Certain Integrals of Generalized Hypergeometric and Confluent Hypergeometric Functions

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Abstract:- In this paper, we aim at establishing certain finite integral formulas for the generalized Gauss hypergeometric and confluent hypergeometric functions. Furthermore, the $F_p^{(\alpha,\beta)}(a,b;c;z)$-function occurring in each of our main results can be reduced, under various special cases, to such simpler functions as the classical Gauss hypergeometric function ${}_2F_1$, Gauss confluent hypergeometric function $\varphi_p^{(\alpha,\beta)}(b;c;z)$ function and generalized hypergeometric function ${}_pF_q$. A specimen of some of these interesting applications of our main integral formulas are presented briefly.

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1 INTRODUCTION AND PRELIMINARIES

In many areas of applied mathematics, various types of special functions become essential tools for scientists and engineers. The continuous development of mathematical physics, probability theory and other areas has led to new classes of special functions and their extensions and generalizations (see, for details, [17] and the references cited therein; see also [16, 18, 19]).
A lot of research work has recently come up on the study and development of the functions, which are more general than the Beta type function $\beta(x,y)$, popularly known as generalized Beta type functions. These functions, as a part of the theory of confluent hypergeometric functions, are important special functions and their closely related ones are widely used in physics and engineering. Moreover, generalized Beta functions [2, 3] have played a pivotal role in the advancement of further research and have proved to be exemplary in nature. The Eulers gamma function $\Gamma(z)$ is one of the most fundamental special functions, because of its important role in various fields in the mathematical, physical, engineering and statistical sciences. Various generalizations of the gamma function can be found in the literature [1, 5, 7, 9, 21].

The following extension of the gamma function is introduced by Chaudhry and Zubair [1]:

$$\Gamma_p(z) = \begin{cases} \int_0^\infty t^{z-1} \exp\left(-t - \frac{p}{t}\right) dt, & \Re(p) > 0; z \in \mathbb{C} \\ \Gamma(z), & \Re(z) > 0 \end{cases}. \tag{1.1}$$

The extension of Euler's beta function is considered by Chaudhry et al. [2] in the following form:

$$\beta_p(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} \exp\left(\frac{-p}{t(1-t)}\right) dt, \quad \Re(p) > 0, \Re(x) > 0, \Re(y) > 0. \tag{1.2}$$

Chaudhry et al. [3] used $\beta_p(x,y)$ to extend the hypergeometric function, known as the extended Gauss hypergeometric function, as follows:

$$F_p(a,b;c;z) = \sum_{n=0}^{\infty} (a)_n \frac{\beta_p(b+n, c-b)}{\beta(b, c-b)} \frac{(z)^n}{n!}, \quad p \geq 0, \Re(c) > \Re(b) > 0, \tag{1.3}$$

where $(a)_n$ denotes the Pochhammer symbol defined as

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1, \ n = 0; \ a \in \mathbb{C}/\{0\} \\ a(a+1)(a+2)\ldots(a+n-1), & n \in \mathbb{N}, a \in \mathbb{C}. \end{cases}$$

The integral representation of Euler's type function is

$$F_p(a,b;c;z) = \frac{1}{\beta(\mathrm{b}, c-b)} \int_0^1 t^{\mathrm{b}-1} \ (1-t)^{\mathrm{c}-b-1}(1-zt)^{-a} \exp\left(\frac{-p}{t(1-t)}\right) dt, \tag{1.4}$$

where $p \geq 0$ and $|\arg(1-z)| < \pi < p; \Re(c) > \Re(b) > 0$.
Also, the extended confluent hypergeometric function is defined as

$$\varphi_p(b;c;z) = \sum_{n=0}^{\infty} \frac{\beta_p(b+n, c-b)}{\beta(b, c-b)} \frac{(z)^n}{n!}, \quad p \geq 0, \ \Re(c) > \Re(b) > 0. \tag{1.5}$$

The transformation formulas, recurrence relations, summation and asymptotic formulas, differentiation properties, the Mellin transforms and some new representations of these extended functions can be found in many earlier work [3, 10, 12, 22, 19].

The generalized Euler's gamma function is defined in [12] as

$$\Gamma_p^{(\alpha,\beta)}(x) = \int_0^\infty t^{x-1} {}_1F_1\left(\alpha; \beta; -t - \frac{p}{t}\right) dt, \ \ \Re(\alpha) > 0, \Re(\beta) > 0, \Re(p) > 0, \Re(x) > 0. \tag{1.6}$$

Recently, Özergin [11] introduced and studied some fundamental properties and characteristics of the generalized Beta type function $\beta_p^{(\alpha,\beta)}(x,y)$ in their Ph.D. Thesis and defined by (see, e.g., [11, p.32]):

$$\beta_{\mathrm{p}}{}^{(\alpha,\beta)}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} \ {}_1F_1\left(\alpha; \beta; \frac{-p}{t(1-t)}\right) \ dt, \tag{1.7}$$

with $\Re(p) \geq 0, \min(\Re(x), \Re(y), \Re(\alpha), \Re(\beta)) > 0$ and $\beta_0^{(\alpha,\beta)}(x,y) = \beta(x,y)$, where $\beta(x,y)$ is a well known Eulers Beta function defined by

$$\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, dt, \quad ((\Re(x)) > 0,\ (\Re(y)) > 0)\,. \tag{1.8}$$

Similarly, by appealing to $\beta_p^{\alpha,\alpha}(x,y)$, Özergin et al. introduced and investigated a further extension of the following potentially useful generalized Gauss hypergeometric functions defined as follows (see, e.g., [12, p.4606, Sec.3]; see also [11, p.39, Ch.4]):

$$F_p^{(\alpha,\beta)}(a,b;c;z) = \sum_{n=0}^{\infty} (a)_n \frac{\beta_p^{(\alpha,\beta)}(b+n, c-b)}{\beta(b, c-b)} \frac{z^n}{n!}, \quad (|z| < 1)\,, \tag{1.9}$$

and

$$_1F_1^{(\alpha,,\beta;p)}(b;c;z) = \sum_{n=0}^{\infty} \frac{\beta_p^{(\alpha,\beta)}(b+n, c-b)}{\beta(b, c-b)} \frac{z^n}{n!}, \quad (|z| < 1)\,, \tag{1.10}$$

corresponding integral representations are given by [12]:

$$F_p^{(\alpha,\beta)}(a,b;c;z) = \frac{1}{\beta(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}\, {}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)(1-zt)^{-a}\, dt, \tag{1.11}$$

for $\Re(p) \geq 0$, and $|\arg(1-z)| < \pi < p$; $\Re(c) > \Re(b) > 0$.

It is obvious to see that [3]:

$$F_p^{(\alpha,\alpha)}(a,b;c;z) = F_p(a,b;c;z)\,, \quad F_0^{(\alpha,\beta)}(a,b;c;z) = {}_2F_1(a,b;c;z)\,;$$
$${}_1F_1^{(\alpha,\alpha;p)}(b;c;z) = {}_1F_1^{p}(b;c;z) = \varphi_p(b;c;z)\,, \quad {}_1F_1^{(\alpha,\beta;0)}(b;c;z) = {}_1F_1(b;c;z)\,. \tag{1.12}$$

where the ${}_2F_1(.)$ is a special case of the well-known generalized hypergeometric series ${}_pF_q(.)$ defined by (see, e.g., [19, Sec.1.5]; see also [20]).

$${}_pF_q\left[\begin{matrix}\alpha_1, ..., \alpha_p; \\ \beta_1, ..., \beta_q;\end{matrix}\ z\right] = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \frac{z^n}{n!} = {}_pF_q(\alpha_1, \ldots, \alpha_p;\ \beta_1, \ldots, \beta_q;\ z)\,, \tag{1.13}$$

where $z \in \mathbb{C}, p \leq q,\ \alpha_i, \beta_j \in \mathbb{C}, \beta_j \neq 0, -1, -2, \ldots,\ (i = 1, 2, \ldots, p,\ j = 1, 2, \ldots, q)$.

2 INTEGRALS INVOLVING GENERALIZED GAUSS HYPERGEO-METRIC AND CONFLUENT HYPERGEOMETRIC FUNCTION

In this section we calculate the $F_p^{(\alpha,\beta)}(a,b;c;z)$-function with some algebraic function.

Theorem 1 *For the generalized Gauss hypergeometric function, we have the following integral*

$$\int_0^1 x^{-\rho}(1-x)^{\rho-\sigma-1} F_p^{(\alpha,\beta)}(a,b;c;kx)\,dx$$
$$= \beta(1-\rho,\rho-\sigma)\sum_{n=0}^{\infty}\frac{(a)_n(1-\rho)_n}{(1-\sigma)_n}\frac{\beta_p^{(\alpha,\beta)}(b+n,c-b)}{\beta(b,c-b)}\frac{k^n}{n!}, \tag{2.1}$$

where, $\Re(p)\geq 0,$ *and* $|\arg(1-kx)|<\pi<p;\ \Re(c)>\Re(b)>0$

Proof Making use of relation (1.9), it gives

$$\mathcal{I}_1 = \int_0^1 x^{-\rho}(1-x)^{\rho-\sigma-1}\sum_{n=0}^{\infty}(a)_n\frac{\beta_p^{(\alpha,\beta)}(b+n,c-b)}{\beta(b,c-b)}\frac{(kx)^n}{n!}\,dx$$

$$=\sum_{n=0}^{\infty}(a)_n\frac{\beta_p^{(\alpha,\beta)}(b+n,c-b)}{\beta(b,c-b)}\frac{(k)^n}{n!}\int_0^1 x^{n-\rho}(1-x)^{\rho-\sigma-1}\,dx$$

$$=\sum_{n=0}^{\infty}(a)_n\frac{\beta_p^{(\alpha,\beta)}(b+n,c-b)}{\beta(b,c-b)}\frac{(k)^n}{n!}\beta(n-\rho+1,\rho-\sigma)$$

$$=\sum_{n=0}^{\infty}(a)_n\frac{\beta_p^{(\alpha,\beta)}(b+n,c-b)}{\beta(b,c-b)}\frac{(k)^n}{n!}\frac{\Gamma(n-\rho+1)\Gamma(\rho-\sigma)}{\Gamma(n-\sigma+1)}$$

$$=\frac{\Gamma(1-\rho)\Gamma(\rho-\sigma)}{\Gamma(1-\sigma)}\sum_{n=0}^{\infty}(a)_n\frac{\beta_p^{(\alpha,\beta)}(b+n,c-b)}{\beta(b,c-b)}\frac{(k)^n}{n!}\frac{(1-\rho)_n}{(1-\sigma)_n}$$

$$=\beta(1-\rho,\rho-\sigma)\sum_{n=0}^{\infty}\frac{(a)_n(1-\rho)_n}{(1-\sigma)_n}\frac{\beta_p^{(\alpha,\beta)}(b+n,c-b)}{\beta(b,c-b)}\frac{k^n}{n!}.$$

This complete the proof of the Theorem 1.

If we set $p=0$ in above result then we obtain the special case of (2.1) in terms of classical Gauss hypergeometric function as given in the following result:

Corollary 1.1

$$\int_0^1 x^{-\rho}(1-x)^{\rho-\sigma-1}\,{}_2F_1(a,b;c;kx)\,dx = \beta(1-\rho,\rho-\sigma)\,{}_3F_2\left(\begin{array}{c}a,b,1-\rho\\1-\sigma\end{array};\,k\right). \tag{2.2}$$

The integral of Gauss Confluent hypergeometric function is given by

Corollary 1.2

$$\int_0^1 x^{-\rho}(1-x)^{\rho-\sigma-1}\,\varphi_p^{(\alpha,\beta)}(b;c;kx)\,dx$$
$$=\sum_{n=0}^{\infty}\beta(1+n-\rho,\rho-\sigma)\,\frac{\beta_p^{(\alpha,\beta)}(b+n,c-b)}{\beta(b,c-b)}\frac{(k)^n}{n!}. \tag{2.3}$$

Moreover, for the generalized hypergeometric function ${}_pF_q$, we have the following corollary:

Corollary 1.3

$$\int_0^1 x^{-\rho}(1-x)^{\rho-\sigma-1}\,{}_pF_q\left[\begin{array}{c}\alpha_1,\ldots,\alpha_p;\\ \beta_1,\ldots,\beta_q;\end{array}kx\right]dx$$
$$=\beta(1-\rho,\rho-\sigma)\,{}_{p+1}F_{q+1}\left[\begin{array}{c}\alpha_1,\ldots,\alpha_p,1-\rho\\ \beta_1,\ldots,\beta_q,1-\sigma\end{array};\,k\right] \tag{2.4}$$

for $x\in\mathbb{C}, p\le q;\ \alpha_j,\beta_j\in\mathbb{C},\beta\ne 0,-1,-2,\ldots;\ i=\left(\overline{1,p}\right), j=\left(\overline{1,q}\right)$.

Theorem 2

$$\int_1^\infty x^{-\rho}(x-1)^{\sigma-1}F_p^{(\alpha,\beta)}(a,b;c;kx)\,dx$$
$$=\beta(\sigma,\rho-\sigma)\sum_{n=0}^{\infty}\frac{(a)_n(\rho)_{-n}}{(\rho-\sigma)_{-n}}\frac{\beta_p^{(\alpha,\beta)}(b+n,c-b)}{\beta(b,c-b)}\frac{k^n}{n!}. \tag{2.5}$$

Proof

$$\mathcal{I}_2=\int_1^\infty x^{-\rho}(x-1)^{\sigma-1}\sum_{n=0}^{\infty}(a)_n\frac{\beta_p^{(\alpha,\beta)}(b+n,c-b)}{\beta(b,c-b)}\frac{(kx)^n}{n!}\,dx$$
$$=\sum_{n=0}^{\infty}(a)_n\frac{\beta_p^{(\alpha,\beta)}(b+n,c-b)}{\beta(b,c-b)}\frac{(k)^n}{n!}\int_1^\infty x^{-\rho+n}(x-1)^{\sigma-1}\,dx.$$

Let $x=1+t$, then we arrive at

$$\mathcal{I}_2=\sum_{n=0}^{\infty}(a)_n\frac{\beta_p^{(\alpha,\beta)}(b+n,c-b)}{\beta(b,c-b)}\frac{(k)^n}{n!}\int_0^\infty t^{\sigma-1}(1+t)^{-\rho+n}\,dt,$$

using the formula

$$\Gamma(\alpha)\Gamma(\beta)=\Gamma(\alpha+\beta)\int_0^\infty x^{\alpha-1}(1+x)^{-(\alpha+\beta)}\,dx, \tag{2.6}$$

then we have

$$\mathcal{I}_2=\sum_{n=0}^{\infty}(a)_n\frac{\beta_p^{(\alpha,\beta)}(b+n,c-b)}{\beta(b,c-b)}\frac{(k)^n}{n!}\frac{\Gamma(\sigma)\Gamma(\rho-\sigma-n)}{\Gamma(\rho-n)}$$

$$= \beta\left(\sigma, \rho-\sigma\right) \sum_{n=0}^{\infty} (a)_n \frac{\beta_p^{(\alpha,\beta)}\left(b+n, c-b\right)}{\beta\left(b, c-b\right)} \frac{(k)^n}{n!} \frac{(\rho)_{-n}}{(\rho-\sigma)_{-n}}.$$

This complete the proof of the Theorem 2.

We can also obtain special cases of Theorem 2 as done in Theorem 1.

Theorem 3

$$\int_0^{\infty} e^{-kx} x^{\rho-1} F_p{}^{(\alpha,\beta)}\left(a, b; c; lx\right) dx$$
$$= \sum_{n=0}^{\infty} \frac{\Gamma\left(\rho\right)}{k^{\rho}} (a)_n \left(\rho\right)_n \frac{\beta_p{}^{(\alpha,\beta)}\left(b+n, c-b\right)}{n! \beta\left(b, c-b\right)} \left(\frac{l}{k}\right)^n, \quad k \neq 0. \tag{2.7}$$

Proof Using (1.11) and $(1-lxt)^{-a} = \sum_{n=0}^{\infty} \frac{(a)_n}{n!} (lxt)^n$, we have

$$\mathcal{I}_3 = \frac{1}{\beta\left(b, c-b\right)} \int_0^{\infty} \int_0^1 t^{b-1} \left(1-t\right)^{c-b-1} e^{-kx} x^{\rho-1} {}_1F_1\left(\alpha; \beta; \frac{-p}{t(1-t)}\right)$$
$$\times \sum_{n=0}^{\infty} \frac{(a)_n}{n!} \left(lxt\right)^n \, dx\, dt, \tag{2.8}$$

$$= \frac{1}{\beta\left(b, c-b\right)} \sum_{n=0}^{\infty} (a)_n \frac{l^n}{n!} \int_0^{\infty} \int_0^1 t^{b+n-1} \left(1-t\right)^{c-b-1} e^{-kx} x^{\rho+n-1} {}_1F_1\left(\alpha; \beta; \frac{-p}{t(1-t)}\right) dx\, dt$$

$$= \sum_{n=0}^{\infty} \frac{(a)_n l^n}{\beta\left(b, c-b\right) n!} \int_0^{\infty} e^{-kx} x^{\rho+n-1} dx \int_0^1 t^{b+n-1} \left(1-t\right)^{c-b-1} {}_1F_1\left(\alpha; \beta; \frac{-p}{t(1-t)}\right) dt$$

$$= \sum_{n=0}^{\infty} \frac{(a)_n l^n}{\beta\left(b, c-b\right) n!} \int_0^{\infty} e^{-kx} x^{\rho+n-1} \beta_p^{(\alpha,\beta)}\left(b+n, c-b\right), dx$$

because,

$$\frac{1}{k} \int_0^{\infty} e^{-\sigma} \left(\frac{\sigma}{k}\right)^{\rho+n-1} d\sigma = \frac{1}{k^{\rho+n}} \int_0^{\infty} e^{-\sigma} \left(\sigma\right)^{\rho+n-1} d\sigma$$

then we have

$$\mathcal{I}_3 = \sum_{n=0}^{\infty} \frac{(a)_n (\rho)_n \Gamma\left(\rho\right)}{k^{\rho}\, n!} \frac{\beta_p^{(\alpha,\beta)}\left(b+n, c-b\right)}{\beta\left(b, c-b\right)} \left(\frac{l}{k}\right)^n.$$

This complete the proof of Theorem 3.

If we set $p = 0$ in (2.7) then we obtain the following corollary:

Corollary 3.1

$$\int_0^{\infty} e^{-kx} x^{\rho-1} {}_2F_1\left(a, b; c; lx\right) dx = \frac{\Gamma(\rho)}{k^{\rho}} {}_3F_2\left(\begin{matrix} a, b; \rho \\ c \end{matrix}; \frac{l}{k}\right), \quad k \neq 0. \tag{2.9}$$

The integral of Gauss Confluent hypergeometric function is given by the following result:

Corollary 3.2

$$\int_0^\infty e^{-kx} x^{\rho-1}\, \varphi_p^{(\alpha,\beta)}(b;c;lx)\, dx = \sum_{n=0}^{\infty} \frac{\Gamma(\rho+n)}{k^\rho\, n!} \frac{\beta_p^{(\alpha,\beta)}(b+n,c-b)}{\beta(b,c-b)} \left(\frac{l}{k}\right)^n, \quad k \neq 0. \tag{2.10}$$

This result is in complete agreement with the result given in [6, p.98].

Next, for the generalized hypergeometric function ${}_pF_q$, we have the following corollary:

Corollary 3.3 *For $x \in \mathbb{C}, p \le q; \alpha_j, \beta_j \in \mathbb{C},\ \beta \neq 0,-1,-2,...;\ \left(i = (\overline{1,p}); j = (\overline{1,q})\right)$, we have*

$$\int_0^\infty e^{-kx} x^{\rho-1}\, {}_pF_q\left[\begin{matrix}\alpha_1,...,\alpha_p; \\ \beta_1,...,\beta_q;\end{matrix}\ lx\right] dx = \frac{\Gamma(\rho)}{k^\rho}\, {}_{p+1}F_q\left[\begin{matrix}\alpha_1,...,\alpha_p,\rho \\ \beta_1,...,\beta_q\end{matrix}\ ;\ \frac{l}{k}\right], \quad k \neq 0. \tag{2.11}$$

Theorem 4

$$\int_0^\infty x^{\rho-1}(x+\beta)^{-\sigma} F_p^{(\delta,\eta)}(a,b;c;kx)\, dx = \beta(\rho,\sigma-\rho)\, \beta^{\rho+n-\sigma} \sum_{n=0}^{\infty} (a)_n (\rho)_n (\sigma-\rho)_{-n} \frac{\beta_p^{(\delta,\eta)}(b+n,c-b)}{\beta(b,c-b)} \frac{k^n}{n!}. \tag{2.12}$$

Proof

$$\mathcal{I}_4 = \int_0^\infty x^{\rho-1}(x+\beta)^{-\sigma} \sum_{n=0}^{\infty} (a)_n \frac{\beta_p^{(\delta,\eta)}(b+n,c-b)}{\beta(b,c-b)} \frac{(kx)^n}{n!}\, dx$$

$$= \sum_{n=0}^{\infty} (a)_n \frac{\beta_p^{(\delta,\eta)}(b+n,c-b)}{\beta(b,c-b)} \frac{(k)^n}{n!} \int_0^\infty x^{\rho+n-1}(x+\beta)^{-\sigma}\, dx.$$

Let $x = \beta t$, then we arrive at

$$\mathcal{I}_4 = \sum_{n=0}^{\infty} (a)_n \frac{\beta_p^{(\delta,\eta)}(b+n,c-b)}{\beta(b,c-b)} \frac{(k)^n}{n!} \beta^{\rho-\sigma+n} \int_0^\infty t^{\rho+n-1}(1+t)^{-\sigma}\, dt,$$

using the relation (2.6), then we have

$$\mathcal{I}_4 = \sum_{n=0}^{\infty} (a)_n \frac{\beta_p^{(\delta,\eta)}(b+n,c-b)}{\beta(b,c-b)} \frac{(k)^n}{n!} \beta^{\rho+n-\sigma} \frac{\Gamma(\sigma+n)\,\Gamma(\sigma-\rho-n)}{\Gamma(\sigma)}$$

$$= \beta(\rho,\sigma-\rho) \sum_{n=0}^{\infty} \beta^{\rho+n-\sigma} (a)_n (\rho)_n (\sigma-\rho)_{-n} \frac{\beta_p^{(\delta,\eta)}(b+n,c-b)}{\beta(b,c-b)} \frac{(k)^n}{n!},$$

then we easily get the R.H.S. of (2.12). This complete the proof of the Theorem 4.

Theorem 5

$$\int_{-1}^{1} (1-x)^{\rho} (1+x)^{\sigma} \; F_p^{(\alpha,\beta)} (a,b;c;kx)\, dx$$
$$= 2^{\rho+\sigma+1} \beta(\rho+1,\sigma+1) \sum_{n=0}^{\infty} \frac{(a)_n \, \beta_p^{(\alpha,\beta)} (b+n,c-b)}{\beta(b,c-b)} F(-n,\rho+1,\rho+\sigma+2;2) \frac{k^n}{n!}. \tag{2.13}$$

Proof

$$\mathcal{I}_5 = \int_{-1}^{1} (1-x)^{\rho} (1+x)^{\sigma} \sum_{n=0}^{\infty} (a)_n \frac{\beta_p^{(\alpha,\beta)} (b+n,c-b)}{\beta(b,c-b)} \frac{(kx)^n}{n!} dx$$

$$= \sum_{n=0}^{\infty} (a)_n \frac{\beta_p^{(\alpha,\beta)} (b+n,c-b)}{\beta(b,c-b)} \frac{(k)^n}{n!} \int_{-1}^{1} (1-x)^{\rho} (1+x)^{\sigma} \; x^n \,\mathrm{dx}.$$

Now, by putting $\frac{1-x}{2} = t \Rightarrow dx = -2\,dt$, and using the integral representation of Gauss hypergeometric series

$$F(a,b;c;x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-tx)^{-a} \; dt, \tag{2.14}$$

then we arrive at the following result after a little simplification:

$$\mathcal{I}_5 = 2^{\rho+\sigma+1} \frac{\Gamma(\rho+1)\Gamma(\sigma+1)}{\Gamma(\rho+\sigma+2)} \sum_{n=0}^{\infty} \frac{(a)_n \, \beta_p^{(\alpha,\beta)} (b+n,c-b)}{\beta(b,c-b)} \frac{(k)^n}{n!}$$
$$\times \; F(-n,\rho+1,\rho+\sigma+2;2),$$

then we obtain the desired result in (2.13). This completes the proof.

The next theorem considers the behavior of the generalized Gauss hypergeometric function using the gamma function.

Theorem 6

$$\lim_{\gamma\to -l} (\Gamma(\gamma))^{-1} F_p{}^{(\alpha,\beta)} (a,b;\gamma;x) = \frac{x^{l+1}}{\Gamma(a)\Gamma(b)\Gamma(-b-l)}$$
$$\times \sum_{r=0}^{\infty} \frac{(a+l+r)!}{(l+r+1)!} \beta_p{}^{(\alpha,\beta)} (b+l+r+1,-b-l)\, x^r. \tag{2.15}$$

Proof Making use of (1.9), we have

$$\mathcal{I}_6 = \sum_{n=0}^{\infty} (a)_n \frac{\beta_p^{(\alpha,\beta)} (b+n,\gamma-b)}{\beta(b,\gamma-b)} \frac{x^n}{n!} \lim_{\gamma\to -l} \frac{1}{\Gamma(\gamma)},$$

by using (1.7), we obtain

$$\mathcal{I}_6 = \sum_{n=0}^{\infty} \frac{(a)_n}{\Gamma(b)} \int_0^1 t^{b+n-1} (1-t)^{-b-1} {}_1F_1 \left(\alpha; \beta; \frac{-p}{t(1-t)} \right) \frac{x^n}{n!} \frac{(1-t)^{-l}}{\Gamma(-l-b)} dt,$$

by setting $n = l + r + 1$ and using (1.7), it yields

$$= \frac{(x)^{l+1}}{\Gamma(b)\Gamma(-l-b)} \sum_{r=0}^{\infty} \frac{(a)_{l+r+1}}{(l+r+1)!} \beta_p^{(\alpha,\beta)} (b+l+r+1, -b-l)\, x^r,$$

then we arrive at the desired result in (2.14).

The special cases of (2.15) for the generalized Gauss hypergeometric function and confluent hypergeometric function are given in the following corollaries:

Corollary 6.1

$$\lim_{\gamma \to -l} (\Gamma(\gamma))^{-1} {}_2F_1(a, b; \gamma; x) = x^{l+1} \frac{(a)_{l+1} (b)_{l+1}}{(l+1)!} {}_2F_1 \left[\begin{array}{c} a+l+1, b+l+1 \\ l+2 \end{array} ; x \right]. \quad (2.16)$$

Corollary 6.2

$$\lim_{\gamma \to -l} (\Gamma(\gamma))^{-1} \varphi_p^{(\alpha,\beta)}(b; \gamma; x) = \frac{x^{l+1}}{\Gamma(-l)} \sum_{r=0}^{\infty} \frac{1}{\Gamma(l+r+2)} \frac{\beta_p^{(\alpha,\beta)}(b+l+r+1, -b-l)}{\beta(b, -l-b)} \frac{x^r}{r!}. \quad (2.17)$$

3 INTEGRALS INVOLVING GAUSS HYPERGEOMETRIC FUNC-TION WITH JACOBI POLYNOMIALS

The Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ [13, p. 254] is defined as following:

$$P_n^{(\alpha,\beta)}(x) = \frac{(1+\alpha)_n}{n!} {}_2F_1 \left[\begin{array}{cc} -n,\ 1+\alpha+\beta+n; & \frac{1-x}{2} \\ 1+\alpha; & \end{array} \right], \quad (3.1)$$

where ${}_2F_1$ is the classical hypergeometric functions; when $\alpha = \beta = 0$, then the polynomial in (3.1) becomes the Legendre polynomial [13, p. 157].
We also have

$$P_n^{(\alpha,\beta)}(1) = \frac{(1+\alpha)_n}{n!}.$$

Theorem 7 *Integral formula involving Gauss hypergeometric function multiplied with Jacobi polynomials is given by*

$$\begin{aligned} &\int_{-1}^{1} x^{\lambda} (1-x)^{\alpha} (1+x)^{\mu} P_n^{(\alpha,\beta)}(x)\ F_p^{(\delta,\eta)}(a, b; c; kx)\, dx \\ &= \frac{(-1)^n\ 2^{\alpha+\mu+1} \beta(\mu+1, n+\alpha+1)}{n!\ (\mu+\beta+1)_n} \sum_{r=0}^{\infty} \frac{(a)_r \beta_p^{(\delta,\eta)}(b+r, c-b)}{\beta(b, c-b)} \frac{k^r}{r!} \\ &\times {}_3F_2 \left[\begin{array}{c} -\lambda - r,\ \mu+\beta+1,\ \mu+1; \\ \mu+\beta+n+1,\ \mu+\alpha+n+2; \end{array} 1 \right]. \end{aligned} \quad (3.2)$$

Proof By using (1.9), we have

$$\mathcal{I}_7 = \int_{-1}^{1} x^{\lambda} (1-x)^{\alpha} (1+x)^{\mu} P_n^{(\alpha,\beta)}(x) \sum_{r=0}^{\infty} (a)_r \frac{\beta_p^{(\delta,\eta)}(b+r, c-b)}{\beta(b,c-b)} \frac{(kx)^r}{r!} dx$$

$$= \sum_{r=0}^{\infty} \frac{(a)_r \beta_p^{(\delta,\eta)}(b+r,c-b)}{\beta(b,c-b)} \frac{(k)^r}{r!} \int_{-1}^{1} x^{\lambda+r} (1-x)^{\alpha} (1+x)^{\mu} P_n^{(\alpha,\beta)}(x)\, dx$$

Next, we use the following formula:

$$\int_{-1}^{1} x^{\lambda} (1-x)^{\alpha} (1+x)^{\mu} P_n^{(\alpha,\beta)}(x)\, dx = (-1)^n \frac{2^{\alpha+\mu+1}\Gamma(\mu+1)\Gamma(n+\alpha+1)\Gamma(\mu+\beta+1)}{n!\,\Gamma(\mu+\beta+n+1)\Gamma(\mu+\alpha+n+2)}$$
$$\times\ {}_3F_2\left[\begin{matrix} -\lambda,\ \mu+\beta+1,\ \mu+1; \\ \mu+\beta+n+1,\ \mu+\alpha+n+2; \end{matrix}\ 1\right], \tag{3.3}$$

where $\alpha > -1$ and $\beta > -1$. Also, ${}_3F_2$ is the special case of generalized hypergeometric series.
Then we arrive at

$$\mathcal{I}_7 = \sum_{r=0}^{\infty} \frac{(a)_r \beta_p^{(\delta,\eta)}(b+r,c-b)}{\beta(b,c-b)} \frac{(k)^r}{r!} \frac{(-1)^n 2^{\alpha+\mu+1}\Gamma(\mu+1)\Gamma(n+\alpha+1)\Gamma(\mu+\beta+1)}{n!\,\Gamma(\mu+\beta+n+1)\Gamma(\mu+\alpha+n+2)}$$
$$\times\ {}_3F_2\left[\begin{matrix} -\lambda-r,\ \mu+\beta+1,\ \mu+1; \\ \mu+\beta+n+1,\ \mu+\alpha+n+2; \end{matrix}\ 1\right],$$

by a little simplification, then we arrive at the desired result in (3.2). This completes the proof.

4 INTEGRALS INVOLVING GAUSS HYPERGEOMETRIC FUNC-TION WITH LEGENDRE FUNCTION

The Legendre functions are the solution of Legendre's differential equation [4, sec.3.1]

$$\left(1-z^2\right)\frac{d^2 f}{dz^2} - 2z\frac{df}{dz} + \left[\nu(\nu+1) - \mu^2\left(1-z^2\right)^{-1}\right] f = 0, \tag{4.1}$$

where z, μ, ν are unrestricted.
If we substitute $f = (z^2-1)^{\frac{1}{2\mu}}\, \nu$, then (4.1) becomes

$$\left(1-z^2\right)\frac{d^2\nu}{dz^2} - 2(\mu+1) z\frac{d\nu}{dz} + [\nu(\mu-\nu)(\mu+\nu+1)] = 0, \tag{4.2}$$

and with $\delta = \frac{1}{2} - \frac{1}{2}z$ as the independent variable the above differential equation becomes as following:

$$\delta(1-\delta)\frac{d^2\nu}{d\delta^2} + (\mu+1)(1-2\delta)\frac{d\nu}{d\delta} + [\nu(\nu-\mu)(\mu+\nu+1)] = 0. \tag{4.3}$$

The solution of (4.1) in the form of Gauss hypergeometric type equation with $a = \mu - \nu$, $b = \mu + \nu + 1$ and $c = \mu + 1$, as follows.

$$f = P_\nu^\mu(z) = \frac{1}{\Gamma(1-\mu)}\left(\frac{z+1}{z-1}\right)^{\frac{1}{2\mu}} F\left[-\nu,\ \nu+1;\ 1-\mu;\ \frac{1}{2}-\frac{1}{2}z\right], \quad |1-z|<2, \tag{4.4}$$

where $P_\nu^\mu(z)$ is known as the Legendre function of the first kind [4].

Next, we derive the integrals with Legendre function.

Theorem 8 *Integral formula involving Gauss hypergeometric function multiplied with Legendre function is given as following:*

$$\begin{aligned}&\int_0^1 x^{\sigma-1}\left(1-x^2\right)^{\frac{\mu}{2}} P_\nu^\mu(x)\ F_p^{(\alpha,\beta)}(a,b;c;kx)\,dx \\ &= \frac{(-1)^\mu\ 2^{-\sigma-\mu}\sqrt{\pi}\Gamma(1+\mu+\nu)}{\Gamma(1-\mu+\nu)}\sum_{n=0}^{\infty}\frac{(a)_n\,\beta_p^{(\alpha,\beta)}(b+n,c-b)}{\beta(b,c-b)}\frac{k^n}{n!} \\ &\times\frac{2^{-n}\Gamma(\sigma+n)}{\Gamma\left(\frac{1}{2}+\frac{\sigma+n}{2}+\frac{\mu}{2}-\frac{\nu}{2}\right)\Gamma\left(1+\frac{\sigma+n}{2}+\frac{\mu}{2}+\frac{\nu}{2}\right)}.\end{aligned} \tag{4.5}$$

Proof

$$\mathcal{I}_8 = \int_0^1 x^{\sigma-1}\left(1-x^2\right)^{\frac{\mu}{2}} P_\nu^\mu(x)\sum_{n=0}^{\infty}\frac{(a)_n\,\beta_p^{(\alpha,\beta)}(b+n,c-b)}{\beta(b,c-b)}\frac{(kx)^n}{n!}\,dx$$

Next, using the formula [4, sec. 3.12] for $\Re(\sigma)>0,\ \ \mu\in\mathbb{N}$.

$$\int_0^1 x^{\sigma-1}\left(1-x^2\right)^{\frac{\mu}{2}} P_\nu^\mu(x)\,dx = \frac{(-1)^\mu\ 2^{-\sigma-\mu}\sqrt{\pi}\,\Gamma(\sigma)\,\Gamma(1+\mu+\nu)}{\Gamma(1-\mu+\nu)\,\Gamma\left(\frac{1}{2}+\frac{\sigma}{2}+\frac{\mu}{2}-\frac{\nu}{2}\right)\Gamma\left(1+\frac{\sigma}{2}+\frac{\mu}{2}+\frac{\nu}{2}\right)}, \tag{4.6}$$

then we obtain

$$\mathcal{I}_8 = \sum_{n=0}^{\infty}\frac{(a)_n\,\beta_p^{(\alpha,\beta)}(b+n,c-b)}{\beta(b,c-b)}\frac{(k)^n}{n!}\ \frac{(-1)^\mu\ 2^{-\sigma-\mu-n}\sqrt{\pi}\Gamma(\sigma+n)\,\Gamma(1+\mu+\nu)}{\Gamma(1-\mu+\nu)\,\Gamma\left(\frac{1}{2}+\frac{\sigma+n}{2}+\frac{\mu}{2}-\frac{\nu}{2}\right)\Gamma\left(1+\frac{\sigma+n}{2}+\frac{\mu}{2}+\frac{\nu}{2}\right)}.$$

This completes the proof.

5 INTEGRALS INVOLVING GAUSS HYPERGEOMETRIC FUNC-TION AND BESSEL MAITLAND FUNCTION

The Bessel Maitland function (also known as Wright generalized Bessel function) defined as following [8]:

$$J_\nu^\mu(z) = \phi(\mu,\nu+1:z) = \sum_{n=0}^{\infty}\frac{1}{\Gamma(\mu n+\nu+1)}\frac{(-z)^n}{n!}. \tag{5.1}$$

Theorem 9

$$\int_0^\infty x^\rho\, J_\nu^\mu(x)\, F_p^{(\alpha,\beta)}(a,b;c;kx)\, dx$$
$$= \sum_{n=0}^{\infty} \frac{(a)_n\, \beta_p^{(\alpha,\beta)}(b+n, c-b)}{\beta(b, c-b)} \frac{\Gamma(\rho+n+1)}{\Gamma(1+\nu-\mu-\mu(\rho+n))} \frac{k^n}{n!}. \tag{5.2}$$

Proof

$$I_9 = \sum_{n=0}^{\infty} \frac{(a)_n\, \beta_p^{(\alpha,\beta)}(b+n, c-b)}{\beta(b, c-b)} \frac{k^n}{n!} \int_0^\infty x^{\rho+n}\, J_\nu^\mu(x)\, dx,$$

Next, using the following formula [15]:

$$\int_0^\infty x^\rho\, J_\nu^\mu(x)\, dx = \frac{\Gamma(\rho+1)}{\Gamma(1+\nu-\mu-\mu\rho)} \qquad (\Re(\rho) > -1, 0 < \mu < 1), \tag{5.3}$$

then we arrive at the desired result in (5.2). This completes the proof.

6 CONCLUDING REMARKS

We have obtained some new integrals involving Gauss hypergeometric and Confluent hypergeometric function. The results obtained here are basic in nature and are likely to find useful applications in the study of simple and multiple variable hypergeometric se-ries which in turn are useful in statistical mechanics, electrical networks and probability theory. Some important results are also given as special cases of our main results.

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