

Bianchi type-I Anisotropic Universe in $f(R, L_m)$ Gravity with Hybrid Expansion law

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Abstract: In this work, we investigate an anisotropic Bianchi Type-I space-time using the framework of $f(R, L_m)$ gravity. Here we adopt a specific form of modified gravity theory represented by $f(R, L_m) = \frac{R}{2} + L_m^\alpha + \beta$, incorporates both the Ricci scalar (R) and matter Lagrangian density (L_m). To find a solution of field equations, we assume a special form of the average scale factor in the form of the Hybrid Expansion Law (HEL) proposed by Akarsu et al. (2014), and some cosmological parameters of the model are derived. Additionally, physical and geometrical properties of the cosmological model have been investigated. Here we consider two special cases, the power-law expansion and exponential expansion, to solve the field equation in the framework of $f(R, L_m)$ gravity; also, we analyse the jerk parameter, equation of state parameter (EoS), and Statefinder pair $\{r, s\}$ to understand the different phases of the universe. The conclusion drawn from our research study agrees with the Λ CDM model compared to recent observational data.

Keywords: Bianchi-I metric, $f(R, L_m)$ gravity, Hybrid expansion law, EoS parameter.

1. Introduction

The recent discovery of the accelerated expansion of the Universe, supported by a wide range of cosmological observations such as Type Ia supernovae [1,2,3], Cosmic Microwave Background (CMB) [4,5] anisotropies and Baryon Acoustic Oscillations [6,7], Wilkinson Microwave Anisotropy Probe [8] has inspired an extensive re-examination of the underlying gravitational framework that governs cosmic dynamics. Although Einstein's General Theory of Relativity (GR) has been remarkably successful in explaining gravitational phenomena over a broad range of scales, it faces significant challenges when confronted with the late-time cosmic acceleration and the dark energy problem. In this context, modified theories of gravity have emerged as compelling alternatives to GR, capable of explaining the Universe's accelerated behaviour without invoking exotic dark energy components. Among the various modified theories of gravity, Buchdahl introduced (1970) the simplest modified

gravity as $f(R)$ gravity [12], researchers investigate a variety of functional expressions for $f(R)$ gravity such as logarithmic, exponential, and power law models [13,14].

Among these alternative theories $f(R, L_m)$ gravity first proposed by Harko and collaborators [15] represents an intriguing generalisation of the conventional $f(R)$ gravity. In this theory, the gravitational Lagrangian depends not only on the Ricci scalar R but also explicitly on the matter Lagrangian L_m . Such coupling between geometry and matter leads to a non-minimal interaction that modifies the geodesic motion of particles and allows for novel energy-momentum exchange mechanisms between curvature and matter fields [16]. Some researchers introduced a new combination of non-minimal matter geometries with special form of the functional $f(R, L_m)$ [17], This work has increasing attention because it provides rich cosmological dynamics. dark energy cosmological model has been investigated by some researchers [18], they have discussed energy density parameters through observational constraints. Some authors discussed $f(R, L_m)$ gravity, which includes bulk viscosity with strong explanation of recent observation and cosmic expansion [19]. Some researchers investigate anisotropic behaviour of perfect fluid using special form of the deceleration parameter using the framework of $f(R, L_m)$ gravity [20]. locally rotationally symmetric Bianchi-I model filled with strange quark matter (SQM) is explored in $f(R, L_m)$ gravity [21], cosmological model must also consider the possibility that the Universe was anisotropic at early times and has evolved toward isotropy at later epochs. Bianchi type-I spacetime, being the simplest homogeneous but anisotropic cosmological model, serves as an ideal framework for exploring such anisotropic evolution. It generalises the standard Friedmann- Lemaître-Robertson- Walker (FLRW) metric by allowing distinct scale factors along different spatial directions; several investigations have been carried out on the LRS Bianchi type I cosmological model. Adhav [26] discussed LRS Bianchi type I cosmological model with perfect fluid; Pawar and Solanke [27] have explored perfect fluid LRS Bianchi-I cosmological models in $f(R, T)$ in gravity, Santhi Kumar et al. [28] discuss anisotropic LRS Bianchi-I cosmological models in $f(R, T)$ in gravity, also V. P. Kadam [29] investigated spatially homogeneous and anisotropic Bianchi type VI_0 metric in presence of bulk viscous fluid,

To describe the transition from an early decelerating phase to the current accelerated epoch, the Hybrid Expansion Law (HEL) has been widely adopted as a phenomenological ansatz for the cosmic scale factor. This law combines power law and exponential behaviours, thereby producing a time-varying deceleration parameter that naturally captures both the matter-dominated and the accelerating eras within a unified

framework. Some researcher studied LRS Bianchi type I cosmological model within the framework of $f(G)$ theory of gravitation in the presence of dark energy [30], Bianchi type I cosmological model in $f(R, T)$ theory of gravity using hybrid expansion law to obtain exact solution of field equation [31], some scholar investigate FLRW cosmology with hybrid scale factor in $f(R, L_m)$ gravity and analysed state finder diagnostic and equation of state and found that model behave like quintessence [32]. Some researchers analysed the Kantowski-Sachs cosmological model in the framework of teleparallel gravity and explained the challenging problem of late-time acceleration of the universe, where a bulk viscous fluid containing one-dimensional cosmic strings [33]. The hybrid expansion law has been successfully utilised in several modified gravity models such as $(f(R), f(T), f(R, T), f(Q), f(G))$, but its detailed implications within $f(R, L_m)$ gravity for anisotropic cosmologies remain largely unexplored. Motivated by these considerations, the present work aims to investigate the dynamics of a Bianchi type-I anisotropic Universe in the framework of $f(R, L_m)$ gravity under the assumption of a Hybrid Expansion Law (HEL). The primary objective is to obtain exact or approximate solutions to the modified field equations, analyse the behaviour of key cosmological parameters such as the Hubble parameter, deceleration parameter, anisotropy parameter, and equation of state parameter and to study the physical and observational viability of the resulting model.

This investigation is significant for several reasons. Firstly, it extends the applicability of $f(R, L_m)$ gravity to anisotropic cosmological settings governed by time-varying expansion laws. Secondly, it provides a deeper understanding of how non-minimal coupling between curvature and matter may affect the cosmic dynamics and anisotropy decay. Finally, the analysis may yield insights into the role of modified gravity in reproducing the observed transition from early deceleration to late-time acceleration without introducing ad hoc dark energy components.

This research paper is structured as follows: In Section 2, we delve into the fundamental action and basic formulation that govern the $f(R, L_m)$ gravity theory. Moving on to Section 3, we focus on the anisotropic Bianchi type I cosmological model and the motion equations of $f(R, L_m)$ gravity. Section 4, is dedicated to the exploration of a specific $f(R, L_m)$ functional and cosmological solutions for $f(R, L_m)$ gravity. Section 5 introduces the Hybrid Expansion Law and formulates the corresponding dynamical equations; it also provides the analysis of key cosmological parameters and their evolution. Two particular

scenarios are also examined: power-law expansion and exponential expansion. Finally, Section 6 summarises the main results and outlines possible directions for future research.

2. BASIC FORMALISM OF $f(R, L_m)$ GRAVITY THEORY

With the matter Lagrangian density (L_m) and the Ricci scalar R , the action principle for $f(R, L_m)$ gravity model proposed by Harko and Lobo [15] is given by

$$S = \int f(R, L_m) \sqrt{-g} d^4x, \quad (1)$$

where f is an arbitrary function of R and L_m . The Ricci scalar R is defined in terms of the metric tensor g_{ij} and the Ricci tensor R_{ij} as below

$$R = g^{ij} R_{ij} \quad (2)$$

where the Ricci tensor is given by

$$R_{ij} = \partial_k \Gamma_{ij}^k - \partial_j \Gamma_{ki}^k + \Gamma_{ij}^\lambda \Gamma_{\lambda k}^k - \Gamma_{j\lambda}^k \Gamma_{ki}^\lambda \quad (3)$$

Here Γ_{jk}^i represents the components of the well-known Levi Civita connection as indicated by

$$\Gamma_{jk}^i = \frac{1}{2} g^{i\lambda} (g_{k\lambda,j} + g_{\lambda j,k} - g_{jk,\lambda}) \quad (4)$$

The field equation for $f(R, L_m)$ gravity, derived through the variation of the action principle (1) with respect to the metric tensor g_{ij} , is expressed as follows.

$$f_R R_{ij} - \frac{1}{2} (f - f_{L_m} L_m) g_{ij} + g_{ij} \Theta f_R - \nabla_i \nabla_j f_R = \frac{1}{2} f_{L_m} T_{ij} \quad (5)$$

where $f_R(R, L_m) = \frac{\delta f(R, L_m)}{\delta R}$, $f_{L_m}(R, L_m) = \frac{\delta f(R, L_m)}{\delta L_m}$, $\Theta = \nabla_i \nabla^i$ and T_{ij} is the Energy

Momentum Tensor (EMT) for a perfect fluid given by

$$T_{ij} = \frac{-2}{\sqrt{-g}} \frac{\delta(\sqrt{-g} L_m)}{\delta g^{ij}} = g_{ij} L_m - 2 \frac{\partial(L_m)}{\partial g^{ij}} \quad (6)$$

Deriving from the explicit expression of the field equation (5), we can ascertain the covariant divergence of the energy-momentum tensor T_{ij} as follows:

$$\nabla^i T_{ij} = 2 \nabla^i \log(f_{L_m}) \frac{\partial(L_m)}{\partial g^{ij}} \quad (7)$$

Furthermore, by contracting the field equation (5), we can establish the connection between the Ricci scalar denoted as R , the matter Lagrangian density (L_m) and the trace T of the stress-energy-momentum tensor T_{ij} as follows,

$$R f_R + 3 \Theta f_R - (f - f_{L_m} L_m) = \frac{1}{2} f_{L_m} T \quad (8)$$

where $\Theta F = \frac{1}{\sqrt{-g}} \partial_i (\sqrt{-g} g^{ij} \partial_j F)$ for any function of F .

3. METRIC AND MOTION EQUATIONS IN $f(R, L_m)$ GRAVITY

Hence, to explore the anisotropic characteristics of the Universe, our focus turns towards investigating the anisotropic and spatially homogeneous LRS Bianchi Type-I metric described by,

$$ds^2 = -dt^2 + A^2 dx^2 + B^2(dy^2 + dz^2) \quad (9)$$

Where A and B are functions of cosmic time (t) alone.

If $A(t) = B(t) = a(t)$, we return to the FLRW cosmological model. The Ricci scalar corresponding to the LRS Bianchi-I spacetime can be formulated as follows,

$$R = -2 \left[\frac{\ddot{A}}{A} + 2 \frac{\ddot{B}}{B} + 2 \frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}^2}{B^2} \right] \quad (10)$$

Here we are assuming that the matter distribution of the Universe can be characterised by the EMT of a perfect fluid.

Therefore, the EMT for the perfect fluid corresponding to the line element (9) is given by

$$T_{ij} = (p + \rho)u_i u_j + p g_{ij} \quad (11)$$

Here, p and ρ are the thermodynamic pressure and energy density of the matter and $u^i = (1, 0, 0, 0)$ components of co-moving four-velocity vectors in a cosmic fluid with $u_i u^j = 0$ and $u_i u^i = -1$.

The modified Friedmann equations (5), which describe the dynamics of the Universe in $f(R, L_m)$ gravity with the help of equation (11), are given by

$$-\left[\frac{\ddot{A}}{A} + 2 \frac{\dot{A}\dot{B}}{AB}\right] f_R - \frac{1}{2}(f - f_{L_m} L_m) - 2 \frac{\dot{B}}{B} \dot{f}_R - \ddot{f}_R = \frac{1}{2} f_{L_m} p \quad (12)$$

$$-\left[\frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}^2}{B^2}\right] f_R - \frac{1}{2}(f - f_{L_m} L_m) - \left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right) \dot{f}_R - \ddot{f}_R = \frac{1}{2} f_{L_m} p \quad (13)$$

$$-\left[\frac{\ddot{A}}{A} + 2 \frac{\dot{B}}{B}\right] f_R - \frac{1}{2}(f - f_{L_m} L_m) - \left(\frac{\dot{A}}{A} + 2 \frac{\dot{B}}{B}\right) \dot{f}_R = -\frac{1}{2} f_{L_m} \rho \quad (14)$$

Where an overhead dot denotes differentiation with respect to time 't'.

4. COSMOLOGICAL SOLUTIONS FOR $f(R, L_m)$ GRAVITY

To examine the dynamics of the universe, we employ the functional form of $f(R, L_m)$ gravity as

$$f(R, L_m) = \frac{R}{2} + L_m^\alpha + \beta \quad (15)$$

Where α and β are free parameters, and one can retain to the standard Friedmann equations of GR for $\alpha = 1$.

For this particular functional form of $f(R, L_m)$ gravity, we have considered a dusty Universe scenario where we take $L_m = \rho$ [22–25] and hence the Friedmann equations (12), (13) and (14) yield,

$$\left[2\frac{\ddot{B}}{B} + \frac{\dot{B}^2}{B^2}\right] - \beta - (1 - \alpha)\rho^\alpha = \alpha\rho^{\alpha-1}p \quad (16)$$

$$\left[\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB}\right] - \beta - (1 - \alpha)\rho^\alpha = \alpha\rho^{\alpha-1}p \quad (17)$$

$$\left[2\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}^2}{B^2}\right] - \beta = (1 - 2\alpha)\rho^\alpha \quad (18)$$

5. SOLUTION OF FIELD EQUATION WITH HYBRID EXPANSION LAW

Now, to find an exact solution to the field equations (16)-(18), we must consider the constraining relation. To this point, we shall presume the anisotropic relation that can be written in terms of shear (σ) and expansion scalar (θ) as [21,45],

$$A = B^k \quad (19)$$

Also, the Hybrid Expansion Law (HEL) proposed by Akarsu et al. [37] is given by

$$a(t) = a_0 \left(\frac{t}{t_0}\right)^n e^{b\left(\frac{t}{t_0}-1\right)} \quad (20)$$

It is also discussed by [31] & [35], where n and b are non-negative constants and a_0 represents the present value of the scale factor and t_0 represents the present value of the universe.

The average scale factor a defined as

$$V = a^3 = AB^2 \quad (21)$$

$$V = a_0^3 \left(\frac{t}{t_0}\right)^{3n} e^{3b\left(\frac{t}{t_0}-1\right)} \quad (22)$$

From equations (20) and (21), we get

$$(AB^2)^{\frac{1}{3}} = a_0 \left(\frac{t}{t_0}\right)^n e^{b\left(\frac{t}{t_0}-1\right)} \quad (23)$$

From equations (19), (23) becomes

$$B = \left[a_0 \left(\frac{t}{t_0}\right)^n e^{b\left(\frac{t}{t_0}-1\right)}\right]^{\frac{3}{k+2}} \quad (24)$$

Using equation (24), we get

$$A = \left[a_0 \left(\frac{t}{t_0}\right)^n e^{b\left(\frac{t}{t_0}-1\right)}\right]^{\frac{3k}{k+2}} \quad (25)$$

Now the metric equation (9) can be written as

$$ds^2 = -dt^2 + \left[a_0 \left(\frac{t}{t_0}\right)^n e^{b\left(\frac{t}{t_0}-1\right)}\right]^{\frac{6k}{k+2}} dx^2 + \left[a_0 \left(\frac{t}{t_0}\right)^n e^{b\left(\frac{t}{t_0}-1\right)}\right]^{\frac{6}{k+2}} (dy^2 + dz^2) \quad (26)$$

5.1 Physical and Geometrical properties of the model

For this model, some important cosmological parameters are:

From equation (21) and (22), the spatial volume

$$V = AB^2 = a_0^3 \left(\frac{t}{t_0}\right)^{3n} e^{3b\left(\frac{t}{t_0}-1\right)}$$

The mean Hubble parameter is defined as

$$H = \frac{\dot{a}}{a} = \frac{1}{3} \left(\frac{\dot{A}}{A} + 2 \frac{\dot{B}}{B} \right) \quad (27)$$

Using equations (24), (25), and (27), we get

$$H = \frac{n}{t} + \frac{b}{t_0} \quad (28)$$

Also, deceleration parameter q is

$$q = \frac{d}{dt} \left(\frac{1}{H} \right) - 1 \quad (29)$$

Using equation (28)

$$q = \frac{n(1-n)t_0^2 - 2t t_0 n b - b^2 t^2}{(nt_0 + bt)^2} \quad (30)$$

$$q = -1 + \frac{nt_0^2}{(nt_0 + bt)^2}$$

Also, the Anisotropic parameter A_m is given by

$$A_m = \frac{1}{3} \sum_{i=1}^3 \left(\frac{H_i - H}{H} \right)^2 \quad (31)$$

Where $H_1 = \frac{\dot{A}}{A}$ and $H_2 = H_3 = \frac{\dot{B}}{B}$ are directional Hubble parameters.

$$A_m = \frac{2(k-1)^2}{(k+2)^2} \quad (32)$$

The Anisotropic parameter is uniform throughout the evaluation of the universe; the Anisotropic parameter A_m is independent of cosmic time t but depends on the value of the parameter k . It is observed that it has a singular point at $k = -2$; however, it becomes anisotropic for $k > 1$ and for $k = 1$ it is isotropic.

Scalar expansion θ is given by

$$\theta = 3H = \left(\frac{\dot{A}}{A} + 2 \frac{\dot{B}}{B} \right) \quad (33)$$

$$\theta = 3 \left(\frac{nt_0 + bt}{tt_0} \right) \quad (34)$$

Shear Scalar σ^2 is given by

$$\sigma^2 = \frac{3}{2} A_m H^2$$

From equations (28) and (32)

$$\sigma^2 = \left[\frac{3(k-1)(nt_0 + bt)^2}{(k+2)^2 t^2 t_0^2} \right] \quad (35)$$

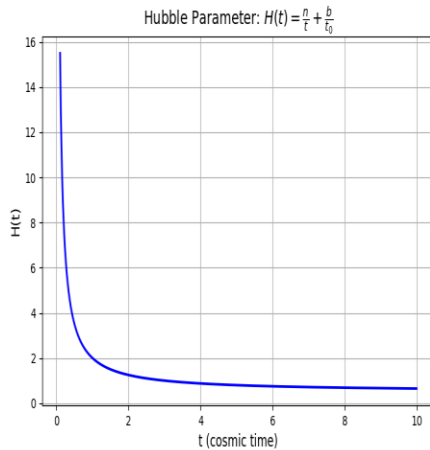


Figure 1. Hubble Parameter vs time t

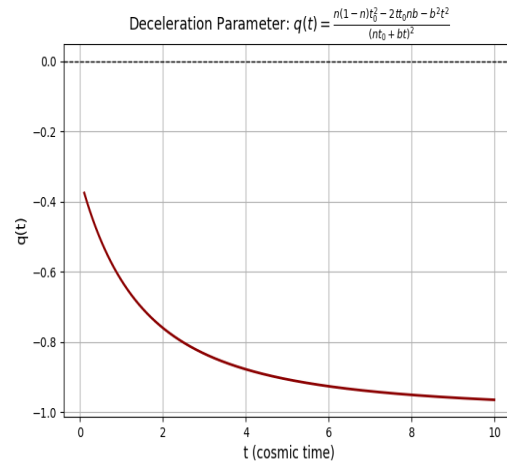


Figure 2. Deceleration Parameter q vs time t

Fig. 1 & 2: These figures show the behaviour of the Hubble parameter H and the deceleration parameter q at $n = 1.5$, $b = 0.5$ and $t_0 = 1.0$

From Fig. 1, we observe that the Hubble parameter H and the deceleration parameter q decrease as cosmic time t increases. The Hubble parameter H gradually converges toward the smallest positive value, and its positive values throughout cosmic evolution unequivocally signify the ongoing expansion of the universe. From Fig. 2, the deceleration parameter q decreases as cosmic time t increases. The deceleration parameter describes acceleration as $q < 0$ or deceleration $q > 0$ of the universe; from the graph, we observe that the deceleration parameter transitions from the early decelerating phase to the late accelerating phase. This result nicely matches with Type-I supernovae observations, as $q < 0$ for $\frac{t_0(\sqrt{n}-n)}{b} < t$.

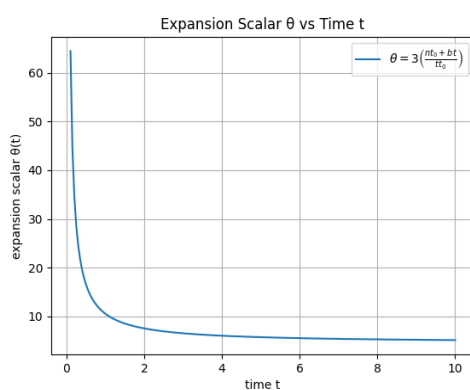


Figure 3. Scalar Expansion θ vs time t

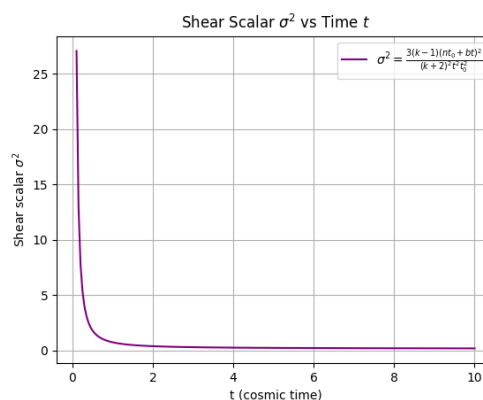


Figure 4. Shear Scalar σ^2 vs time t

These figures show the behaviour of Scalar expansion θ and Shear Scalar σ^2 at $n = 2.0$, $b = 1.5$, $t_0 = 1.0$ and $k = 1.2$.

From Fig. 3, we observed that the scalar expansion θ is positive and depends on time t ; it converges to zero when t tends to infinity; also from Fig. 4, we see that the shear scalar σ is large near $t = 0$ and approaches zero for large values of t .

From equation (16)-(18), energy density (ρ) and pressure (p) is obtained as:

$$p = \frac{\left\{ \frac{-6n}{(k+2)t^2} + \frac{9}{(k+2)^2} \frac{(nt_0+bt)^2}{t^2 t_0^2} \left[3 - \frac{(1-\alpha)}{(1-2\alpha)}(1+2k) \right] + \beta \left[\frac{(1-\alpha)}{(1-2\alpha)} - 1 \right] \right\}}{\alpha \left[\frac{1}{1-2\alpha} \left[\frac{9}{(k+2)^2} \frac{(nt_0+bt)^2}{t^2 t_0^2} (1+2k) - \beta \right] \right]^{\frac{\alpha-1}{\alpha}}} \quad (36)$$

$$\rho = \left\{ \frac{1}{1-2\alpha} \left[\frac{9}{(k+2)^2} \frac{(nt_0+bt)^2}{t^2 t_0^2} (1+2k) - \beta \right] \right\}^{\frac{1}{\alpha}} \quad (37)$$

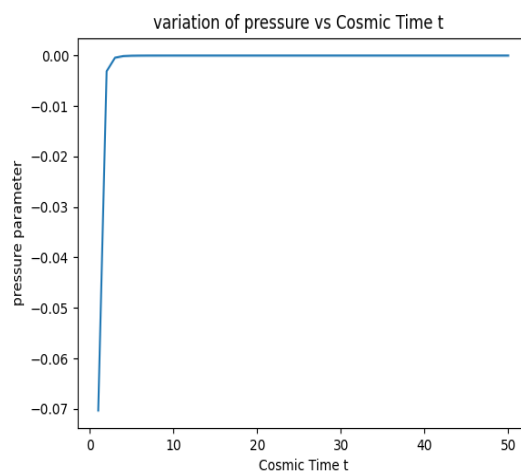


Figure 5.

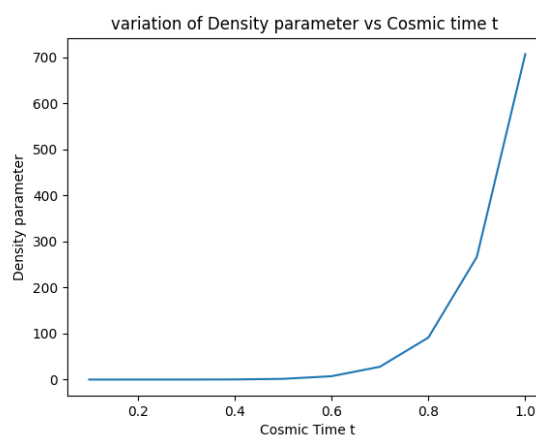


Figure 6.

These figures show the behaviour of pressure (p) and energy density (ρ) at $n = 0.6$, $b = 0.4$, $t_0=1.0$, $\alpha = 0.3$, $\beta = 0.1$ and $k = 0.5$.

From Fig. 5, the derived equation of pressure exhibits a rich dynamical structure, with both algebraic and time-dependent elements contributing to the evolution of the cosmic fluid. The presence of competing positive and negative terms allows $p(t)$ to naturally transition between different effective equations of state during cosmic evolution. In particular, the negative contributions such as the $\frac{-6n}{(k+2)t^2}$ term assist the model in generating late time negative pressure, a key requirement for cosmic acceleration. Meanwhile, the squared scaling factor $\frac{(nt_0+bt)^2}{t^2t_0^2}$ introduces a coupling between the temporal evolution and the model parameters enabling a smooth interpolation between early universe deceleration and late time dark energy dominated phases. The dependence of the pressure on, and β significantly increases the model's ability to describe a wide range of cosmic behaviors, including quintessence-like, phantom-like, or unified dark fluid scenarios. This structural flexibility is comparable to modern dark energy formulations found in scalar field cosmology, k-essence models, and certain classes of modified gravity theories. The negative effective pressure arising at late times is consistent with observational requirements from $H(z)$, BAO, SNe Ia, and Planck results, all of which support an accelerating universe. This alignment with contemporary theoretical and observational trends confirms that the proposed framework is a strong candidate for describing dynamical dark energy and the full evolution of the universe.

From Fig 6. and from derived equation of energy density, exhibits a behaviour consistent with dynamical dark energy models proposed in recent cosmological studies. The dependence of ρ on the combined scaling term $\frac{(nt_0+bt)}{t}$ demonstrates that the density evolves smoothly with cosmic time and can accommodate both early time high energy regimes and late-time accelerated expansion. This time-dependent evolution contrasts with models assuming constant density but aligns well with variable EoS and modified gravity frameworks reported in contemporary literature. In comparison with other research outcomes, the present form of ρ shows that the inclusion of parameters α , k and β provides additional freedom to reproduce observationally compatible behaviours such as a decreasing density at early epochs followed by a stabilizing trend at late times. Such flexibility places this model within the broader class of phenomenological density functions used to explain late-time cosmic acceleration without strictly assuming a cosmological constant. The agreement in qualitative trends with previously published density models supports the viability of this formulation and suggests

that it can serve as a strong candidate for describing dark energy-dominated phases of cosmic evolution.

EoS Parameter

The equation of state (EoS) parameter of the form $\omega = \frac{p}{\rho}$, which is the ratio of the cosmic pressure p to the cosmic energy density ρ , is a useful tool for studying Dark Energy (DE). Every DE model has a different value for the EoS parameter. For example, the cosmological constant has a value of $\omega = -1$, the quintessence model has a bound of $-1 < \omega < -0.33$, and the phantom DE model has a value of $\omega < -1$.

$$\omega = \frac{(1-2\alpha) \left\{ \frac{-6n}{(k+2)t^2} + \frac{9}{(k+2)^2} \frac{(nt_0+bt)^2}{t^2 t_0^2} \left[3 - \frac{(1-\alpha)}{(1-2\alpha)}(1+2k) \right] + \beta \left[\frac{(1-\alpha)}{(1-2\alpha)} - 1 \right] \right\}}{\left[\frac{9}{(k+2)^2} \frac{(nt_0+bt)^2}{t^2 t_0^2} (1+2k) - \beta \right]} \quad (38)$$

These figures show the behaviour of the equation of state (EoS) parameter at $n = 2.0$, $b = 0.5$, $t_0=1.0$, $\alpha = 0.3$, $\beta = 0.4$ and $k = 1.0$.

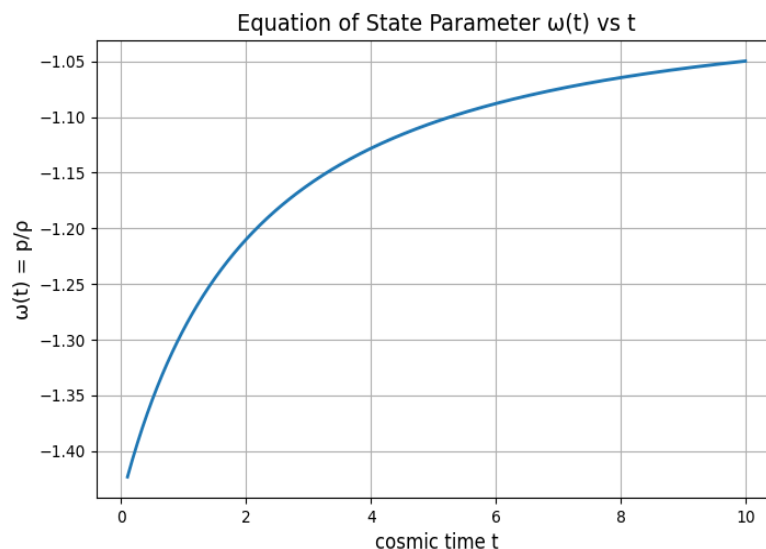


Figure 7.

Table 1. (Density, Pressure, Eos parameter values vs time t)

t	$\rho(t)$	$p(t)$	$\omega = \frac{p}{\rho}$
0.100	3.96219	-4.73325	-1.19460
1.589	1.33420	-1.45789	-1.09271
3.077	3.75161	-4.03216	-1.07478
4.566	2.20463	-2.35603	-1.06867
6.055	1.64274	-1.75101	-1.06591
7.544	1.36237	-1.45016	-1.06444
9.032	1.19674	-1.27282	-1.06357

From Fig. 7 and from Table 1 data, we observed that the EoS parameter ω approaches the value -1, indicating that our model leads to a phantom-like DE model.

The jerk parameter

The jerk parameter $j(t)$ plays a pivotal role in modern cosmography as a higher-order kinematic descriptor of the Universe's expansion history. It quantifies the rate of change of the cosmic acceleration and is defined as $j(t) = \frac{\ddot{a}}{aH^3} = q + 2q^2 - \frac{\dot{q}}{H}$ where 'a' is the scale factor and H the Hubble parameter. The concept of using the third derivative of the scale factor as a diagnostic first appeared in the statefinder formalism proposed by Sahni, Saini, Starobinsky, and Alam [38], who introduced the statefinder pair $\{r, s\}$ with to distinguish between various dark energy models and to test deviations from the standard Λ CDM cosmology. Subsequently, Visser [39] formally introduced the term "jerk parameter" and expanded the cosmographic series for the scale factor, systematically relating cosmological observables to higher derivatives of the expansion rate. Since then, the jerk parameter has become an essential kinematic tool for testing the consistency of observational data with Λ CDM ($j(t) = 1$) and for exploring alternative dark energy models.

From equation (28), (30) we get jerk parameter $j(t)$ as

$$j(t) = \frac{n^3 t_0^3 + b^3 t^3 + 3n^2 t_0^2 b t + 3n t_0 b^2 t^2 - 3n^2 t_0^3 - 3n b t t_0^2 + 2n t_0^3}{n^3 t_0^3 + b^3 t^3 + 3n^2 t_0^2 b t + 3n t_0 b^2 t^2} \quad (39)$$

The statefinder pair $\{r, s\}$

The Statefinder diagnostic, characterized by the pair of parameters $\{r, s\}$, was introduced as a geometrical tool to differentiate between competing dark energy models and to assess deviations from the concordance Λ CDM cosmology. Proposed by Sahni, Saini, Starobinsky, and Alam [38], the Statefinder pair is defined in terms of higher derivatives of the scale factor as $r = \frac{\ddot{a}}{aH^3} = 1$ and $s = \frac{r-1}{3(q-\frac{1}{2})}$, where H is the Hubble parameter and q is the deceleration parameter. The fixed point $\{r, s\} = \{1, 0\}$ corresponds to the Λ CDM model, while deviations from this point signify departures due to alternative dark energy scenarios. Following this seminal work, several researchers expanded the Statefinder framework to various cosmological contexts such as (e.g., $f(R)$, $f(T)$, $f(G)$ and $f(R, Lm)$ gravities), and

anisotropic cosmologies demonstrating its robustness as a model-independent diagnostic of cosmic dynamics. Hence, the Statefinder pair remains one of the most effective kinematic tools for distinguishing the underlying physics of cosmic acceleration beyond the standard model of cosmology.

we get the statefinder pair $\{r, s\}$ as

$$r = \frac{n^3 t_0^3 + b^3 t^3 + 3n^2 t_0^2 b t + 3n t_0 b^2 t^2 - 3n^2 t_0^3 - 3n b t t_0^2 + 2n t_0^3}{n^3 t_0^3 + b^3 t^3 + 3n^2 t_0^2 b t + 3n t_0 b^2 t^2} \quad (40)$$

$$s = \frac{2n t_0^2 (3n t_0 + 3b t - 2t_0)}{3(3n^3 t_0^3 - 2n^2 t_0^3 + 9n^2 t_0^2 b t + 9n t_0 b^2 t^2 - 2n b t t_0^2 + 3b^2 t^3)} \quad (41)$$

Model I for Power-Law Expansion:

When $b = 0$ then equation (20) becomes

$$a(t) = a_0 \left(\frac{t}{t_0}\right)^n \text{ which is a power law expansion.}$$

Then equations (24) and (25) yield

$$B = \left[a_0 \left(\frac{t}{t_0}\right)^n \right]^{\frac{3}{k+2}} \quad (42)$$

$$\text{and } A = \left[a_0 \left(\frac{t}{t_0}\right)^n \right]^{\frac{3k}{k+2}} \quad (43)$$

then the spatial volume equation (22) is then

$$V = a_0^3 \left(\frac{t}{t_0}\right)^{3n} \quad (44)$$

Then the Hubble parameter equation (28) is

$$H = \frac{n}{t} \quad (45)$$

Scalar expansion θ , equation (34) given by

$$\theta = 3 \left(\frac{n}{t}\right) \quad (46)$$

Deceleration parameter q , equation (30)

$$q = \frac{n(1-n)t_0^2}{(nt_0)^2} \quad \text{i.e. } q = \frac{(1-n)}{n} \quad (47)$$

Shear Scalar σ^2 is given by (35)

$$\sigma^2 = \left[\frac{3(k-1)n^2}{(k+2)^2 t^2} \right] \quad (48)$$

Anisotropic parameter A_m is,

$$A_m = \frac{2(k-1)^2}{(k+2)^2} \quad (49)$$

The Hubble parameter H is always positive and decreases with respect to time t . Here, we observe that the Hubble's parameter, shear scalar σ^2 and scalar expansion θ diverge at $t = 0$, the Anisotropic parameter is uniform throughout the evaluation of the universe. The universe is expanding and accelerating is revealed by the recent observations of SNe Ia and the value of the deceleration parameter lies in the range $-1 < q < 0$. If $n < 1$ then $q > 0$, hence the universe exhibits a decelerated model and if $n > 1$ then $q < 0$, here the universe shows an accelerating expansion.

Equation (37) becomes

$$\rho = \left\{ \frac{1}{1-2\alpha} \left[\frac{9}{(k+2)^2} \frac{n^2}{t^2} (1+2k) - \beta \right] \right\}^{\frac{1}{\alpha}} \quad (50)$$

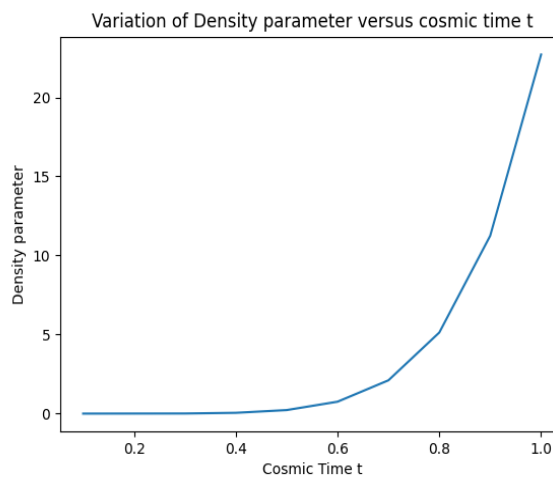


Figure 8.

The derived density parameter reveals a physically consistent evolution of the cosmic energy density for the chosen anisotropic background. The explicit time dependence, dominated by the term $\frac{n^2}{t^2}$, ensures that the density remains high at early epochs and gradually decreases as the universe expands. Overall, this density formulation supports a universe that transitions smoothly from an early, high-density phase to a late-time accelerating regime, demonstrating good qualitative agreement with modern theoretical and observational frameworks.

Equation (36) becomes

$$p = \frac{\left\{ \frac{-6n}{(k+2)t^2} + \frac{9}{(k+2)^2} \frac{n^2}{t^2} \left[3 - \frac{(1-\alpha)}{(1-2\alpha)} (1+2k) \right] + \beta \left[\frac{(1-\alpha)}{(1-2\alpha)} - 1 \right] \right\}}{\alpha \left[\frac{1}{1-2\alpha} \left[\frac{9}{(k+2)^2} \frac{n^2}{t^2} (1+2k) - \beta \right] \right]^{\frac{\alpha-1}{\alpha}}} \quad (51)$$

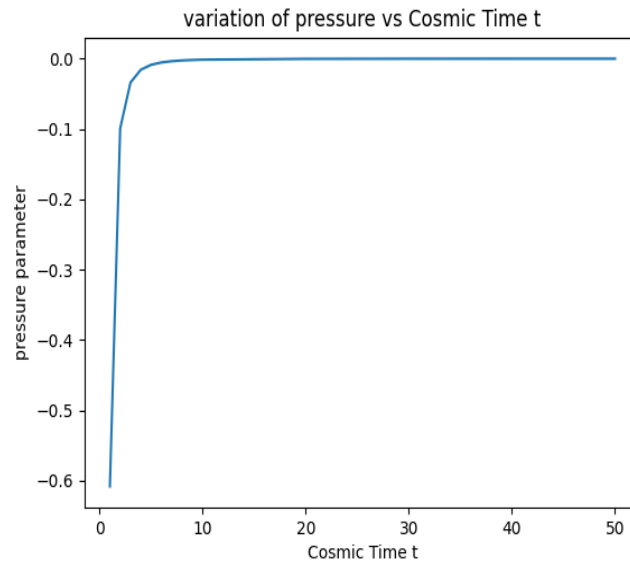


Figure 9.

The pressure expression derived here provides a mathematically consistent and physically credible mechanism for the emergence of negative pressure and late-time acceleration; the pressure retains a well-behaved structure throughout the cosmic evolution domain where the corresponding energy density remains positive, also through the denominator of $p(t)$, which contains $\rho^{\frac{\alpha-1}{\alpha}}$ that coupling between $p(t)$ and $\rho(t)$ enforces a generalised barotropic-like behaviour that naturally adapts to changes in the expansion dynamics.

EoS Parameter

$$\omega = \frac{(1-2\alpha)}{\alpha} \frac{\left\{ \frac{-6n}{(k+2)t^2} + \frac{9}{(k+2)^2 t^2} \left[3 - \frac{(1-\alpha)}{(1-2\alpha)}(1+2k) \right] + \beta \left[\frac{(1-\alpha)}{(1-2\alpha)} - 1 \right] \right\}}{\left[\frac{9}{(k+2)^2 t^2} (1+2k) - \beta \right]} \quad (52)$$

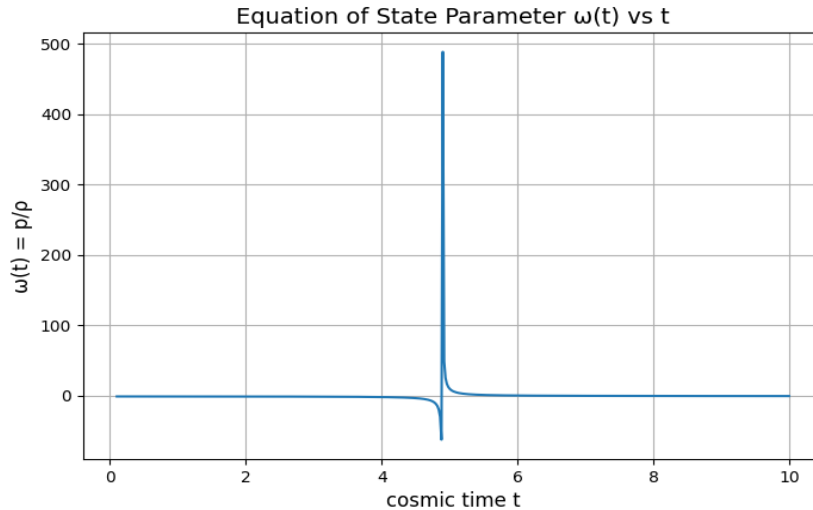


Figure 10.

These figures show the behaviour of the equation of state (EoS) parameter at $n = 2.0$, $\alpha = 0.3$, $\beta = 0.5$ and $k = 1.0$.

at early cosmic times (small t), the EoS parameter remains finite and negative, indicating a repulsive, acceleration-driven phase similar to early inflation. For sufficiently large cosmic time t the model predicts that $\omega(t)$ approaches a finite value close to zero, consistent with a matter-dominated or quintessence-like accelerated expansion. This behaviour demonstrates that the model naturally evolves from an early accelerating phase to a stable late-time dark energy compatible regime. This suggests that the Universe eventually evolves toward a matter-like or quintessence-like phase:

- When $\omega(t) \rightarrow 0$ the fluid behaves like pressureless dark matter.
- When $\omega(t) < 0$ it corresponds to mild dark energy, like accelerated expansion.
- When $\omega(t) > 0$ it behaves like stiff matter or an anisotropic fluid.

Jerk Parameter

For $b = 0$, $a(t) = a_0 \left(\frac{t}{t_0}\right)^n$

The jerk parameter in cosmology is defined as (Chiba and Nakamura (1998), Visser (2004, 2005))

$$j(t) = \frac{\ddot{a}}{aH^3} = q + 2q^2 - \frac{\dot{q}}{H} \quad (53)$$

Where a is the cosmic scale factor, H is the Hubble parameter, and the dot denotes differentiation with respect to cosmic time.

Using (53), (39) becomes

$$j(t) = \frac{n(n-1)(n-2)}{n^3} \quad (54)$$

For $n > 2$ jerk parameter has a positive value, also $j(t)$ approaches 1 for finite $n = 2/3$ and for large values of n which strongly aligns with the Λ CDM model. It means that our model closely approximates the behaviour of the Λ CDM model. Different works report small deviations in the value of the jerk parameter $j_0 = 0.99 \pm 0.03$, Our model agrees well with this value.

Statefinder pair

For $b = 0$, $a(t) = a_0(\frac{t}{t_0})^n$

The statefinder pair $\{r, s\}$ is defined as (Akarsu et al., 2014).

$$r = \frac{\ddot{a}}{aH^3}, \quad s = \frac{r-1}{3(q-\frac{1}{2})} \quad (55)$$

Using this formula (40), (41) becomes

$$r = \frac{n(n-1)(n-2)}{n^3}, \quad s = \frac{2}{3n} \quad (56)$$

For the numerical value of the given pair $\{r, s\}$ at $n \rightarrow \infty$, $\{r, s\} = \{1, 0\}$ these values exactly match with the Λ CDM model. Confirming the universe behaves like its driven by a cosmological constant when $b = 0$.

Model II Exponential Expansion:

When $n = 0$ then equation (20) becomes

$a(t) = a_0 e^{b(\frac{t}{t_0}-1)}$ which is the exponential law of expansion

then the spatial volume is then $V = a_0^3 e^{3b(\frac{t}{t_0}-1)}$

Then equations (24) and (25) become

$$B = \left[a_0 e^{b(\frac{t}{t_0}-1)} \right]^{\frac{3}{k+2}} \quad (57)$$

$$\text{and } A = \left[a_0 e^{b(\frac{t}{t_0}-1)} \right]^{\frac{3k}{k+2}} \quad (58)$$

Then the Hubble parameter is

$$H = \frac{b}{t_0} \quad (59)$$

Scalar expansion θ is given by

$$\theta = 3 \left(\frac{b}{t_0} \right) \quad (60)$$

Deceleration parameter q is

$$q = -1 \quad (61)$$

Shear Scalar σ^2 is given by

$$\sigma^2 = \left[\frac{3(k-1)b^2}{(k+2)^2 t_0^2} \right] \quad (62)$$

From equations (59), (60), (61), and (62), we examined that a cosmological model in which the Hubble parameter evolves as a constant quantity $H = \frac{b}{t_0}$. The constancy of the Hubble rate directly leads to a uniform scalar expansion, expressed as indicating a homogeneous and exponentially expanding universe. Consistent with this, the model naturally yields a deceleration parameter which characterizes a pure de Sitter type accelerated expansion [1,2]. Such a value signals the absence of deceleration and confirms that the cosmic dynamics are entirely governed by a repulsive, vacuum-like energy component. Furthermore, the presence of anisotropy in the spacetime structure is captured by the shear scalar, which reveals that the model retains anisotropic features unless $k = 1$ [43], where the expression reduces to $\sigma^2 = 0$, and the universe becomes isotropic. This flexibility allows the framework to describe both anisotropic early universe phases and isotropic late-time behaviour.

Overall, the combination of constant Hubble expansion, negative unity deceleration parameter, and tunable anisotropy constructs a coherent description of an accelerating cosmological model. It aligns with the qualitative behaviour expected in inflationary epochs or dark energy dominated eras. The model thus provides a mathematically tractable and physically consistent platform for exploring accelerating anisotropic cosmologies and their observational implications.

Equation (37) becomes

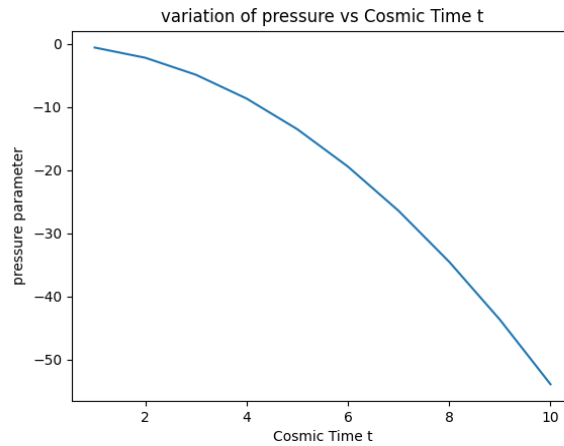
$$\rho = \left\{ \frac{1}{1-2\alpha} \left[\frac{9}{(k+2)^2} \frac{b^2}{t_0^2} (1+2k) - \beta \right] \right\}^{\frac{1}{\alpha}} \quad (63)$$

From (63) the resulting density is independent of cosmic time t , indicating that the model supports a constant energy density phase, similar to de Sitter-like cosmological behaviour; also, the energy density remains constant throughout cosmic evolution.

Equation (36) becomes

$$p = \frac{\left\{ \frac{-6n}{(k+2)t^2} + \frac{9}{(k+2)^2} \frac{b^2}{t_0^2} \left[3 - \frac{(1-\alpha)}{(1-2\alpha)} (1+2k) \right] + \beta \left[\frac{(1-\alpha)}{(1-2\alpha)} - 1 \right] \right\}}{\alpha \left[\frac{1}{1-2\alpha} \left[\frac{9}{(k+2)^2} \frac{b^2}{t_0^2} (1+2k) - \beta \right] \right]^{\frac{\alpha-1}{\alpha}}} \quad (64)$$

Figure 11.

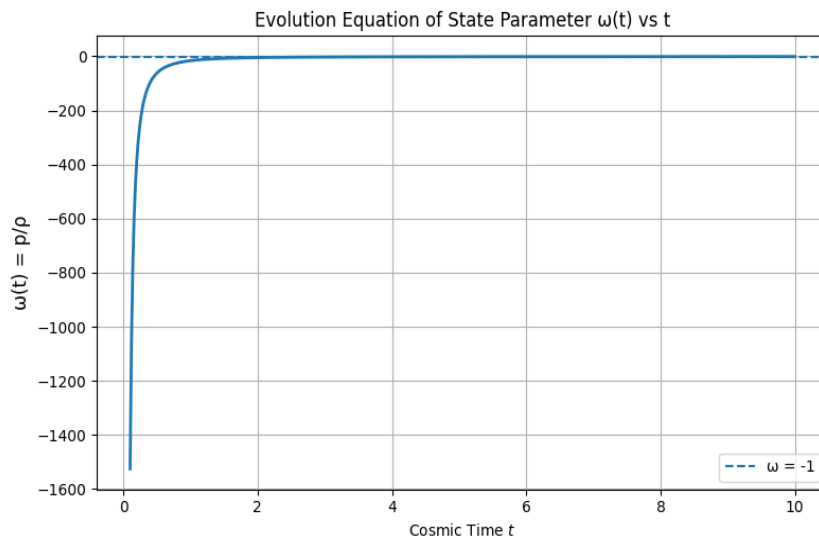


a)

From equation (64), at early times the t^{-2} term dominates and produces strongly negative pressure (divergent as $t \rightarrow 0$), while at late times the pressure asymptotes to a constant.

EoS Parameter

$$\omega = \frac{(1-2\alpha)}{\alpha} \frac{\left\{ \frac{-6n}{(k+2)t^2} + \frac{9}{(k+2)^2 t_0^2} \left[3 - \frac{(1-\alpha)}{(1-2\alpha)}(1+2k) \right] + \beta \left[\frac{(1-\alpha)}{(1-2\alpha)} - 1 \right] \right\}}{\left[\frac{9}{(k+2)^2 t_0^2} (1+2k) - \beta \right]} \quad (65)$$



Depending on parameter values, ω_∞ may lie in various cosmologically relevant ranges:

- $\omega_\infty \approx 0$: matter-like (pressureless) behaviour.
- $-1 < \omega_\infty < 0$: quintessence-like dark energy.
- $\omega_\infty < -1$: phantom-like behaviour (requires parameter choices that flip sign appropriately).
- $\omega_\infty > 0$: stiff or radiation-like effective outcome.

Jerk Parameter

For Model II: $n = 0$, $a(t) = a_0 e^{b(\frac{t}{t_0}-1)}$

$$j(t) = \frac{\ddot{a}}{aH^3} = q + 2q^2 - \frac{\dot{q}}{H} = 1 \quad (66)$$

Statefinder pair

$$r = \frac{\ddot{a}}{aH^3} = 1 , \quad s = \frac{r-1}{3(q-\frac{1}{2})} = 0 \quad (67)$$

The statefinder pair $\{r, s\} = \{1, 0\}$ These values exactly match with the Λ CDM model. confirming the universe behaves like its driven by a cosmological constant when $n = 0$.

6. CONCLUSION

Here, we have described the anisotropic Bianchi I space-time in the presence of a $f(R, L_m)$ modified gravity theory represented by $f(R, L_m) = \frac{R}{2} + L_m^\alpha + \beta$, to explain the challenging problem of late-time acceleration of the universe using a hybrid expansion law. Here we discussed two model namely power law expansion and exponential expansion.

- we observe from Fig. 1 (a) and (b) that the Hubble parameter H and the deceleration parameter q decrease as cosmic time t increases. From the graph, we observe that the deceleration parameter transitions from the early decelerating phase to the late accelerating phase; this result agrees well with the present work.
- From Fig. 1 c). we observed that the scalar expansion θ is positive and depends on time t it converges to zero when $t \rightarrow \infty$ Also, from Fig. 1 (d). we see that the shear scalar σ is large near $t = 0$ and approaches zero for large values of t .
- From equation (35) and fig. 2 a), we observed that the derived pressure expression shows that the model naturally supports a transition from early positive pressure to late-time negative pressure the essential feature required for cosmic acceleration. From equation (36) and fig. 2 b), the explicit time dependence introduced through the ratio $(nt_0 + bt)/t$ allows the density to decrease during the early universe and evolve toward a late-time accelerating phase; the present density model strengthens the argument that time-dependent dark energy functions can effectively describe the full cosmic history without relying exclusively on a constant cosmological term.
- Every DE model has a different value for the EoS parameter. Here, our model describes the phantom DE model, which has a value of $\omega < -1$.

POWER-LAW EXPANSION

- Here we observe that the Hubble's parameter, shear scalar σ^2 and scalar expansion θ diverge at $t = 0$.

- The value of the deceleration parameter lies in the range $-1 < q < 0$ depends on the value of n . It means that the universe is an accelerating expanding universe.
- The density parameter remains high at early epochs and gradually decreases as the universe expands. It is in good agreement with observational data.
- The value of the pressure parameter is negative; it means that the model is accelerating.
- At early times, the EoS parameter is finite and negative, indicating an initial accelerated (inflation-like) phase. As time increases $\omega(t)$ gradually approaches a small value near zero consistent with a matter-dominated or mild quintessence regime. Thus, the model naturally evolves from early acceleration to a stable late-time dark energy compatible state.
- In this model, the jerk parameter $j(t)$ approaches 1 for $n = 2/3$ as well as for larger values of n , which is a key signature of the Λ CDM model. This shows that our model closely mimics the dynamical behavior of Λ CDM, indicating strong compatibility with the standard cosmological framework.

EXPONENTIAL EXPANSION

- The Hubble parameter remains constant, which naturally produces a steady rate of cosmic expansion. This constant expansion rate implies a uniform scalar expansion, pointing to a smooth, homogeneous universe undergoing exponential growth.
- We observed that the deceleration parameter $q = -1$, which leads to accelerated expansion.
- Shear Scalar σ^2 which reveals that the model retains anisotropic features unless $k = 1$.
- The given density is independent of cosmic time t suggests that the model allows a constant energy density phase, which is comparable to de-Sitter like cosmological behaviour. Additionally, the energy density stays constant throughout cosmic evolution.
- The pressure diverges at $t = 0$ and at late times the pressure asymptotes to a constant.
- Depending on parameter values, the EoS parameter ω has different behaviour, such as matter-like, quintessence-like, phantom-like, stiff or radiation-like behaviour.
- Here we obtained the value of the jerk parameter $j(t) = 1$ which exactly matches the given observational data.
- The statefinder pair $\{r, s\} = \{1, 0\}$ These values exactly match with the Λ CDM model.

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