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CAYLEY-HAMILTON THEOREM FOR SQUARE AND RECTANGULAR MATRICES AND BLOCK MATRICES

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Abstract: **The main aim of the paper is to introduce cayley-Hamilton Theorem and also to explain its extension for the square and rectangular matrics. In this paper C-H Theorem extension for block matrics has also explained.**

Keywords**: Cayley- Hamilton Theorem, topology, Matrices, square, rectangular , block.**

I. INTRODUCTION

Defintion 1.1. If A is an n×n matrix ,then the characteristic polynomial of A is defined to be $P_A(x)$= det(xI-A). This is a polynomial in x of degree n with leading term x^n . the constant term c_0 of a polynomial q(x) is interpreted as c_0 I in q(A).

Theorem 1.2 (Cayley – Hamilton Theorem). If A is an n×n matrix ,then $p_A(A)=0$, the zero matrix.

Theorem 1.3 If q≠0 is a quaternion of the form q-= a+bi+cj+dk with a,b,c,d, being real , then $q^2 - 2aq + (a^2 + b^2 + c^2 + d^2) = 0$

$$q^{-1} = \frac{\bar{q}}{|q|}$$

$$= \frac{a-bi-cj-dk}{a^2+b^2+c^2+d^2}$$

$$= \frac{2a}{a^2+b^2+c^2+d^2} - \frac{a+bi+cj+dk}{a^2+b^2+c^2+d^2}$$

$$= \frac{1}{a^2+b^2+c^2+d^2}(2a - q)$$

$$\Rightarrow a^2 + b^2 + c^2 + d^2 = 2aq - q2$$

$$\Rightarrow q2 - 2aq + (a^2 + b^2 + c^2 + d^2) = 0$$

If one represents a quaternion q= a+bi+cj+dk as a matrix,

$$A = \begin{bmatrix} a+bi & c+di \\ -c+di & a-bi \end{bmatrix},$$

$P_A(A) = A^2 - 2aA + (a^2 + b^2 + c^2 + d^2)I = 0$, and the polynomial given in Theorem 1.3 is characteristic polynomial of A

II. GENERALIZATION OF CAYLEY HAMILTON THEOREM

Theorem 2.1 (Cayley-Hamilton Theorem). For any n × n Matrix A, $P_A(A)=0$.

Proof. Let D(x) be the matrix with polynomial entries D(x)= adj(xI_n-A), So D(x)(xI-A)= det(xI_n-A)I_n. Since each entry in D(x) is the determinant of an (n-1) X((n-1) submatrix of (xI_n-A), each entry of D(x) is a polynomial of degree less than or equal to n-1. It folowws that there exist matrices D_0, D_1,………D_{n-1} with entries from C such that D(x) = $D_{n-1} x^{n-1}$+……+ D_1 x +D_o . Then the matrix equation follows

$$\det(xI_n\text{-}A)\, I_n = (x I_n\text{-}A)\, \mathrm{adj}(xI_n\text{-}A) = (xI_n\text{-}A)D(x)$$

Substituting $p_A(x)$= det(xI_n-A), (and using the fact that scalars commute with matrix)

$$X^n I_n + b_{n-1}x^{n-1}I_n + \ldots\ldots\ldots\ldots + b_1 x I_n + b_0 I_n$$

$$= p_A(x)\, I_n = \det(xI_n\text{-}A)I_n$$

$$=(xI_n\text{-}A)\mathrm{adj}(xI_n\text{-}A)$$

$$=(xI_n\text{-}A)(x^{n-1}D_{n-1} + \ldots\ldots + xD_1 + D_o)$$

$$=x^{nD}{}_{n-1} - x^{n-1}AD_{n-1} + x^{n-1}D_{n-2} - x^{n-2}AD_{n-2} + \ldots\ldots + xD_0 - AD_o$$

$=x^{nD}{}_{n-1}+ x^{n-1}(-AD_{n-1}+ D_{n-2}) +\ldots\ldots.+(-AD_1+D_o) -AD_o$

Since two polynomials are equal if and only if their coefficients are equal , the coefficient matrices are equal ; that is , $I_n=D_{n-1}$, $b_{n-1}I_n=(-AD_{n-1}+D_{n-2}),\ldots\ldots,b_1I_n=-(AD_1+D_0)$, and $b_0I_n=-AD_0$. This means that A may be substituted for the variable x in the equation (2.1) to conclude

$P_A(A) = A^n+ b_{n-1} A^{n-1}+\ldots\ldots+ b_1A+b_0I_n$

$=A^nD_{n-1}+A^{n-1}(-AD_{n-1} +D_{n-2})+\ldots\ldots+A(-AD_1+D_0)-AD_0$

$=A^nD_{n-1}-A^nD_{n-1}+A^{n-1}D_{n-2}- A^{n-1}D_{n-2}+\ldots\ldots\ldots AD_0-AD_0$

=0 This proves the theorem

III. CAYLEY-HAMILTON THEOREM FOR SQUARE AND RECTANGULAR MATRICES

Let $C^{n\times m}$ be the set of complex $(n\times m)$ matrices.

Theorem 1. (Cayley-Hamilton theorem). Let

$$p(s) = \det[I_n s - A] = \sum_{i=0}^{n} a_i s^i \;(a_n = 1) \quad (3.1)$$

be the characteristic polynomial of A, where I_n is the $(n\times n)$ identity matrix. Then

$$p(A) = \sum_{i=0}^{n} a_i A^i = 0_n \quad (3.2)$$

Where 0_n is the $(n\times n)$ matrix.

The classical Cayley-Hamilton theorem can be extended to rectangular matrices as follows [16]

Theorem 2. (Cayley-Hamilton theorem for rectangular matrices).

Let

$$A = [A_1 \quad A_2] \in C^{m\times n}, A_1 \in C^{m\times m}, A_2 \in C^{m\times(n-m)}, \quad (n > m) \quad (3.3)$$

and

$$p_{A_1} = \det[I_m s - A_1] = \sum_{i=0}^{m} a_i s^i \;(a_m = 1) \quad (3.4)$$

be the characteristic polynomial of A_1.

Then

$$\sum_{i=0}^{m} a_{m-i}[A_1^{n-i} \quad A_1^{n-i-1}A_2] = 0_{mn} \quad (3.5)$$

Where 0_{mn} is the $(m\times n)$ matrix.

Theorem 3. Let ,

$$A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \in C^{m\times n}, \quad m > n$$

and let the characteristic polynomial of A_1 have the form . Then

$$\sum_{i=0}^{n} a_{n-i} \begin{bmatrix} A_1^{m-i} \\ A_2 A_1^{m-i-1} \end{bmatrix} = 0_{mn} \quad (3.6)$$

IV. CAYLEY-HAMILTON THEOREM FOR BLOCK MATRIX

The classical Cayley-Hamilton theorem can be also extended for block matrices.

Theorem 4. (Cayley-Hamilton theorem for block matrices).

Let

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1m} \\ \vdots & \ddots & \vdots \\ A_{m1} & \cdots & A_{mm} \end{bmatrix} \in C^{mn\times mn} \quad (4.1)$$

where $A_{ij} \in C^{n\times n}$are commutative i.e.,

$$A_{ij}A_{kl} = A_{kl}A_{ij} \text{ for all } i,j,k,l = 1,2,\ldots.,m \quad (4.2)$$

Let

$$P(S) = det[I_m \otimes -A] = S^m + D_1S^{m-1} + \cdots + D_{m-1}S + D_m \quad (4.3)$$

be the matrix polynomial of A , where $S \in C^{n\times n}$ is the block matrix having eigenvalue of A, $\otimes$ denotes the Kronecker product of matrix .

Then

$$P(A) = \sum_{i=0}^{m}[I_m \otimes D_{m-i}]A^i = 0 \quad (D_0 = I_n) \quad (4.4)$$

The matrix (4.3) is obtained by developing the determinant of the matrix $[I_n \otimes S - A]$, considering its commuting block as scalar entries.

Theorem 5. (Cayley- Hamilton Theorem for rectangular block matrices)

Let $\bar{A} = [A_1 \quad A_2] \in C^{mn\times(mn+p)}$ and let matrix charactristics polynomial of *A* have the form *(4.3.2),* then

$$\sum_{i=0}^{m}[I_m \otimes D_{m-i}][A^{i+1} \quad A^i A_2] = 0 \quad (D_0 = I_n) \quad (4.5)$$

Theorem 6.

Let

$$\bar{A} = \begin{bmatrix} A \\ A_2 \end{bmatrix} \in C^{(mn+p)\times mn}, A \in C^{mn\times mn}, A_2 \in C^{p\times mn}$$

and let the matrix characteristic polynomial of A have the form , then

$$\sum_{i=0}^{m}\begin{bmatrix} A \\ A_2 \end{bmatrix}[I_m \otimes D_{m-i}]A^i = 0 \quad (D_0 = I_n)$$

V. CHARACTERSTICS

The Cayley –Hamilton theorem is one of the most powerful and classical matrix theory theorem. Many application derive their results from this theorem. To understand the scope of this theorem , alternate proofs were used. Each proof helped to understand how intertwined areas of mathematics are with respect to matrices and the characteristics polynomial.

VI. APPLICATION OF CAYLEY– HAMILTON THEOREM

A very common application of the Cayley- Hamilton Theorem is to use it to find A^n usually for the large powers of n. However many of the techniques involved require the use of the eigen values of A.

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