Published by :
http://www.ijert.org

International Journal of Engineering Research & Technology (IJERT)
ISSN: 2278-0181
Vol. 9 Issue 10, October-2020

Chaos in Associativity of A Finite Quaternion Algebra

T. Srinivasarao
Asst. Professor
Dept. Of Math.
Adikavi Nannaya University
Rajahmundry

K. Geetha Lakshmi
Asst. Professor
Dept. Of App.Math
UCST, ANUR

***Abstract*: when an associative algebra is defined over another associative algebra, there is every chance of the associative property holding. Keeping this in view, if the set of Quaternions taking coefficients from real numbers in the form** $\mathbf{Q}=\left\{a_0+a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k} / a_0,a_1,a_2,a_3\in\mathbf{R}\right\}$ **which makes a Skew Field under the addition of quaternions defined by**

$$\left(a_0+a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}\right)+\left(b_0+b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}\right)$$

$$=\left(a_0+b_0\right)+\left(a_1+b_1\right)\mathbf{i}+\left(a_2+b_2\right)\mathbf{j}+\left(a_3+b_3\right)\mathbf{k}$$ **and the multiplication defined by**

$$\left(a_0+a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}\right)\times\left(b_0+b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}\right)=$$

$$=\left(a_0b_0-a_1b_1-a_2b_2-a_3b_3\right)+\left(a_0b_1+a_1b_0+a_2b_3-a_3b_2\right)\mathbf{i}+\left(a_0b_2+a_2b_0+a_3b_1-a_1b_3\right)\mathbf{j}$$
$$+\left(a_0b_3+a_3b_0+a_1b_2-a_2b_1\right)\mathbf{k}.$$

Also, $\mathbf{Z}_p=\left\{[0],[1],[2],\ldots,[p-1]\right\}$ **is the set of residue classes modulo *p* forms a field under the addition modulo *p* defined by** $[a]+_p[b]=[a+b]\bmod p$ **and multiplication modulo *p* is defined by** $[a]\times_p[b]=[ab]\bmod p$**. Since the elements of** $\mathbf{Z}_p$ **are *p* a finite number, the formed field is a finite field. When the Quaternions are defined over this finite field, then a chaotic situation except the closed property can be observed.**

The present discussion is, showing that it is not necessary always that the composite algebra formed by the quaternion skew field defined over the field of residue classes modulo *p* is not associative.

CHAPTER 1:

Definition1.1: $\mathbf{Q}_p=\left\{q_{\alpha_0\alpha_1\alpha_2\alpha_3}=\alpha_0+\alpha_1\mathbf{i}+\alpha_2\mathbf{j}+\alpha_3\mathbf{k} / \alpha_0,\alpha_1,\alpha_2,\alpha_3\in\mathbf{Z}_p,\mathbf{i},\mathbf{j},\mathbf{k}\text{ are unit vectors}\right\}$

then α_i can take any value between 0 and $p-1$ integers. $0\le i\le 3$. This allows the cardinality of $\mathbf{Q}_p$ is $\left|\mathbf{Q}_p\right|=p^4$.

Definition 1.2: $\oplus$ is defined on $\mathbf{Q}_p$ by $q_{\alpha_0\alpha_1\alpha_2\alpha_3}\oplus q_{\beta_0\beta_1\beta_2\beta_3}$

$$=\left(\alpha_0+_p\beta_0\right)+\left(\alpha_1+_p\beta_1\right)\mathbf{i}+\left(\alpha_2+_p\beta_2\right)\mathbf{j}+\left(\alpha_3+_p\beta_3\right)\mathbf{k}$$

See that $\alpha_i+_p\beta_i\in\mathbf{Q}_p\forall 0\le i\le 3$ which verifies that $\left(\mathbf{Q}_p,\oplus\right)$ is a closed set.

Definition 1.3: $\otimes$ is defined on $\mathbf{Q}_p$ by $q_{\alpha_0\alpha_1\alpha_2\alpha_3}\otimes q_{\beta_0\beta_1\beta_2\beta_3}$

$$=\left[\left(\alpha_0\times_p\beta_0\right)+_p(-)\left(\alpha_1\times_p\beta_1\right)+_p(-)\left(\alpha_2\times_p\beta_2\right)+_p(-)\left(\alpha_3\times_p\beta_3\right)\right]$$
$$+\left[\left(\alpha_0\times_p\beta_1\right)+_p\left(\alpha_1\times_p\beta_0\right)+_p\left(\alpha_2\times_p\beta_3\right)+_p(-)\left(\alpha_3\times_p\beta_2\right)\right]\mathbf{i}$$
$$+\left[\left(\alpha_0\times_p\beta_2\right)+_p\left(\alpha_2\times_p\beta_0\right)+_p\left(\alpha_3\times_p\beta_1\right)+_p(-)\left(\alpha_1\times_p\beta_3\right)\right]\mathbf{j}$$
$$+\left[\left(\alpha_0\times_p\beta_3\right)+_p\left(\alpha_3\times_p\beta_0\right)+_p\left(\alpha_1\times_p\beta_2\right)+_p(-)\left(\alpha_2\times_p\beta_1\right)\right]\mathbf{k}$$

Note that $(-)\left(\alpha_i\times_p\beta_j\right)=p-\left(\alpha_i\beta_j\right)\bmod p$

Since $0\le\left(\alpha_i\beta_j\right)\bmod p\le p-1$, it follows $0\le p-\left(\alpha_i\beta_j\right)\bmod p\le p-1$.

While $\alpha_i\times_p\beta_j\in\mathbf{Q}_p\forall 0\le i,j\le 3$ and $\left(\alpha_i\times_p\beta_j\right)+_p\left(\alpha_s\times\beta_t\right)\in\mathbf{Q}_p\forall 0\le i,j,s,t\le 3$ which verifies that $\left(\mathbf{Q}_p,\otimes\right)$ is a closed set.

CHAPTER 2:

To substantiate the idea of non – associativity in this closed set, here is an example.
Example 2.1: suppose p = 5 while p stands for the prime number, and write the residue classes modulo 5.

International Journal of Engineering Research & Technology (IJERT)
ISSN: 2278-0181
Vol. 9 Issue 10, October-2020

$\mathbf{Z}_5 = \{[0],[1],[2],[3],[4]\}$ is the set of residue classes modulo 5 upon which the addition modulo 5 is defined by $\mathbf{x} +_5 \mathbf{y} = (\mathbf{x}+\mathbf{y}) \bmod 5$ which means, the remainder obtained when $\mathbf{x} + \mathbf{y}$ is divided by 5 while $\mathbf{x}$ and $\mathbf{y}$ are integers.

Also, the multiplication modulo 5 is defined by $\mathbf{x} \times_5 \mathbf{y} = (\mathbf{x}\times\mathbf{y}) \bmod 5$. With these two operations, $(\mathbf{Z}_5, +_5, \times_5)$ forms a ring without divisors of zero.

$\mathbf{Q}_5 = \{q_{\alpha_0\alpha_1\alpha_2\alpha_3} = \alpha_0 + \alpha_1\mathbf{i} + \alpha_2\mathbf{j} + \alpha_3\mathbf{k} \,/\, \alpha_0,\alpha_1,\alpha_2,\alpha_3 \in \mathbf{Z}_5, \mathbf{i},\mathbf{j},\mathbf{k} \text{ are unit vectors}\}$

By the rule of permutation, each component of $q_{\alpha_0\alpha_1\alpha_2\alpha_3}$ can take 5 choices 0 through 4.

So, the total number of members of $\mathbf{Q}_5$ possible is $5^4 = 625$

See that 0 can be represented by $0\times5^3 + 0\times5^2 + 0\times5^1 + 0\times5^0 = 0000$, 27 can be represented by $0\times5^3 + 1\times5^2 + 0\times5^1 + 2\times5^0 = 0102$ and 624 can be represented by $= 4444$ while $4\times5^3 + 4\times5^2 + 4\times5^1 + 4\times5^0 = 624$.

So, the members of $\mathbf{Q}_5$ can be the base 5 number system or the Penta- base system.

Definition 2.1: $q_{\alpha_1\alpha_2\alpha_3\alpha_4} \oplus q_{\beta_1\beta_2\beta_3\beta_4} = (a_0 +_5 b_0) + (a_1 +_5 b_1)\mathbf{i} + (a_2 +_5 b_2)\mathbf{j} + (a_3 +_5 b_3)\mathbf{k}$, and use $\mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = \mathbf{k}\cdot\mathbf{k} = -1$, $\mathbf{i}\cdot\mathbf{j} = \mathbf{k}$, $\mathbf{j}\cdot\mathbf{i} = -\mathbf{k}$, $\mathbf{i}\cdot\mathbf{k} = -\mathbf{j}$, $\mathbf{k}\cdot\mathbf{i} = \mathbf{j}$, $\mathbf{j}\cdot\mathbf{k} = \mathbf{i}$, $\mathbf{k}\cdot\mathbf{j} = -\mathbf{i}$ for all $0 \le i, j \le 624$ indices.

Definition 2.2: $q_{a_0a_1a_2a_3} \otimes q_{b_0b_1b_2b_3} = (a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \otimes (b_0 + b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k})$

$$= \{(a_0 \times_5 b_0) +_5 (-)(a_1 \times_5 b_1) +_5 (-)(a_2 \times_5 b_2) +_5 (-)(a_3 \times_5 b_3)\}$$
$$+ \{(a_0 \times_5 b_1) +_5 (a_1 \times_5 b_0) +_5 (a_2 \times_5 b_3) +_5 (-)(a_3 \times_5 b_2)\}\mathbf{i} +$$
$$+ \{(a_0 \times_5 b_2) +_5 (a_2 \times_5 b_0) +_5 (a_3 \times_5 b_1) +_5 (-)(a_1 \times_5 b_3)\}\mathbf{j} +$$
$$+ \{(a_0 \times_5 b_3) +_5 (a_3 \times_5 b_0) + (a_1 \times_5 b_2) +_5 (-)(a_2 \times_5 b_1)\}\mathbf{k}$$

Observe that $(-)(a_3 \times_5 b_2)$ is the inverse addition modulo 5 and not the inverse multiplication modulo 5.

$q_{426} = [3] + [2]\mathbf{i} + [0]\mathbf{j} + [1]\mathbf{k}$, $q_{188} = [1] + [2]\mathbf{i} + [2]\mathbf{j} + [3]\mathbf{k}$ and $q_{32} = [0] + [1]\mathbf{i} + [1]\mathbf{j} + [2]\mathbf{k}$

$$q_{426} \otimes q_{188} = \{[3] + [2]\mathbf{i} + [0]\mathbf{j} + [1]\mathbf{k}\} \otimes \{[1] + [2]\mathbf{i} + [2]\mathbf{j} + [3]\mathbf{k}\}$$
$$= ([3] +_5 (-)[4] +_5 (-)[0] +_5 (-)[3]) + ([1] +_5 [2] +_5 [0] +_5 (-)[2])\mathbf{i}$$
$$+ ([1] +_5 [0] +_5 [2] +_5 (-)[1])\mathbf{j} + ([4] +_5 [1] +_5 [4] +_5 (-)[0])\mathbf{k}$$
$$= [1] + [1]\mathbf{i} + [2]\mathbf{j} + [4]\mathbf{k}$$
$$= q_{164}$$

$$\{q_{426} \otimes q_{188}\} \otimes q_{32} = q_{164} \otimes q_{32}$$
$$= \{[1] + [1]\mathbf{i} + [2]\mathbf{j} + [4]\mathbf{k}\} \otimes \{[0] + [1]\mathbf{i} + [1]\mathbf{j} + [2]\mathbf{k}\}$$
$$= ([0] +_5 (-)[1] +_5 (-)[2] +_5 (-)[3]) + ([1] +_5 [0] +_5 [4] +_5 (-1)[4])\mathbf{i}$$
$$+ ([1] +_5 [0] +_5 [4] +_5 (-)[2])\mathbf{j} + ([2] +_5 [0] +_5 [1] +_5 (-)[2])\mathbf{k}$$

See that $(-)[2] = (5-2) \bmod 5 = [3]$

$$= ([0] +_5 [4] +_5 [3] +_5 [2]) + ([1] +_5 [0] +_5 [4] +_5 [1])\mathbf{i} + ([1] +_5 [0] +_5 [4] +_5 [3])\mathbf{j}$$
$$+ ([2] +_5 [0] +_5 [1] +_5 [3])\mathbf{k}$$

$$\{q_{426} \otimes q_{188}\} \otimes q_{32} = [4] + [1]\mathbf{i} + [3]\mathbf{j} + [1]\mathbf{k}$$
$$= q_{501} \qquad \text{...... (2.1)}$$

Take the other side of the equation.

$$q_{188} \otimes q_{32} = \{[1] + [2]\mathbf{i} + [2]\mathbf{j} + [3]\mathbf{k}\} \otimes \{[0] + [1]\mathbf{i} + [1]\mathbf{j} + [2]\mathbf{k}\}$$
$$= ([0] +_5 (-)[2] +_5 (-)[2] +_5 (-)[1]) + ([1] +_5 [0] +_5 [4] +_5 (-1)[3])\mathbf{i}$$

International Journal of Engineering Research & Technology (IJERT)
ISSN: 2278-0181
Vol. 9 Issue 10, October-2020

$$+\left([1]+_5[0]+_5[3]+_5(-1)[4]\right)\mathbf{j}+\left([2]+_5[0]+_5[2]+_5(-)[0]\right)\mathbf{k}$$

$$=[0]+[2]\mathbf{i}+[0]\mathbf{j}+[4]\mathbf{k}$$

$$=q_{54}$$

$$q_{426}\otimes\{q_{188}\otimes q_{32}\}=q_{426}\otimes q_{54}$$

$$=\{[3]+[2]\mathbf{i}+[0]\mathbf{j}+[1]\mathbf{k}\}\otimes\{[0]+[2]\mathbf{i}+[0]\mathbf{j}+[4]\mathbf{k}\}$$

$$=\left([0]+_5(-)[4]+_5(-)[0]+_5(-)[4]\right)+\left([1]+_5[0]+_5[0]+_5(-)[0]\right)\mathbf{i}$$

$$+\left([0]+_5[0]+_5[2]+_5(-)[3]\right)\mathbf{j}+\left([2]+_5[0]+_5[0]+_5(-)[0]\right)\mathbf{k}$$

$$=[2]+[1]\mathbf{i}+[4]\mathbf{j}+[2]\mathbf{k}$$

$$=q_{297} \qquad \text{...... (2.2)}$$

(2.1) and (2.2) confirm that associative law does not hold in the Quaternion set under the said operations.

Theorem: 3.1. $\left(\mathbf{Q}_p,\otimes\right)$ is not an associative algebra.

Proof: consider $\left(q_{\alpha_0\alpha_1\alpha_2\alpha_3}\otimes q_{\beta_0\beta_1\beta_2\beta_3}\right)\otimes q_{\gamma_0\gamma_1\gamma_2\gamma_3}=$

$$=\left\{\left(\alpha_0\beta_0 \bmod p\right)\gamma_0 \bmod p+\sum_{i=1}^{3}\left[p-\left(p-\alpha_i\beta_i \bmod p\right)\gamma_i \bmod p\right]\right\}+q_1\mathbf{i}+q_2\mathbf{j}+q_3\mathbf{k}$$

$$\neq\left\{\alpha_0\left(\beta_0\gamma_0 \bmod p\right)\bmod p+\sum_{i=1}^{3}\left[p-\alpha_i\left(p-\beta_i\gamma_i \bmod p\right)\bmod p\right]\right\}+r_1\mathbf{i}+r_2\mathbf{j}+r_3\mathbf{k} \text{ where}$$

$$q_i,r_j\in\mathbf{Q}_p \text{ for } 1\le i,j\le 3$$

Observe that $a-(b-c)\neq(a-b)-c$ either in a finite field or in an infinite field where $-x$ stands for the additive inverse of x.

$$=q_{\alpha_0\alpha_1\alpha_2\alpha_3}\otimes\left(q_{\beta_0\beta_1\beta_2\beta_3}\otimes q_{\gamma_0\gamma_1\gamma_2\gamma_3}\right)$$

REFERENCES:

[1] Hamilton, W.R.(1844). "Letter" *London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science* Vol.xxv.pp.489-496
[2] Eves, Howard (1976), An Introduction to the History of Mathematics (4th ed.) New York. Holt, Rinehart and Winston, ISBN 0-03-089539-1
[3] Altmann, Simon L.(December 1989). "Hamilton, Rodrigues, and the quaternion scandal". Mathematics Magazine. 62(5):306. JSTOR 2689481
[4] Nagell, T. "Residue Classes and Residue Systems." *Introduction to Number Theory* New York: Wiley, pp.69-71, 1951
[5] *An Introduction to the Theory of Numbers, 5th edition,* Oxford University Press, G.H.Hardy and E.M.Wright, p – 351, 1979
[6] H.Davenport, *Multiplicative Number Theory*, 2nd edition, Springer – Verlag, New York, 1980