Abstract: The main aim of the paper is to introduce Cayley-Hamilton Theorem and also to explain its extension for the square and rectangular matrices. In this paper C-H Theorem extension for block matrices has also explained.

Keywords: Cayley-Hamilton Theorem, topology, Matrices, square, rectangular, block.

I. INTRODUCTION

Definition 1.1. If A is an n×n matrix, then the characteristic polynomial of A is defined to be \( p(A) = \det(xI-A) \). This is a polynomial in x of degree n with leading term \( x^n \). The constant term \( c_0 \) of a polynomial \( q(x) \) is interpreted as \( c_0 \) in \( q(A) \).

Theorem 1.2 (Cayley–Hamilton Theorem). If A is an n×n matrix, then \( p(A) = 0 \), the zero matrix.

Theorem 1.3. If \( q \neq 0 \) is a quaternion of the form \( q = a + bi + cj + dk \) with \( a, b, c, d \) being real, then \( q^2 = 2aq + (a^2 + b^2 + c^2 + d^2)I = 0 \), and the polynomial given in Theorem 1.3 is characteristic polynomial of A

II. GENERALIZATION OF CAYLEY HAMILTON THEOREM

Theorem 2.1 (Cayley-Hamilton Theorem). For any \( n \times n \) Matrix A, \( p(A) = 0 \).

Proof. Let \( D(x) \) be the matrix with polynomial entries \( D(x) = \det(xI_n-A) \). So \( D(x)(xI-A) = \det(xI_n-A)I_n \). Since each entry in \( D(x) \) is the determinant of an \( (n-1) \times (n-1) \) submatrix of \( (xI_n-A) \), each entry of \( D(x) \) is a polynomial of degree less than or equal to \( n-1 \). It follows that there exist matrices \( D_0, D_1, \ldots, D_{n-1} \) with entries from \( C \) such that
\[
D(x) = D_{n-1}x^{n-1} + \cdots + D_1x + D_0.
\]
Then the matrix equation follows
\[
det(xI_n-A)I_n = (xI_n-A)adj(xI_n-A) = (xI_n-A)D(x)
\]
Substituting \( p_A(x) = \det(xI_n-A) \), (and using the fact that scalars commute with matrix)
\[
X^nI_n + b_{n-1}x^{n-1}I_n + \cdots + b_1xI_n + b_0I_n
= p_A(x)I_n = \det(xI_n-A)I_n
= (xI_n-A)adj(xI_n-A)
= (xI_n-A)(x^{n-1}D_{n-1} + \cdots + xD_1 + D_0)
= x^{n-1}A_{n-1} + x^{n-2}A_{n-2} + \cdots + xD_0 + D_0
\]
Where \( b_{n-1}I_n = (-AD_{n-1} + D_{n-2}), \ldots, b_1I_1 = (-AD_1 + D_0), \) and \( b_0I_0 = -AD_0 \). This means that \( A \) may be substituted for the variable \( x \) in the equation (2.1) to conclude

\[
P_A(A) = A^n + b_{n-1}A^{n-1} + \ldots + b_1A + b_0I_n
\]

\[
= A^nD_{n-1} + A^{n-1}(AD_{n-1} + D_{n-2}) + \ldots + A(AD_1 + D_0) - AD_0
\]

\[
= 0
\]

This proves the theorem.

### III. CAYLEY-HAMILTON THEOREM FOR SQUARE AND RECTANGULAR MATRICES

Let \( C^{n \times m} \) be the set of complex \((n \times m)\) matrices.

**Theorem 1.** (Cayley-Hamilton theorem). Let

\[
p(s) = \det[I_n s - A]
\]

\[
= \sum_{i=0}^{n} a_i s^i \quad (a_n = 1)
\]

(3.1)

be the characteristic polynomial of \( A \), where \( I_n \) is the \((n \times n)\) identity matrix. Then

\[
p(A) = \sum_{i=0}^{n} a_i A^i = 0_n
\]

(3.2)

Where \( 0_n \) is the \((n \times n)\) matrix.

The classical Cayley-Hamilton theorem can be extended to rectangular matrices as follows [16]

**Theorem 2.** (Cayley-Hamilton theorem for rectangular matrices).

Let

\[
A = \begin{bmatrix} A_{11} & \cdots & A_{1m} \\ \vdots & \ddots & \vdots \\ A_{m1} & \cdots & A_{mm} \end{bmatrix} \in C^{m \times mn}
\]

(4.1)

where \( A_{ij} \in C^{n \times n} \) are commutative i.e.,

\[
A_{ij}A_{kl} = A_{kl}A_{ij} \quad \text{for all } i,j,k,l = 1,2,\ldots,m
\]

(4.2)

**Let**

\[
P(S) = \det[I_m \otimes S - A] = S^n + D_1S^{n-1} + \ldots + D_{n-1}S + D_n
\]

(4.3)

be the matrix polynomial of \( A \), where \( S \in C^{n \times n} \) is the block matrix having eigenvalue of \( A \), \( \otimes \) denotes the Kronecker product of matrix.

Then

\[
P(A) = \sum_{i=0}^{m} [I_m \otimes D_{m-i}] A^i = 0 \quad (D_0 = I_n)
\]

(4.4)

The matrix (4.3) is obtained by developing the determinant of the matrix \([I_n \otimes S - A]\), considering its commuting block as scalar entries.
Theorem 5. (Cayley- Hamilton Theorem for rectangular block matrices)

Let \( \bar{A} = [A_1, A_2] \in \mathbb{C}^{nm \times (mn+p)} \) and let matrix characteristic polynomial of \( A \) have the form (4.3.2), then

\[
\sum_{i=0}^{m} [I_m \otimes D_{m-i}][A^{i+1} A_1 A_2] = 0 \quad (D_0 = I_n)
\]

Theorem 6.

Let

\[
\bar{A} = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \in \mathbb{C}^{(mn+p) \times mn}, A \in \mathbb{C}^{mn \times mn}, A_2 \in \mathbb{C}^{p \times mn}
\]

and let the matrix characteristic polynomial of \( A \) have the form, then

\[
\sum_{i=0}^{m} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} [I_m \otimes D_{m-i}] A^i = 0 \quad (D_0 = I_n)
\]

V. CHARACTERISTICS

The Cayley–Hamilton theorem is one of the most powerful and classical matrix theory theorem. Many application derive their results from this theorem. To understand the scope of this theorem, alternate proofs were used. Each proof helped to understand how intertwined areas of mathematics are with respect to matrices and the characteristics polynomial.

VI. APPLICATION OF CAYLEY– HAMILTON THEOREM

A very common application of the Cayley- Hamilton Theorem is to use it to find \( A^n \) usually for the large powers of \( n \). However many of the techniques involved require the use of the eigen values of \( A \).

REFERENCES


