-closed in (Y, σ) for every closed set V of (X, τ)

Theorem 2.4[4]: (i) Every β*closed set is gsp closed(resp. η̂*-closed) set.

(ii) β*closed set is independent of gclosed(resp. ω-closed) set.

(iii) In a topological space X if the set of all β*-open sets is closed under any union then β*cl(A) is a β*-closed set for every subset A of X.

3. β*-CLOSED MAPS:

Definition -3.1.1: A map f: X→Y is said to be β*-closed, if the image of every closed set of X is β*-closed in Y.

Definition -3.1.2: A map f: X→Y is said to be semi β* closed if the image of every semi closed set of X is β*-closed in Y.

Remark – 3.1.3 If f: X→Y is closed (semi β*-closed), then f is β*-closed since every closed (resp,semi closed) sets are β*-closed .However the converses are not true. The following examples prove them.

Example-3.1.4: Let X=Y={a, b, c}, τ={φ, {a},{b}, {a, b},X} and σ={φ, {a}, {b, c},Y}. The identity map f: (X, τ)→(Y, σ) is β*-closed but not closed. Since {c} is closed in X but f({c})= {c} is not closed in Y.

Example – 3.1.5: Let X=Y={a, b, c}, τ={φ, {a},{b}, {a, b},X} and σ={φ,{a}, Y}. The identity map f: (X, τ)→(Y, σ) is β*-closed but not semi β*-closed. Since {a} is semi closed in X but f({a})= {a} is not β* closed in Y.

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Proposition – 3.1.6: Every β*-closed map f: (X, τ)→(Y, σ) is a gsp closed (resp η̂* closed) map.

Proof: Since every β*-closed set is a gsp closed set (resp. η̂* closed set) the proof follows by Theorem 2.4(i).

However the converses are not true. It can be seen through the following example.

Example -3.1.7 : Let X={a, b, c} = Y, τ={φ, {a}, {a, b},x} and σ={φ, {a}, {b, c}, Y}.Let f: (X, τ)→(Y, σ) be the identity map. Then f is a η̂* closed and a gsp closed map but f is not β*-closed, since for the closed set {c}, f({c})={c} is not β*-closed in Y.

Proposition – 3.1.8: The concept of g closed map (resp. ω-closed map) and β*-closed map are independent.

Proof: Follows from Theorm 2.4(ii) using the fact that β*-closed sets are independent of g closed sets and ω-closed set.

Theorem – 3.1.9: A surjective map f: X→Y is β*-closed if and only if for each subset S of Y and each open set U containing $f^{-1}(S)$, there exists a β*-open set V of Y such that $S \subset V$ and $f^{-1}(V) \subset U$.

Proof: Suppose f is β*-closed. Let S be any subset of Y and U be an open set of X containing $f^{-1}(S)$. If we let $V=(f(U^c))^c$ then V is β*open in Y containing S and $f^{-1}(V) \subset U$.Conversely let F be any closed set of X. Let $B = (f(F))^c$,then we have $f^{-1}(B) \subset F^c$ and F^c is open in X. By hypothesis there exists a β*-open set V of Y such that $B \subset V$ and $f^{-1}(V) \subset F^c$ and so $F \subset (f^{-1}(V))^c = f^{-1}(V^c)$. Therefore, we obtain $f(F) = V^c$. Since V^c is β*closed, f(F) is β*-closed in Y. This gives f is β*-closed.

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Remark -3.1.10: The following example shows that composition of two β*-closed maps is not β*-closed.

Example -3.1.11: Let X=Y=Z= {a, b, c}, τ={φ, {a},{b, c}, X} and σ={φ, {a},{b}, {a, b },Y} and η ={φ,{a}, {a, b}, z}. Let f: (X, τ)→(Y, σ) and g:(Y,σ)→(Z,η) be identity maps. Then f and g are β*-closed maps but g∘f: X→Z is not β*-closed. Since {a} is closed in X but gof({a}) = g(f({a})) = g({a}) ={a} is not β*-closed in Z.

Proposition – 3.1.12: If f: X→Y and g: Y→Z are β*-closed maps with Y as a T_{β^*} space then gof: X→Z is a β*-closed map.

Proof: Obvious from the definitions.

3.2: β* - OPEN MAPS

Here β*-open maps in topological spaces have been introduced and also obtained the characterizations of these maps.

Definition –3.2.1: A map f: X→Y is said to be β*-open map if the image f(A) is β* open in Y for every open set A in X.

Theorem -3.2.2: Every β*-open map is a gsp open (resp.$\hat{\eta}$*) map but not conversely.

Proof: Since every β*-open sets is a gsp open set (resp.$\hat{\eta}$* open set) the proof follows.

Example-3.2.3: Let X=Y= {a,b,c}, τ={φ,{a}, {a,b}, X} and σ={φ, {a,b}, Y}. Let f:X→Y be the identity map. Then f is β*-open.

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Theorem – 3.2.4: For any bijection f: X→ Y the following statements are equivalent.

1. f^{-1}: Y→X is β*-continuous
2. f is a β*-open map and
3. f is a β*-closed map.

Proof: 1⇒2: Let U be an open set of X. By assumption $(f^{-1})^{-1}(U)$ = f(U) is β*-open in Y and so f is a β*-open map.

2⇒3: Let V be a closed set of X. Then V^c is open in X. By assumption $f(V^c)$ = $(f(v))^c$ is β*-open in Y and therefore f(V) is β*-closed in Y. Hence f is a β*-closed map.

3⇒1: Let V be a closed set of X. By assumption f(V) is β*-closed in Y. But f(V)= $(f^{-1})^{-1}(V)$ and therefore f^{-1} is β* continuous.

Theorem – 3.2.5: Let f: $X \to Y$ be a mapping. If f is a β*-open mapping, then for each $x \in X$ and for each neighbourhood U of x in X, there exists a β* neighbourhood W of f(x) in Y such that $W \subset f(U)$.

Proof: Let $x \in X$ and U be an arbitrary neighbourhood of x. Then there exists an open set V in X such that $x \in V \subseteq U$. By assumption f(V) is a β*-open set in Y. Further $f(x) \in f(V) \subseteq f(U)$. Clearly f(U) is a β*-neighbourhood of f(x) in Y and so the theorem follows if we take W=f(V).

Theorem 3.2.6: A function f: $X \to Y$ is β*-open if and only if for any subset B of Y and for any closed set S containing $f^{-1}(B)$, there exists a β*-closed set A of Y containing B such that $f^{-1}(A) \subseteq S$.

Proof: Similar to the proof of theorem-3.1.9

3.3.Ultra β*-regular and ultra β*-normal spaces.

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Definition – 3.3.1 A space X is said to be ultra β*-regular if for each closed set, F of X and each point $x \notin F$ there exist disjoint β*-open sets U and V such that $F \subset U$ and $x \in V$.

Theorem – 3.3.2: In a topological space X, assume that $\beta^*o(\tau)$ is closed under any union. Then the following statements are equivalent.

a) X is ultra β*-regular.
b) For every point x of X and every open set V containing x, there exists a β* - open set A such that $x \in A \subset \beta^*cl(A) \subset V$.

Proof:

$a \Rightarrow b$. Let $x \in X$ and V be an open set containing x. Then V^c is closed and $x \notin V^c$ By (a) there exist disjoint β*-open sets A and B such that $x \in A$ and $V^c \subset B$. that is $B^c \subset V$. Since every open set is β*-open, V is β*-open. Since B is β*-open, B^c is β*-closed. Therefore $\beta^*cl(B^c) \subset V$. Also since $A \cap B = \varphi$. $A \subset B^c$. Therefore $x \in A \subset \beta^*cl(A) \subset \beta^*cl(B^c) \subset V$. Hence, $x \in A \subset \beta^*cl(A) \subset V$.

b⇒a: Let F be a closed set and $x \notin F$. This implies that F^c is an open set containing x. By (b) there exists a β*-open set A such that $x \in A \subset \beta^*cl(A) \subset F^c$. That is $F \subset (\beta^*cl(A))^c$. By Theorem 2.4(iii) β*cl(A) is β*-closed. Hence $(\beta^*cl(A))^c$ is β*-open. Therefore, A and $(\beta^*cl(A))^c$ are the required β* - open sets.

Theorem – 3.2.3: Assume that β*o(τ) is closed under any union. If f: X→Y is a continuous open β*-closed surjective map and X is a regular space, then Y is ultra β*-regular.

Proof: Let $y \in Y$ and V be an open set containing y of Y. Let x be a point of X such that y=f(x). Since f is continuous, $f^{-1}(V)$ is open in X. Since X is regular there exists an open set U such that $x \in U \subset cl(U) \subset f^{-1}(V)$. Hence $y=f(x) \in f(U) \subset f(cl(U)) \subset V$. Since f is a β*-closed map f(cl(U)) is a β*-closed set contained in the open set V. Therefore $\beta^*cl(f(cl(U))) = f(cl(U)) \subset V$. This implies that $y \in f(U) \subset \beta^*cl(f(U)) \subset \beta^*cl(f(cl(U))) \subset V$. Since f is an open map and U is open in X, f(U) is open Y. Since every open set is β*-open, f((U) is β*-open in Y. Thus for every point y of Y and every open set V containing y, there exist a β*-open set f(U) such that $y \in f(U) \subset \beta^*cl(f(U)) \subset V$. Hence by theorem – 5.2.16 Y is ultra β*-regular.

Definition-3.2.4: A space X is said to be ultra β*-normal if for each pair of disjoint closed sets A and B of X there exist disjoint β*-open sets U and V such that $A \subset U$ and $B \subset V$.

Theorem -3.2.5: Assume that β*o(τ) is closed under any union. If f: X→Y is a continuous β*-closed surjection and X is a normal space, then Y is ultra β*-normal.

Proof: Let A and B be disjoint closed sets of Y. Since X is normal there exist disjoint open sets U and V of X such that $f^{-1}(A)\subset U$ and $f^{-1}(B)\subset V$. By theorem 3.1.10, there exist β*-open sets G and H such that $A\subset G$, $B\subset H$ and $f^{-1}(G)\subset U$ and $f^{-1}(H)\subset V$. Then we have $f^{-1}(G)\cap f^{-1}(H) = \varphi$ and hence $G\cap H = \varphi$. Since G is β*-open, $A \subset G$ implies $A\subset \beta^*$ int (G) Similarly $B\subset\beta^*$ int (H). Therefore β* int (G)∩β* int (H) = φ. Thus Y is ultra β* normal.

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