Chase-Repulsion analysis for (2+1)-Dimensional Lotka-Volterra System

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***Abstract*—Homogeneous balance method (HB) is applied to the (2+1)-dimensional Lotka-Volterra system to construct exact solutions. A homogeneous system of equations for the quasi-solution is solved. The travelling wave quasi-solution leads to the solitary wave solution of the system. According to the different system parameters, three ecosystems; predator-prey, symbiosis and competition are analyzed and plotted. The chase-repulsion relationship is cleared in the two opposite soliton waves for the predator-prey case. The inhibit and favor between species are obvious in competition and symbioses cases.**

Keywords*— *Lotka-Volterra equations; Homogeneous balance method; predator-prey dynamics; Mathematical ecology.

I. INTRODUCTION

Investigation of the interaction between species plays an important role in mathematical ecology [1-8], technology evolution [9-11] and other applications [12, 13]. There are six interaction modes between species in an ecosystem [14], namely, neutralism, competition, symbioses, commensalism, amensalism and predation (predator-prey).

Predator-prey dynamics has been extensively studied by Lotka-Volterra models [15-18]. Many authors have focused on solving and analyzing the Lotka-Volterra models. Among them, Alam and Tunc [19], constructed many families of exact solutions of nonlinear Predator-prey equations using the exp (-φ(ξ)) expansion method. Kraenkel *et al* [20], Applied the $\frac{G'}{G}$ expansion method to drive exact solutions to a predator-prey with Allee effect prey per capita growth rate. They discussed the system for two different wave speeds and reported three different solutions for each wave velocity. In [21], the Khasminskii's theory of periodic solution, was applied to identify that the system fulfils a nontrivial positive T -periodic solution. Numerical simulations in different cases were analyzed [22-24]. The existence of traveling wave solutions for a diffusive predator–prey system were discussed by using the original Wazewskii's theory [25]. Different pattern formations for the Predator-prey system were discussed in [26-28].

The plenteous number of mathematical methods of solutions of differential equations (DEs) [15, 29-43], authorized scientists to pointedly elucidate the different mathematical models. Some of these methods are; singular manifold method [29, 30], Hirota's bilinear method [31, 35], transformed rational function method [39, 41], exponential function method [36], Darboux transformation [32, 40], conservation laws and symmetry methods [42, 43], homotopy analysis transformation method [33, 34], direct algebraic method [37], $\frac{G'}{G}$ expansion method [38] and homogeneous balance method [44-47] . This work is motivated to solve and analyze the (2+1)-dimensions Lotka-Volterra system. The homogenous balance method is applied to find the exact soliton solution of this system. The paper is arranged as follows. In section II, the homogenous balance method is described. In section III, the method is applied to solve the Lotka-Volterra system in (2+1)-dimensions. Section IV, is devoted to analyze and discuss the results. The paper ends with conclusions in section V.

II. DESCRIPTION OF THE HOMOGENEOUS BALANCE METHOD

The homogeneous balance method [44-47], is a systematic and effective for finding explicit solitary wave solutions. Consider a system of partial differential equations (PDE);

$$\begin{cases} p_1(\varphi, \varphi_x, \varphi_t, \varphi_y, \varphi_{xx}, \varphi_{yy}, \dots) = 0 \\ p_2(\psi, \psi_x, \psi_t, \psi_y, \psi_{xx}, \psi_{yy}, \dots) = 0 \end{cases} \quad (1)$$

Where p_1 and p_2 are polynomials in φ, ψ and their partial derivatives. A function $\xi = \xi(x, y, t)$ is considered as a quasi-solution of the system (1) if there are functions $f = f(\xi)$ and $g = g(\xi)$, of only one argument so that, a nominated linear combination of;

$$1, f(\xi), f_x(\xi), f_t(\xi), f_y(\xi), f_{xx}(\xi), f_{yy}(\xi), f_{xy}(\xi) \dots \quad (2)$$

and; $1, g(\xi), g_x(\xi), g_t(\xi), g_y(\xi), g_{xx}(\xi), g_{yy}(\xi), g_{xy}(\xi) \dots$ (3)

Are solutions of the system (1). The HB method is recapped in the prosecuting steps;

Step 1: Choose the solution of (1) as a linear combination of (2) and (3), satisfying the balance between the highly nonlinear and the highest order derivative terms in the system (1).

Step (2): Substitute the combination picked out in *step 1* into the system (1). Collect all terms with the highest degree of ξ *(x, y, t)* and set their coefficients to zero. Secure a system of ordinary differential equations (ODEs) in $f(\xi)$ and $g(\xi)$, solve this system to find $f(\xi)$, $g(\xi)$ and relations between their nonlinear derivatives.

Step (3): Replace for nonlinear derivatives of $g(\xi)$ and f(ξ), then collect all the terms with the same order of $f, f', f'', f''', \dots, g, g', g'', g''', \dots$, and set their coefficients to zero. Get a homogeneous system of ODEs in ξ *(x, y, t)*. According to the homogeneous property of this system of equations ξ *(x, y, t)* can be predicted as an exponential function.

Step (4): Substiute $f(\xi)$, $g(\xi)$ and ξ *(x, y, t)* in the linear combination from *step(1)*, then the solution of the system (1) is obtained.

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III. SOLITARY WAVE SOLUTION OF (2+1)-DIMENSIONAL LOTKA-VOLTERRA SYSTEM

The (2+1) dimensional Lotka-Volterra system is represented as;

$$d_1\varphi_{xx} + d_1\varphi_{yy} - \varphi_t + r\varphi\left(1 - \frac{\varphi}{k}\right) - \alpha_1\varphi\psi = 0 \quad (4)$$

$$d_2\psi_{xx} + d_2\psi_{yy} - \psi_t - \mu\psi + \alpha_2\varphi\psi = 0 \quad (5)$$

Where $\varphi(x, y, t)$ and $\psi(x, y, t)$ are the species densities, d_1 and d_2 are specific diffusion rate, r is the population intrinsic rate of growth, k is the carrying capacity, μ is the per capita injury rate and α's represent consumption interaction coefficient between species.

In this section the homogeneous balance method is applied to find the exact solution of the Lotka-Volterra system (4)-(5). Choose the solution of this system as a linear combination in the form;

$$\varphi(x,y,t) = \frac{\partial^{\beta_1} f(\xi)}{\partial x^{\beta_1}} \quad (6)$$

$$\psi(x,y,t) = \frac{\partial^{\beta_2} g(\xi)}{\partial x^{\beta_2}} \quad (7)$$

The balance between the highly nonlinear and the highest order derivative terms in the system (4)-(5), results in $\beta_1 = \beta_2 = 2$. Then the linear combination (6) and (7) is written as;

$$\varphi(x,y,t) = f''\xi_x^2 + f'\xi_{xx} \quad (8)$$

$$\psi(x,y,t) = g''\xi_x^2 + g'\xi_{xx} \quad (9)$$

Substitute (8) and (9) into (4);

$$\left(-\frac{r}{k}f''^2 + d_1 f^{(4)} - \alpha_1 f''g''\right)\xi_x^4 + \left(-\alpha_1 g''f'\xi_{xx} + d_1 f^{(4)}\xi_y^2 - f'''\xi_t + 6d_1 f'''\xi_{xx} - \frac{2r}{k}f''f'\xi_{xx} - \alpha_1 f''g'\xi_{xx} + d_1 f'''\xi_{yy} + rf''\right)\xi_x^2 + \left(4d_1 f'''\xi_y\xi_{xy} + 4d_1 f''\xi_{xxx} + 2d_1 f''\xi_{xyy} - 2f''\xi_{xt}\right)\xi_x + rf'\xi_{xx} + d_1 f'\xi_{4x} + d_1 f'''\xi_y^2\xi_{xx} + d_1 f''\xi_{yy}\xi_{xx} + 3d_1 f''\xi_{xx}^2 + 2d_1 f''\xi_{xy}^2 + 2d_1 f''\xi_y\xi_{xxy} - \frac{r}{k}f'^2\xi_{xx}^2 - f'\xi_{xxt} - \alpha_1 f'g'\xi_{xx}^2 - f''\xi_t\xi_{xx} + d_1 f'\xi_{xxyy} = 0$$

(10)

Substitute (8) and (9) into (5);

$$\left(\alpha_2 f''g'' + d_2 g^{(4)}\right)\xi_x^4 + \left(d_2 g^{(4)}\xi_y^2 + \alpha_2 f'g''\xi_{xx} + \alpha_2 f''g'\xi_{xx} + 6d_2 g'''\xi_{xx} + d_2 g'''\xi_{yy} - g'''\xi_t - \mu g''\right)\xi_x^2 + \left(4d_2 g'''\xi_{xy}\xi_y + 4d_2 g''\xi_{xxx} + 2d_2 g''\xi_{xyy} - 2g''\xi_{xt}\right)\xi_x + d_2 g'''\xi_y^2\xi_{xx} + 2\alpha_2 g'f'\xi_{xx}^2 + d_2 g'''\xi_{xx}^2 + d_2 g''\xi_{yy}\xi_{xx} + 2d_2 g''\xi_y\xi_{xxy} + 2d_2 g''\xi_{xy}^2 - g''\xi_t\xi_{xx} - \mu g'\xi_{xx} + d_2 g'\xi_{4x} + d_2 g'\xi_{xxyy} - g'\xi_{xxt} = 0$$

(11)

Setting the coefficient of $\xi_x^4 = 0$, after setting $\xi_x = -\xi_y$, yielding a system of ordinary differential equations;

$$\begin{cases} -\frac{r}{k}f''^2 + 2d_1 f^{(4)} - \alpha_1 f''g'' = 0 \\ \alpha_2 f''g'' + 2d_2 g^{(4)} = 0 \end{cases} \quad (12)$$

The solutions of this ODE system are;

$$f = C_1 ln\xi, \ \ C_1 = \frac{12d_2}{\alpha_2} \quad (13)$$

$$g = C_2 ln\xi, C_2 = -\frac{12rd_2}{\alpha_1\alpha_2 k} - \frac{12d_1}{\alpha_1} \quad (14)$$

The relations between the nonlinear derivatives of $g(\xi)$ and $f(\xi)$ are summarized as;

$$\begin{cases} f'^2 = -C_1 f'', \ \ f'f'' = -\frac{1}{2}C_1 f''' \\ g'^2 = -C_2 g'', \ \ g'g'' = -\frac{1}{2}C_2 g''' \\ g'f'' = f'g'' = -\frac{1}{2}C_1 g''' = -\frac{1}{2}C_2 f''' \\ f'g' = -C_1 g'' = -C_2 f'' \end{cases} \quad (15)$$

By using (12) and (15) the equations (10) and (11) are simplified as;

$$\left(\alpha_1 C_2 f'''\xi_{xx} - f'''\xi_t + 6d_1 f'''\xi_{xx} + \frac{r}{k}C_1 f'''\xi_{xx} + d_1 f'''\xi_{yy} + rf''\right)\xi_x^2 + \left(4d_1 f'''\xi_y\xi_{xy} + 4d_1 f''\xi_{xxx} + 2d_1 f''\xi_{xyy} - 2f''\xi_{xt}\right)\xi_x + rf'\xi_{xx} + d_1 f'\xi_{4x} + d_1 f''\xi_{yy}\xi_{xx} + 3d_1 f''\xi_{xx}^2 + 2d_1 f''\xi_{xy}^2 + 2d_1 f''\xi_y\varphi_{xxy} + \frac{r}{k}C_1 f''\xi_{xx}^2 - f'\xi_{xxt} + \alpha_1 C_2 f''\xi_{xx}^2 - f''\xi_t\xi_{xx} + d_1 f'\xi_{xxyy} = 0$$

(16)

$$\left(-\alpha_2 C_1\, g'''\xi_{xx} + 6d_2 g'''\xi_{xx} + 6d_2 g'''\xi_{yy} - g'''\xi_t - \mu g''\right)\xi_x^2 + \left(4d_2 g'''\xi_{xy}\xi_y + 4d_2 g''\xi_{xxx} + 2d_2 g''\xi_{xyy} - 2g''\xi_{xt}\right)\xi_x + d_2 g''\xi_y^2\xi_{xx} - \alpha_2 C_1 g''\xi_{xx}^2 + 3d_2 g''\xi_{xx}^2 + d_2 g''\xi_{yy}\xi_{xx} + 2d_2 g''\xi_y\xi_{xxy} + 2d_2 g''\xi_{xy}^2 - g''\xi_t\xi_{xx} - \mu g'\xi_{xx} + d_2 g'\xi_{4x} + d_2 g'\xi_{xxyy} - g'\xi_{xxt} = 0$$

(17)

setting the coefficients of f''', f'' and f' in (16) and the coefficients $g''', g''and$ g' in (17) equal to zero; yields a system of partial differential equations for ξ (x, y, t);

$$\left(\alpha_1 C_2\xi_{xx} - \xi_t + 6d_1\xi_{xx} + \frac{rC_1}{k}\xi_{xx} + d_1\xi_{yy}\right)\xi_x^2 + 4d_1\xi_y\xi_{xy}\xi_x = 0 \quad (18)$$

$$r\xi_x^2 + \left(4d_1\xi_{xxx} + 2d_1\xi_{xyy} + 2\xi_{xt}\right)\xi_x - d_1\xi_{yy}\xi_{xx} + 3d_1\xi_{xx}^2 + 2d_1\xi_{xy}^2 + 2d_1\xi_y\xi_{xxy} + \frac{rC_1}{k}\xi_{xx}^2 + \alpha_1 C_2\xi_{xx}^2 + \xi_t\xi_{xx} = 0$$

(19)

$$d_1\xi_{4x} - r\xi_{xx} - \xi_{xxt} + d_1\xi_{xxyy} = 0 \quad (20)$$

$$\left(-\alpha_2 C_1\,\xi_{xx} + 6d_2\xi_{xx} - d_2\xi_{yy} - \xi_t\right)\xi_x^2 + d_2\xi_y^2\xi_{xx} + 4d_2\xi_{xy}\xi_y\xi_x = 0 \quad (21)$$

$$-\mu\xi_x^2 + \left(4d_2\xi_{xxx} - 2d_2\xi_{xyy} - 2\xi_{xt}\right)\xi_x - 2\alpha_2 C_1\xi_{xx}^2 - d_2\xi_{yy}\xi_{xx} + 2d_2\xi_y\xi_{xxy} + 2d_2\xi_{xy}^2 - \varphi_t\varphi_{xx} + 3d_1\xi_{xx}^2 = 0$$

(22)

$$-\mu\xi_{xx} + d_2\xi_{4x} - d_2\xi_{xxyy} - \xi_{xxt} = 0 \quad (23)$$

To solve the homogeneous system (18)-(23), assume that

$$\xi(x,y,t) = 1 + e^{\alpha x + \beta y + \gamma t + \delta} \quad (24)$$

Where α, β, γ are to be determined and δ is a constant. Substitute (24) into the system (18)-(24), considering that $\xi_x = -\xi_y$. It is found that $\xi(x,y,t)$ satisfies this system of equations when, $\gamma = \alpha^2 d_1$, $\frac{d_1}{d_2} = 2$, $\mu = 2\alpha^2 d_2$ and $\alpha = -\beta$.

Then the solution of the Lotka-Volterra system (4)-(5) is;

$$\varphi(x,y,t) = \frac{C_1\alpha^2}{4}\text{sech}^2\left(\frac{1}{2}(\alpha x + \beta y + \gamma t + \delta)\right) \quad (25)$$

$$\psi(x,y,t) = \frac{C_2\alpha^2}{4}\text{sech}^2\left(\frac{1}{2}(\alpha x + \beta y + \gamma t + \delta)\right) \quad (26)$$

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IV. RESULTS AND DISCUSSION

This section is motivated to plot and discuss the solutions of the system (4)-(5). The dynamics of the species densities are varied according to the system parameters. It is clear that the relation between species is affected by the signs of the interaction coefficients (α's). Table 1 illustrate three relations between species for different (α's) signs.

TABLE I. THE EFFECT OF THE INTERACTION COEFFICIENTS' SIGNS IN THE RELATION BETWEEN SPECIES.

Relation between species	***Sign of α_1***	***Sign of α_2***	***Sign of interaction term in equation (4)***	***Sign of interaction term in equation (5)***
Prey-Predator	+	+	–	+
Symbiosis	–	+	+	+
Competition	+	–	–	–

The prey-predator relation is presented in *Fig. 1*, for $t = 0$, $t = 5$ and $t = 10$ with the parameters $\alpha_1 = \alpha_2 = 50$, $d_1 = 10$, $d_2 = 5$, $k = 10$, $r = 30$, $\mu = 10$, $\alpha = 1$, $\beta = -1$, $\gamma = -1$ and $\delta = 1$. The densities evolve in two opposite solitary waves, which confirms the chase-repulsion relation between species. The increase in the prey density reveals a decrease in predator density and vice versa. The two dimensional plot for this case is shown in *Fig.2*, at y =2.

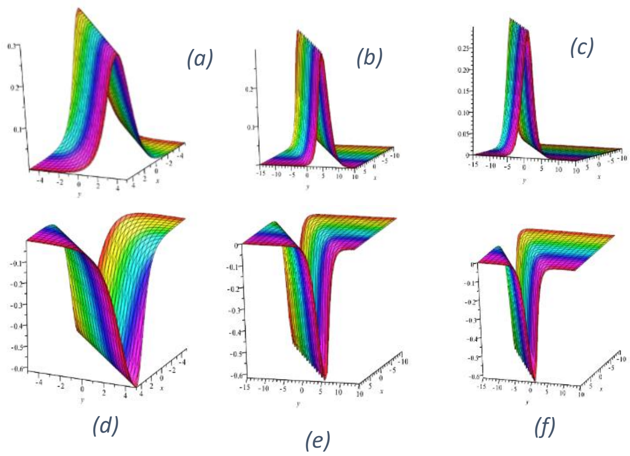

Fig. 1: (a, b and c) is the prey evolution and (d, e and f) is the predator evolution for α1 = α2 = 50, d1 =10 , d2 = 5, k = 10, r = 30, μ = 10, α = 1, β = -1, γ =-1 and δ = 1 at t = 0, t = 5 and t =10.

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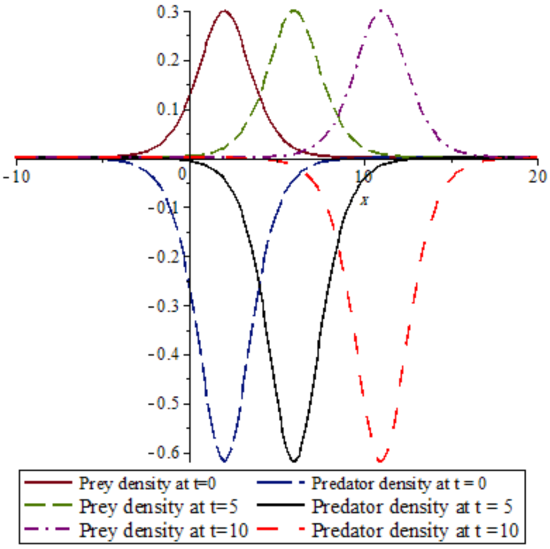

Fig. 2: The 2D-prey- predator evolution at α1=α2=50, d1=10, d2=5, k=10, r=30, μ=10, α=1, β=–1, γ=–1, δ=1 and y=2 for t =0, t =5 and t =10.

The symbiosis relation between species appears when α_1 is negative. *Fig.3*, represents the symbiosis relation between φ and ψ *for* $\alpha_1 = -100$, $\alpha_2 = 50$, $d_1 = 10$, $d_2 = 5$, $k = 1$, $r = 30$, $\mu = 10$, $\alpha = 1$, $\beta = -1$, $\gamma = -1$, $\delta = 1$, $t = 2$ and $y = 1$. The solitary waves show that both species favor each other.

The competition relation between φ and ψ is illustrated in *Fig.4*, for $\alpha_1 = 100$, $\alpha_2 = -50$, $d_1 = 100$, $d_2 = 50$, $k = 1$, $r = 30$, $\mu = 10$, $\alpha = 1$, $\beta = -1$, $\gamma = -1$, $\delta = 1$, $t = 2$ and $y = 1$. The solution shows that Both species inhibit each other.

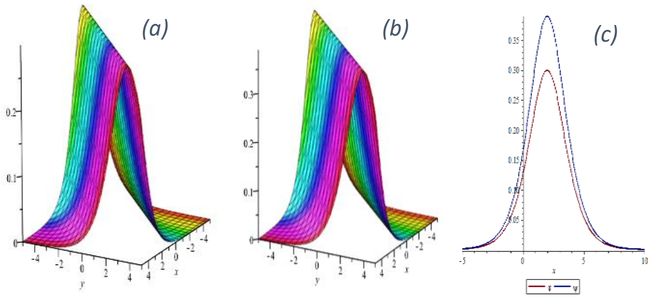

Fig. 3: The symbiosis relation between φ and ψ for $\alpha_1 = -100$, $\alpha_2 = 50$, $d_1 = 10$, $d_2 = 5$, $k = 1$, $r = 30$, $\mu = 10$, $\alpha = 1$, $\beta = -1$, $\gamma = -1$, $\delta = 1$, $t = 2$ and y=1,(a) represents φ, (b) represents ψ and (c) is the 2D solitons.

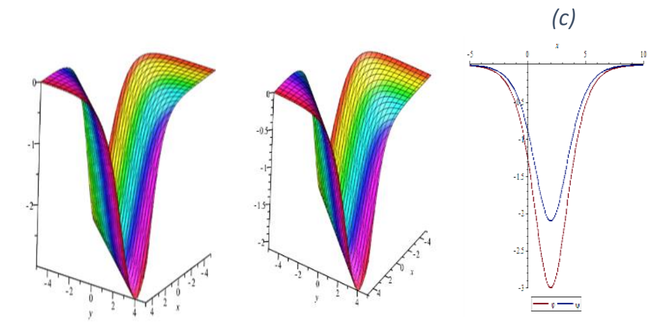

Fig. 4: The competition relation between φ and ψ for $\alpha_1 = -100$, $\alpha_2 = -50$, $d_1 = 100$, $d_2 = 50$, $k = 1$, $r = 30$, $\mu = 10$, $\alpha = 1$, $\beta = -1$, $\gamma = -1$, $\delta = 1$, $t = 2$ and y=1,(a) represents φ, (b) represents ψ and (c) is the 2D solitons.

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V. CONCLUSIONS

Homogeneous balance method is effective in detecting the solitary wave solution of the Lotka-Volterra system. Three interaction relations are discussed and plotted. The chase-repulsion relation between species is conspicuous in the two opposite solitary waves of the prey-predator case. The dynamics of the system is discussed for the different parameter values. The symbiosis and competition relations manifested at different signs of the interaction coefficients α_1 and α_2.

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