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Best Simultaneous Approximation in 2-Normed Almost Linear Space

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ABSTRACT

In this paper we establish some of the results of best simultaneous approximation in linear 2- normed space in the context of 2- normed almost linear space.

1 .INTRODUCTION

In (1) Gliceria Godini introduced the concept "almost linear space" which is defined as " A non empty set X together with two mappings s: XxX → X and m:RxX→ X Where s(x ,y)=x +y and m(λ, x) = λ x is said to be an almost linear space if it satisfies the following properties.

For every x, y, z Є X and for every λ, μ Є R

i) x +y Є X ,
ii) (x + y) +z = x+(y + z) ,
iii) x +y = y + x ,
iv) There exists an element 0 ЄX such that x+0=x ,
v) 1 x = x ,
vi) λ(x + y)=λ x+ λ y ,
vii) 0 x =0 ,
viii) λ(μ x)=(λ μ)x ,ix) (λ + μ)x=λ x +μ x for λ ≥0,μ≥0.

In (1 & 4) Gliceria Godini also introduced the concept" normed almost linear space" which is defined as "an almost linear space X together with |||.||| : X→ R is said to be normed almost linear space if it satisfies the following properties

i) ||| x |||=0 if and only if x=0,
ii) ||| λx |||=IλI ||| x ||| ,
iii) ||| x-z |||≤|||x-y|||+|||y-z||| for every x,y ЄX and λ ЄR.

The concept of linear 2-normed space has been initially investigated by S. Gähler(17) and has been extensively by Y.J.Cho, C.Diminnie, R.Freese and many other, which is defined as

" a linear space X over R with dim>1 together with II .II is called Linear 2-normed space if II . II satisfy the following properties

i) II x, y II>0 and II x , y II=0 if and only if x and y are linearly dependent,
ii) II x , y II = II y , x II,
iii) II λx ,y II = IλI II x , y II, and
iv) II x , y+z II = II x , z II + II y ,z II for every x,y,z ЄX and λ ЄR.

In (23) T.Markandeya naidu and Dr D.Bharathi introduced a new concept called 2-normed almost linear space and established some of the results of best approximation in 2-nomed almost linear space.In (22) S.Elumalai & R.Vijayaragavan established some of the results of best simultaneous approximation in linear space in the context of linear2-normed space. In this paper we extend some of the results of best simultaneous approximation on linear2-normed space in to 2-normed almost linear space.

2. PRELIMINARIES

Definition 2.1. Let X be an almost linear space of dimension> 1 and

||| .|||: X x X → R be a real valued function. If ||| . ||| satisfy the following properties

i) ||| α, β |||=0 if and only if α and β are linearly dependent,
ii) ||| α, β ||| = ||| β, α |||,
iii) ||| aα , β ||| = IaI ||| α, β |||,
iv) **||| α, β-δ ||| ≤ ||| α, β-γ ||| + ||| α, γ-δ |||** for every α, β, γ, δ ∈ X and a ∈ R.

then (X,|||.|||) is called **2-normed almost linear space**.

Definition 2.2. Let X be a 2-normed almost linear space over the real field R and G a non empty subset of V_x . For a bounded sub set A of X let us define

rad_G(A) $=inf_{g\epsilon G} sup_{a\epsilon A}$ ||| x , a-g ||| for every x ϵ X\$V_x$ **2.1**

and

$cent_G$(A) = $g_0 \epsilon$ G: $sup_{a\epsilon A}$||| x , a-g_0 ||| = rad_G(A) for every x ϵ X\V_x. **2.2**

The number rad_G(A) is called the chebyshev radius of A with respect to G and an element $g_0 \epsilon\, cent_G$(A) is called a **best simultaneous approximation** or chebyshev centre of A with respect to G.

Definition 2.3. When A is a singleton say A={a}, a ϵ X\$\bar{G}$ then rad_G(A) is the distance of a to G, denoted by dist(a,G) and defined by dist(a,G)= $inf_{g\epsilon G}$ ||| x,a-g||| for every x ϵ X\$V_x$
2.3

and $cent_G$(A) is the set of all best approximations of a out of G denoted by P_G(a) and defined by

P_G(a)={ $g_0 \epsilon$G:|||x,a-g_0|||=dist(a,G),

for every x ϵ X\V_x}
2.4

It is well known that for any bounded subset A of X we have

rad_G(A)= $rad_G(C_0(A))$= $rad_G(\bar{A})$

$cent_G$(A)= $cent_G(C_0(A))$= $cent_G(\bar{A})$

Where $C_0(A)$ stands for the convex hull of A and $\bar{A}$ stands for the closure of A.

Definition 2.4. Let X be a 2-normed almost linear spaces and ϕ≠ G ⊂ V_x. We difine

R_x(G) ⊂ X
in the following way
$a\epsilon R_x$(G) if for each g ϵG *there exists* $v_g\, \epsilon\, V_x$ such that the following conditions are hold

i) ||| x, a-g ||| = ||| x, v_g-g ||| for each $v_g \epsilon V_x$ **2.5**

ii) ||| x, a- v ||| ≥ ||| x, v_g- v ||| for every x ϵ X\\V_x. **2.6**

We have $V_x \subset R_x$(G).

If $G_1 \subset G_2$ then $R_x(G_2)$ $\subset R_x(G_1)$.

Definition 2.5. Let X be a 2-normed almost linear space. The set G is said to be proximinal if P_G(a) is nonempty for each aϵ X\\V_x .

3.Main Results

Theorem 3.1 Let (X, ||| . |||) be a 2- normed almost linear space. Let G ⊂ X and A be a bounded subset of X. Then the function Ψ(h,g) defined by sup ||| h, a-g |||, h ϵ X\\V_x , g ϵ G, a ϵ A is a continuous function on X.

Proof: For any a ϵ A and g ,g' ϵ G we have
||| h, a-g ||| ≤ ||| h, a-g' ||| +
||| h, $g' - $ g ||| , h ϵ X\\$V_x$.
Then
$sup_{a\epsilon A}$ ||| h, a-g ||| ≤
$sup_{a\epsilon A}$\{ ||| h, a-g' ||| + ||| h, $g' -$ g ||| \}
Now if ||| h, $g' -$ g ||| ≤ ε then
Ψ(h,g) ≤ Ψ(h, g'') + ε
By interchanging g and g'' we obtain
Ψ(h, g') ≤ Ψ(h,g) + ε that implies
| Ψ(h,g) - Ψ(h, g') |< ε
That is Ψ(h,g) is continuous on X.

Theorem 3.2 Let (X, ||| . |||) be a 2- normed almost linear space. Let G be a finite dimensional subspace of X. Then there exist a best simultaneous approximation g ϵ G to any given compact subset A of X.

Proof: Since A is compact there exist a constant ' t ' such that

||| a , b ||| ≤ t for all a ϵ A and b ϵ X.

Now we define the subset H of G as

G ≡ G(0,2t) then

inf. $sup_{a\epsilon A}$ ||| b, a-h ||| =

$inf_{g\epsilon G}$ $sup_{a\epsilon A}$ ||| b, a-h ||| , b ϵ X\\V_x ≤ t

Since h is compact the continuous function Ψ(h,b) attains its minimum over H for some g ϵ G.

which is the best simultaneous approximation to A.

Theorem 3.3 Let (X, ||| . |||) be a 2- normed almost linear space and let G be a convex subset of X and A ⊂ X . If g, g' ϵ G are two best simultaneous approximations to A by elements of G. Then
g'' = ⅄ g + (1- ⅄) g',0≤ ⅄ ≤ 1 is also best simultaneous approximation to A.

Proof: For x ϵ X\\V_x,

$sup_{a\epsilon A}$ ||| x, a-g'' |||

= $sup_{a\epsilon A}$ ||| x, a-⅄ g + (1- ⅄) g' |||

= $sup_{a\epsilon A}$ ||| x, ⅄(a-g) + (1- ⅄) (a-g') |||

≤ $sup_{a\epsilon A}$ ⅄ ||| x, a-g ||| +

(1-⅄) ||| x, a- g' |||

≤ ⅄ $sup_{a\epsilon A}$ ||| x, a-g ||| +

$(1-\lambda) sup_{a\epsilon A}$ ||| x, a- g' |||

$= \lambda\, d(A,G)_x + (1-\lambda)\, d(A,G)_x$

$= d(A,G)_x$

3.1

$d(A,G)_x = inf_{g\epsilon G}\ sup_{a\epsilon A}$ ||| x, a-g' |||

$\leq sup_{a\epsilon A}$ ||| x, g-g' ||| **3.2**

$d(A,G)_x = sup_{a\epsilon A}$ ||| x, a-g'' |||

This proves the result.

Theorem 3.4 Let (X,||| . |||) be a strictly convex 2-normed almost linear space. Let G be a finite dimensional subspace of X. Then there exists one and only one best simultaneous approximation from the element G by any given compact subset A of X.

Proof : The existence of a best simultaneous approximation follows from the Theorem3.2.

Suppose $g'$and g'' $(g' \neq g'')$ are best simultaneous approximations to A then for x ϵ X\\V_x,

$inf_{g\epsilon G}\ sup_{a\epsilon A}$ ||| x,a-g |||

$=sup_{a\epsilon A}$ ||| x, a-g' |||

$=sup_{a\epsilon A}$ ||| x, a-g'' |||

=k **3.3**

Then by theorem (3.3),

$\frac{g'+g''}{2}$ is also

the best simultaneous approximation.

That is

$sup_{a\epsilon A}$ ||| x, a-$\frac{g'+g''}{2}$ ||| = k **3.4**

Since A is compact there exist an element a_0 such that

$sup_{a\epsilon A}$ ||| x, a-$\frac{g'+g''}{2}$ |||

$=sup_{a\epsilon A}$ ||| x, a_0 -$\frac{g'+g''}{2}$ |||

=k **3.5**

*From eq.*3.3 ||| x, a_0-g' ||| ≤ k and ||| x, a_0-g'' ||| ≤ k .

Then by strict convexity we have

||| x, a_0-g'+a_0-g'' ||| < 2 k

That is ||| x, a_0 -$\frac{g'+g''}{2}$ ||| < k

This contradicts eq.3.5.

Hence the proof.

Theorem 3.5 Let G be a closed and convex subset of a uniformly convex 2-Banach space X.Then for any compact subset A of X there exist unique best approximation to A from the element of G.

Proof:

Let k=$inf_{g\epsilon G}\ sup_{a\epsilon A}$ ||| x,a-g |||: x ϵ X\\$V_x$ and $\{g_n\}$ be any sequence of elements in G Such that

$lim_{n\to\infty} sup_{a\epsilon A}$ ||| x,a-g_n ||| =k.

Also $k_m = sup_{a\epsilon A}$ ||| x,a-g_m ||| ,m≥1 and x ϵ X\\V_x.

Then $k_m \geq$ k which implies

||| x, $\frac{a-g_m}{k_m}$ |||≤1 for a ϵ A **3.6**

Now we consider $\frac{1}{2}\left[\frac{g_m}{k_m} + \frac{g_n}{k_n}\right] = \frac{k_n g_m + k_m g_n}{2 k_m k_n}\left[\frac{k_m + k_n}{k_m + k_n}\right]$

Let $y_{m,n} = \frac{k_n g_m + k_m g_n}{k_m + k_n}$.

since G is convex, $y_{m,n}$ ϵ G.

Hence $sup_{a\epsilon A}$ ||| x,a-$y_{m,n}$ |||≥k and $sup_{a\epsilon A}$ ||| x,$\frac{k_n+k_m}{2k_m k_n}$ a - $\frac{1}{2}$ $(\frac{g_m}{k_m}+\frac{g_n}{k_n})$ ||| = $sup_{a\epsilon A}$ ||| x,a-$y_{m,n}$ ||| $(\frac{k_n+k_m}{2k_m k_n})$≥ k$(\frac{k_n+k_m}{2k_m k_n})$. Since A is compact sub set of X there exist an a ϵ A such that

$$||| x, \frac{a-g_m}{k_m}+\frac{a-g_n}{k_n} |||\geq k(\frac{k_n+k_m}{k_m k_n}).$$

By eq.3.6 and the uniform convexity of the 2-norm it follows that

for given $\varepsilon > 0$ there exits an N such that ||| x, $\frac{a-g_m}{k_m}+\frac{a-g_n}{k_n}$ ||| $< \varepsilon$ for

m,n $>$ N and x ϵ X\V_x.

Since k_m → k as m → ∞ we can easily see that the sequence $\{g_n\}$ is a Cauchy sequence hence it converges to some g ϵ G as G is closed subset of X.

This provides that G is a best simultaneous approximation.

Assume that there exist two best simultaneous approximations

g_1 and g_2 .

Then there exist sequences $\{g_n\}$ and $\{g_m\}$ such that $g_n \rightarrow g_1$ as n → ∞

and $g_m \rightarrow g_2$ as m → ∞ .

Again $lim_{n\to\infty}$ $sup_{a\epsilon A}$ ||| x,a-g_n ||| =k = $lim_{n\to\infty}$ $sup_{a\epsilon A}$ ||| x,a-g_m ||| .

This implies that $sup_{a\epsilon A}$ ||| x,a-g_1 ||| = $sup_{a\epsilon A}$ ||| x,a-g_2 ||| and hence

$g_1 = g_2$.

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