Certain Type of Special Function Associate by Pathway Fraction Integral Operator

Hemlata Saxena[1], Himanshu Sharma[2], Harsha Babani[3]
Department of Mathematics,
Career Point University, Kota

Abstract:- **In the present paper we consider product of some special function associated with the pathway functional integral operator. This operator generalizes of the classical Riemann-Liouville fractional integral operator. The results derived here are quite general in nature and their several known and new special cases are also obtained here.**

Keywords:- Pathway fractional integral operator, Fox's H-function, Generalized Mittag-Leffler function, G-function.

1. INTRODUCTION

The Pathway fractional integral operator introduced by Nair [9] is defined in the following manner

$$\left(P_{0_+}^{(\eta,\alpha)} f\right)(x) = x^{\eta} \int_0^{\left[\frac{x}{a(1-\alpha)}\right]} \left[1 - \frac{a(1-\alpha)t}{x}\right]^{\frac{\eta}{1-\alpha}} f(t)\, dt \ldots (1.1)$$

where $f(x) \in L(a, b), \eta \in C, R(\eta) > 0, a > 0$ and 'pathway parameter' $\alpha < 1$.

The pathway model introduced by Mathai [6] and further studied by the Mathai and Haubold [7], [8]. For real scalar α, the pathway model for scalar random variables is represented by the following probability density function (p. d. f.).

$$f(x) = c|x|^{\gamma-1}\left[1 - a(1-\alpha)|x|^{\delta}\right]^{\frac{\beta}{1-\alpha}} \ldots (1.2)$$

$-\infty < x < \infty, \delta > 0, \beta \geq 0, [1 - a(1-\alpha)|x|^{\delta}]^{\frac{\beta}{1-\alpha}} > 0, \gamma > 0$ where C is the normalizing constant and α is called the pathway parameter. For real α, the normalizing constant is as follows:

$$c = \frac{1}{2}\frac{\delta\left[a(1-\alpha)^{\frac{\gamma}{\delta}} \Gamma\left(\frac{\gamma}{\delta} + \frac{\beta}{1-\alpha} + 1\right)\right]}{\Gamma\left(\frac{\gamma}{\delta}\right) \Gamma\left(\frac{\beta}{1-\alpha} + 1\right)}, \alpha < 1 \quad .. (1.3)$$

$$= \frac{1}{2}\frac{\delta\left[a(1-\alpha)^{\frac{\gamma}{\delta}} \Gamma\left(\frac{\beta}{1-\alpha}\right)\right]}{\Gamma\left(\frac{\gamma}{\delta}\right) \Gamma\left(\frac{\beta}{\alpha-1} - \frac{\gamma}{\delta}\right)}, \quad \text{for } \frac{1}{\alpha-1} - \frac{\gamma}{\delta} > 0, \alpha > 1 \quad \ldots (1.4)$$

$$= \frac{1}{2}\frac{\delta(a\beta)^{\frac{\gamma}{\delta}}}{\Gamma\left(\frac{\gamma}{\delta}\right)}, \quad \text{for } \alpha \to 1 \quad \ldots (1.5)$$

for $\alpha < 1$ it is a finite range density with $\left[1 - a(1-\alpha)|x|^{-\beta}\right]^{\frac{\beta}{1-\alpha}}$ and (1.2) remains in the extended generalized type-1 beta family.

$$f(x) = c|x|^{\gamma-1}\left[1 + a(\alpha-1)|x|^{\delta}\right]^{\frac{\beta}{\alpha-1}} \ldots (1.6)$$

Provided that $-\infty < x < \infty, \delta > 0, \beta \geq 0, \alpha > 1$ which is the extended generalized type-2 beta model for real x. It includes the type-2 beta density, the F-density, the Cauchy density and many more.

Here we consider only the case of pathway parameter $\alpha < 1$. For $\alpha \to 1$, (1.2) and (1.6) take the exponential from, since

$$\lim_{\alpha\to 1} c|x|^{\gamma-1}\left[1 - a(1-\alpha)|x|^{\delta}\right]^{\frac{\beta}{1-\alpha}}$$

$$= \lim_{x\to 1} c|x|^{\gamma-1}\left[1 + a(\alpha-1)|x|^{\delta}\right]^{-\frac{\eta}{\alpha-1}}$$

$$= c|x|^{\gamma-1} e^{-a\eta|x|^{\delta}} \quad \ldots (1.7)$$

This include the generalized Gamma, the Weibull, the Chi-square, the Laplace, Maxwell-

Boltzmann and other related densities.

When $\alpha \to 1_-$, $\left[1 - \frac{a(1-\alpha)t}{x}\right]^{\frac{\eta}{1-\alpha}} \to e^{\frac{a\eta t}{x}}$ U, the operator (1.1) reduces to the Laplace integral transform of f with parameter $\frac{a\eta}{x}$:

$$\left(P_{0_+}^{(\eta,\alpha)} f\right)(x) = x^{\eta} \int_0^{\infty} e^{-\frac{a\eta}{x}} f(t)\, dt = x^{\eta} L_r\left(\frac{a\eta}{x}\right) \ldots (1.8)$$

when $\alpha = 0, a = 1$ then replacing η by $\eta - 1$ in (1.1) the integral operator reduces to the

Riemann-Liouville fractional integral operator.

The following generalized M-series was introduced by Sharma and Jain [10]:

$${}_{\rho}^{\alpha',\beta'}M_{\sigma}\left(a_1, \ldots \ldots a_{\rho};\, b_1, \ldots \ldots b_{\sigma}; z\right) = M(z)$$

The pathway density in (1.2) for α<1, includes the extended type-1 beta density, the triangular density, the uniform density and many other p.d.f. for α>1, we have

$$ {}_{\rho}M_{\sigma}^{\alpha',\beta'}(z) = \sum_{k=0}^{\infty} \frac{(a'_1)_k \dots (a'_\rho)_k}{(b'_1)_k \dots (b'_\sigma)_k} \frac{z^k}{\Gamma(\alpha' k+\beta')} \quad \dots (1.9)$$

$$= \psi_1(k)$$

where $z, \alpha', \beta' \epsilon\, c, Re(\alpha') > 0, \forall\, z \text{ if } \rho \le \sigma, |z| < \alpha'^{\alpha'}$ for other details see [10]

Fox H-function[4] was studied by Skibiński [13] and defined as a following manner:

$$H_{P,Q}^{M,N}[Z] = H_{P,Q}^{M,N}\left[Z \left| \begin{matrix} (e_P, E_P) \\ (f_Q, F_Q) \end{matrix} \right.\right] = \sum_{h=1}^{N}\sum_{v=0}^{\infty} \frac{(-1)^v X(\xi)}{v! E_h}\left(\frac{1}{z}\right)^{\xi}$$
... (1.10)

where $\xi = \frac{e_h - 1 - v}{E_h}$ and (h=1, 2,……….N)

and

$$X(\xi) = \frac{\{\prod_{j=1}^{M}\Gamma(f_j + F_j\xi)\}\{\prod_{j=1, j\ne h}^{N}\Gamma(1 - e_j - E_j\xi)\}}{\{\prod_{j=M+1}^{Q}\Gamma(1 - f_j - F_j\xi)\}\{\prod_{j=N+1}^{P}\Gamma(e_j + E_j\xi)\}}$$

$$= \psi_2(\xi)$$

for convergence condition and other details see[4],[11].

A generalized Mittag-Leffler function studied by Shukla & Prajapati [12] in the following manner-

$$E_{\alpha,\beta}^{\gamma,q}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}}{\Gamma(\alpha n+\beta)} \frac{(z)^n}{n!} \rho, \beta, \gamma \epsilon C, R(\rho) > 0, R(\beta) > 0 \quad \dots (1.11)$$

where $\alpha, \beta, \gamma\, \varepsilon\, C, Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0$ and $q\, \varepsilon (0,1) U\, N$. This is a generalization of the exponential function exp(z), the confluent hyper geometric function $\phi(\gamma, \alpha, z)$. The Mittag –Leffler function E_α (z), the wimenens's function $E_{\alpha,\beta}(z)$ and the function $E^{\gamma}_{\alpha,\beta}(z)$defined by Prabhakar.

$$(\gamma)_{qn} = \frac{\Gamma(\gamma + qn)}{\Gamma(\gamma)}$$

Denotes the generalized Pochhamamer symbol which in particular reduces to $q^{qn}\prod_{r=1}^{q}\left[\frac{n+r-1}{q}\right] n$ if $q \epsilon N$.

The G-functions is defined by Lorenzo & Hartley [5]:

$$G_{q,\mu,r}[a,z] = z^{rq-\mu-1}\sum_{n=0}^{\infty}\frac{(r)_n (az^q)^n}{\Gamma[(1+n)]\Gamma[(nq+rq-\mu)]} \quad \dots (1.12)$$

2. MAIN RESULTS

Theorem: 1

Let $\eta, \gamma, \delta, \beta, T_1, T_2 > 0, Re(\eta) > 0, Re(\gamma) > 0, Re(\omega) > 0, Re\left(1 + \frac{\eta}{1-\alpha}\right) > max,\ [0, -Re(\omega)], b, c, \epsilon R, \alpha < 1, Re\left[\omega + \delta\frac{f_j}{F_j}\right] > 0,\ Re\left[\omega + \beta\frac{b_j}{B_j}\right] |arg\, c| < \frac{1}{2}T_1\pi, |arg\, b| < \frac{1}{2}T_2\pi, \rho \le \sigma \text{ and } |d| < \alpha'^{\alpha'}, \beta^* > 0,\ j = 1, \dots\dots\dots, Q, j' = 1, \dots\dots\dots, q$

Then

$$P_{0+}^{(\eta,\alpha)}\left[t^{\omega-1}\ {}_{\rho}M_{\sigma}^{\alpha',\beta'}[dt^{-\beta^*}]\ H_{P,Q}^{M,N}\left[ct^{\delta}\left|\begin{matrix}(e_P, E_P)\\(f_Q, F_Q)\end{matrix}\right.\right] E_{\alpha,\beta}^{\gamma,k}(bt^{\beta})\right](x)$$

$$= \psi_1(k)\frac{d^k x^{\eta+\omega-\beta^* k}\Gamma\left(1 + \frac{\eta}{1-\alpha}\right)}{[a(1-\alpha)]^{\omega-\beta^* k}} H_{P,Q}^{M,N}\left[\frac{cx^{\delta}}{[a(1-\alpha)]^{\delta}}\left|\begin{matrix}(e_P, E_P)\\(f_Q, F_Q)\end{matrix}\right.\right]$$

$$\cdot\ {}_2\psi_2\left[\frac{bx^{\beta}}{a(1-\alpha)^{\beta}}\left|\begin{matrix}\omega - \beta^* k - \delta\xi - \beta & & (\gamma, \delta)\\ (\rho, \beta) & & (1 + \omega + \frac{\eta}{1-\alpha} - \delta\xi - \beta * k, \beta)\end{matrix}\right.\right]$$
... (2.1)

Proof: Making use (1.9), (1.10), (1.11) and (1.1)in the theorem 1 then interchanging the order of integration and summation by means of beta function we at once arrive at the desired result (2.1).

Theorem: 2

Let $\eta, \omega\ \epsilon\ C, Re(\beta) > 0, Re(\delta) > 0, Re\left(1 + \frac{\eta}{1-\alpha}\right) > 0, Re(\rho) > 0, \alpha < 1, b\epsilon R,$

$$Re\left[\omega + \delta\frac{f_j}{F_j}\right] > 0, Re\left[\omega + \beta\frac{b_j}{B_j}\right] > 0\ |arg\, c| < \frac{1}{2}T_1\pi, \rho \le \sigma, |arg\, b| < \frac{1}{2}T_2\pi,$$

$$\beta^* > 0, T_1, T_2 > 0, \rho \le \sigma, |d| < \alpha'^{\alpha'}, j = 1, \dots\dots\dots, Q, j' = 1, \dots\dots\dots, q$$

Then

$$P_{0+}^{(\eta,\alpha)}\left[t^{\omega-1}\ {}_{\rho}M_{\sigma}^{\alpha',\beta'}[dt^{-\beta^*}]\ H_{P,Q}^{M,N}\left[ct^{\delta}\left|\begin{matrix}(e_P, E_P)\\(f_Q, F_Q)\end{matrix}\right.\right] G_{q,\delta,\gamma}[\omega t]\right](x)$$

$$= \psi_1(k)\frac{d^k x^{\eta+\omega-\beta^* k+qr-\delta\xi}\Gamma\left(1 + \frac{\eta}{1-\alpha}\right)}{[a(1-\alpha)]^{\omega-\beta^* k+qr-\delta\xi}} H_{P,Q}^{M,N}\left[\frac{cx^{\delta}}{[a(1-\alpha)]^{\delta}}\left|\begin{matrix}(e_P, E_P)\\(f_Q, F_Q)\end{matrix}\right.\right] G_{q,\delta-\frac{\eta}{1-\alpha}-1,\gamma}\left[\frac{\omega x}{a(1-\alpha)}\right]$$
... (2.2)

Proof: Making use (1.9), (1.10), (1.12) and (1.1)in the theorem 1 then interchanging the order of integration and summation by means of beta function we at once arrive at the desired result (2.2).

Special Cases:

1. If we take k=1, $\alpha \rightarrow \beta$ *and* $\beta \rightarrow \omega$ in theorem-1 then we get the known result of [3, eq. (2.2)].
2. If we take $\beta^* \rightarrow 0, \delta \rightarrow 0, \omega \rightarrow \rho$ and k=1 in theorem-1 then we get the known result of [9, eq. (25) p.244].
3. If we take $\beta^* \rightarrow 0, \delta \rightarrow 0, \omega \rightarrow \rho$ *and* $\gamma = k = 1$ in theorem-1 then we get the known result of [9, eq. (26) p.245].
4. If we take $\beta^* \rightarrow 0, \delta \rightarrow 0, \omega = 1$ e and in theorem-2 then we get the known result of [10, eq. (3.3)].

CONCLUSION

The pathway fractional integral operator is expected to have wide application in statistical distribution theory. It could help to extend some classical statistical distribution to wider classes of distribution. During the last three decades fraction calculus has been applied to almost every field of science, engineering and mathematics.

REFERENCES:

1. Chaurasia, V.B.L. and Ghiya, Neeti, Pathway fractional integral operator pertaining to special functions, Global J.Sci., Front. Res., 10(6)(Ver.1.0), (2010),pp.79-83.
2. Chaurasia, V.B.L. and Gill,Vinod, Pathway fractional integral operator involving H-functions (Communicated).
3. Chaurasia V.B.L. and Singh J, Pathway fractional integral operator associated with certain special functions, Global journals inc. (USA), Vol-12, Issue-9 (2012).
4. Fox, C., The G and H-functions as symmetrical Fourier kernels, Trans. Amer. Math.Soc., 98 (1961),pp.395-429.
5. Lorenzo C.F. and Hartley T.T., Generalized functions for the fractional calculus, NASA Tech. Pub.209424 (1999) pp.1-17.
6. Mathai, A.M., A pathway to matrix variate gamma and normal densities, Linear Algebra and its Applications, 396 (2005), pp.317-328.
7. Mathai, A.M. and Haubold, H.J., on generalized distribution and pathways, Phy. Letters, 372(2008), pp.2109-2113.
8. Mathai, A.M. and Haubold, H.J., Pathway models, superstatisties, trellis statistics and a generalized measure of entropy, Physica, A375 (2007),pp.110-122.
9. Nair, Seema S., Pathway fractional integration operator, Fract. Cal. Appl. Anal., 12(3) (2009),pp.237-259.
10. Saxena H., Jaimini B.B., On certain type of generalized Mittag-Leffler function and generalized G-function via Pathway fractional integral operator (2015),communicate.
11. Sharma, Manoj and Jain, Renu, A note on a generalized M-series as a special function of fractional calculus, Fract. Cal. Appl. Anal., 12(4) (2009), pp.449-452.
12. Shukla A.K., Prajapati J.C., Some remarks on generalized Mittag-Leffler function, Proyecciones, vol.-28, pp.27-34 (2009).
13. Skibiński, P., Some expansion Theorems for the H-function, Ann. Polon. Math., 23 (1970),pp.125-138.