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Best Approximation in 2-Normed Almost Linear Space

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ABSTRACT

In this paper we introduce a new concept called "2-normed almost linear space" and establish some of the results of best approximation on normed almost linear space in the context of 2- normed almost linear space.

1 .INTRODUCTION

In (1) Gliceria Godini introduced the concept "almost linear space" which is defined as " A non empty set X together with two mappings s: XxX → X and m:RxX→ X Where s(x ,y)=x +y and m(λ, x) = λ x is said to be an almost linear space if it satisfies the following properties.

For every x, y, z ∈ X and for every λ, μ ∈ R i) x +y ∈ X, ii) (x + y) +z = x+(y + z) ,iii) x +y = y + x, iv) There exists an element 0 ЄX such that x+0=x , v) 1 x = x ,
vi) λ(x + y)=λ x+ λ y , vii) 0 x =0 , viii) λ(μ x)=(λ μ)x
and ix) (λ + μ)x=λ x +μ x for λ ≥0,μ≥0.

In (1 & 4) Gliceria Godini also introduced the concept "normed almost linear space" which is defined as an almost linear space X together with III . III : X→ R is said to be normed almost linear space if it satisfies the following properties

i) III x III=0 if and only if x=0,
ii) III λx III=IλI III x III ,
iii) III x-z III≤IIIx-yIII+IIIy-zIII for every x,y ЄX and λ ЄR.

The concept of linear 2-normed space has been initially investigated by S. Gӓhler(17) and has been extensively by Y.J.Cho,C.Diminnie, R.Freese and many other, which is defined as a linear space X over R with dim>1 together with II .II is called Linear 2-normed space if II . II satisfy the following properties

i) II x, y II>0 and II x , y II=0 if and only if x and y are linearly dependent,
ii) II x , y II=II y , x II ,
iii) II λx ,y II= IλI II x , y II and
iv) II x , y+z II=II x , z II + II y ,z II for every x,y,z ЄX and λ ЄR.

In (2 & 3) G.Godini established some results of best approximation on normed linear space in the context of normed almost linear space. In this paper we extend some of the results of best approximation on normed almost linear space in to 2-normed almost linear space.

2. PRELIMINARIES

Definition 2.1. Let X be an almost linear space of dimension> 1 and III .III: X x X → R be a real valued function. If III . III satisfy the following properties

i) III α, β III=0 if and only if α and β are linearly dependent,
ii) III α, β III = III β, α III ,
iii) III aα , β III = IaI III α, β III ,
iv) **III α, β-δ III ≤ III α, β-γ III + III α, γ-δ III** for every α, β, γ, δ ∈ X and a ∈ R.

then (X,III.III) is called **2-normed almost linear space**.

Definition 2.2. Let X be a 2-normed almost linear space over the real field R and G a non empty subset of V_x . For a bounded sub set A of X let us define

rad_G(A) $=inf_{g\epsilon G}\, sup_{a\epsilon A}$ III x , a-g III for every x ϵ X\\$V_x$ and **2.1**

$cent_G$(A) = $g_0\epsilon$ G: $sup_{a\epsilon A}$ III x , a-g_0 III = rad_G(A) for every x ϵ X\\V_x. **2.2**

The number rad_G(A) is called the chebyshev radius of A with respect to G and an element $g_0\epsilon\ cent_G$(A) is called a best simultaneous approximation or chebyshev centre of A with respect to G.

Definition 2.3. When A is a singleton say A= {a}, a ϵ X\\$\bar{G}$ then rad_G(A) is the distance of a to G, denoted by dist(a,G) and defined by

dist(a,G)=$inf_{g\epsilon G}$ IIIx,a-gIII for every xϵ X\\V_x **2.3**

and $cent_G$(A) is the set of all best approximations of 'a' out of G denoted by P_G(a) and defined by

P_G(a)={ $g_0\epsilon$ G : IIIx,a-g_0III=dist(a,G), for every x ϵ X\\V_x} **2.4**

Definition 2.4. Let X be a 2-normed almost linear space. The set G is said to be proximinal if P_G(a) is nonempty for each aϵ X\\V_x .

It is well known that for any bounded subset A of X we have rad_G(A)= $rad_G(C_0(A))$= $rad_G(\bar{A})$

$cent_G$(A)= $cent_G(C_0(A))$= $cent_G(\bar{A})$

Where $C_0(A)$ stands for the convex hull of A and $\bar{A}$ stands for the closure of A.

Definition 2.5. Let X be a 2-normed almost linear spaces and ϕ≠ G ⊂ V_x. We difine R_x(G) ⊂ X in the following way

$a\epsilon R_x$(G) if for each g ϵG *there exists* $v_g\ \epsilon\ V_x$ such that the following conditions are hold

i) III x, a-g III = III x, v_g-g III for each $v_g\ \epsilon\ V_x$ **2.5**

ii) III x, a- v III ≥ III x, v_g- v III for every x ϵ X\\V_x. **2.6**

We have $V_x \subset R_x$(G).

If $G_1 \subset G_2$ then $R_x(G_2) \subset R_x(G_1)$.

3. MAIN RESULTS

Theorem 3.1 Let X be a 2-normed almost linear spaces and G a bounded weakly compact subset of V_x. Then G is proximinal in X.

Proof : Let G be a bounded compact subset of V_x.

By definition of d(a,G), there exist a sequence $\{g_n\}, n = 1\ to\ \infty$ in G such that

$\lim_{n\to\infty}$ III x, a-g_n III = dist (a, G)

Since G is bounded for some $\varepsilon > 0$ there exist N such that

III x, a-g_n III ≤ dist(a,G)+ ε for n≥N.

≤ M for every n.

Where M=max(M_1, M_2), $M_1 = d$(a,G)+ ε and M_2=max III x, a-g_nIII for n≤N.

Now III x, g_nIII≤IIIx,a-g_nIII+IIIx,aIII

≤M+IIIx,aIII

This implies that $\{g_n\}$ *is bounded* and therefore converges weakly to g in G.

Hence we have IIIx,a-g III≤ $\lim_{n\to\infty}$ IIIx,a-g_nIII = dist(a,G)

But III x, a-g III ≥ dist(a,G)

Therefore we have III x, a-g III= dist (a, G) and so 'g' is a best approximation to 'a' from G.

Thus G is Proximinal in X.

Definition3.2 Let X be 2-normed almost linear space and G a non-empty subset of V_x.

Let $T_G$be the sub set of X defined in the following way.

a ϵ T_G if for each g ϵG and r_i> 0 i=1,2, the relations III x , a-g III < $r_1 + r_2$ and B_X (a, r_2)∩ G is non-empty implies B_X (a, r_1)∩ B_X (a, r_2)∩G is no-empty.

We observe that by definition of T_G,

G is a subset of T_G.

Theorem 3.3 Let X be 2- normed almost linear space and G is a non-empty subset of V_x.Then for each a ϵ T_G we have P_G(a) is non-empty.

Proof: Let g_1 ϵ G and a ϵ T_G.

Let r_1=$\frac{1}{2}$ and r_2 = d(a,G)+ $\frac{1}{2}$

Then we have III x, a-g_1III < $\frac{1}{2}$ + (d(a,G) + $\frac{1}{2}$) and B (a, r_2))∩ G ≠ ϕ.

Since a ϵ T_G we get that B(g_1, r_1) ∩ B(a, r_2))∩ G ≠ ϕ.
Let us choose g_2 ϵ (B(g_1, r_1) ∩ B(a, r_2)) ∩ G
Then lll x, g_1-$g_2$lll< $\frac{1}{2}$ and lll x, a-$g_2$lll<dist(a,G)+ $\frac{1}{2}$
Let r_1=$\frac{1}{2^2}$ and r_2=$dist(a,G) + \frac{1}{2^2}$
Again we have lll x, a-$g_2$lll< $\frac{1}{2^2} + dist(a,G) + \frac{1}{2^2}$ and B (a, r_2) ∩ G ≠ ϕ.

Since a ϵ T_G we have (B(g_2, r_1) ∩ B(a, r_2))∩ G ≠ ϕ.

Choose g_3 ϵ (B(g_2, r_1) ∩ B(a, r_2))∩ G. We get

lllx, g_2-$g_3$lll< $\frac{1}{2^2}$ and lllx, a-$g_3$lll < dist(a,G)+ $\frac{1}{2^2}$.
By continuing the above process at 'n' stages we get
lll x, g_n-g_{n+1}lll< $\frac{1}{2^n}$ and
lll x, a-g_{n+1}lll<dist(a, G)+ $\frac{1}{2^n}$ **3.1**
By eq. (3.1) it follows that $\lim_{n\to\infty}$ lll x, a-g_{n+1}lll= dist(a,G) and $\{g_n\}$ is a Cauchy sequence.
Since G is complete $\{g_n\}$ contains a sub-sequence say $\{g_n'\}$ which converges to g_0 in G.
Now lim lll x, a-g_n' lll = dist(a,G) implies
lll x, a-$g_0$lll = dist(a,G).
Hence g_0 ϵ p_G(a) implies that p_G(a) ≠ ϕ.

Theorem: 3.4 Let X be a 2-normed almost linear spaces and G ⊂ V_x. If V_x *is strictly convex* with respect to G, then for each a ϵR_x(G), the set P_G(a) contains atmost one element. If in addition G is reflexive then for each a ϵR_x(G), the set P_G(a) is singleton.

Proof: Let a ϵR_x(G) and suppose ∃ g_1, g_2 ϵ G such that lll x, a-g_i lll = dist(a,G), i=1,2
Then lll x, a-($\frac{g_1+g_2}{2}$) lll = dist(a,G).
since a ϵR_x(G), for the element ($\frac{g_1+g_2}{2}$) ϵ G,
∃ $v_0 \epsilon V_x$ such that
lll x, a-($\frac{g_1+g_2}{2}$) lll = lll x, v_0-($\frac{g_1+g_2}{2}$) lll and
lll x, a-$g_1$lll≥ lll x, v_0-g_i lll, i=1,2.
Then dist(a,G) = lll x, v_0-($\frac{g_1+g_2}{2}$) lll
≤(lll x, v_0-g_1 lll+ lll x, v_0-g_2 lll)/2
≤ dist(a, G).
And so lll x, v_0-g_1 lll= lll x, v_0-g_2 lll
= lll x, v_0-($\frac{g_1+g_2}{2}$) lll
Since (v_0-g_1)-(v_0-g_2)= $g_2 - g_1$ ϵG and V_x is strictly convex with respect to G it follows that $g_1 = g_2$.

If G is reflexive then by theorem (3.1)
G is proximinal in X.

Definition: 3.5 Let X be a 2-normed almost linear spaces and G is subset of V_x. We shall assign to each aϵR_x(G) a non-empty subset D_G(a) is subset of V_x in the following way

for gϵG , let D_g(a) = {v_g : $v_g \epsilon V_x$ } satisfying (i) and (ii) of definition (2.4).
Since aϵR_x(G), the set D_g(a) is non-empty.

Lemma 3.6 Let X ϵR_x(G), and gϵ G. then for each V_g ϵ D_g(a) we have
lll x, a-glll = lll x, v_g –g lll
=$sup_{b\epsilon D_G(a)}$lll x , b-g lll **3.2**
Consequently, the set D_G(a) is the non empty bounded subset of V_x, which is removable with respect to G. if a ϵV_x, then D_g(a) = {a}.

Proof:- Let a ϵR_x(G), gϵG and v_g ϵ D_g(a)
By (i) of definition (2.4) we have
lll x, a-glll = lll x, v_g-glll .
Let b ϵ D_g(a). By (ii) of definition (2.4) we have
lll x, a-glll ≥ lll x, b-glll.
From this it follows that
lll x, v_g-glll = lll x, a-glll ≥ lll x, b-glll.
Hence equation (3.2) follows since v_g ϵ D_g(a).
Let now a ϵV_x is subset of R_x(G) and $v_0 \epsilon$ D_G(a).
Now by (ii) of definition (2.4) for v = a ϵV_x
we have
0= lll x, a-alll ≥ lll x, v_0-alll
This implies a=v_0. Hence D_G(a)={a}.

Theorem:3.7 Let X be a 2- normed almost linear spaces, ϕ ≠ G_1 ⊂ G ⊂ V_x and let a ϵR_x(G) we have

dist(a,G_1)=rad_{G_1}(D_G(a)) **3.3**
and $P_{G_1}(a) = Cent_{G_1}$(D_G(a)) **3.4**

Proof:- Let gϵG_1, since aϵR_x(G) and G_1 ⊂ G by lemma(3.6)We have
lll x, a-$g_1$lll = $sup_{b\epsilon D_G(a)}$lll x, b-$g_1$lll
Now taking the infimum in both sides over all g_1 ϵ G
We get $inf_{g\epsilon G_1}$ lllx,a-glll = $inf_{g\epsilon G_1}$ $sup_{b\epsilon D_G(a)}$lll b-glll
By definition rad_G (A) we have
rad_G (A) =$inf_{g\epsilon G}$ $sup_{a\epsilon A}$ lll x, a-g lll for every
x ϵ X\V_x **3.5**

By definition dist(a,G)=$inf_{g\epsilon G}$ IIIx,a-gIII **3.6**

Now by equations (3.5 & 3.6)

we get dist(a, G_1) = $rad_{G_1}(D_G$(a)).

Then it follows that $P_{G_1}(a) = Cent_{G_1}(D_G$(a)).

Theorem 3.8 Let G be a one dimensional chebyshev sub-space of V_x . Then P_G(a) is a singleton for each a ϵR_x(G) .

Proof: Clearly G is proximinal in X, since G is one dimensional sub-space of V_x.

Let now a ϵR_x(G) and suppose there exist

$g_1, g_2 \epsilon\ P_G$(a), $g_1 \neq g_2$.

For $(\frac{g_1+g_2}{2}) \epsilon\ G, let\ v_0 \epsilon V_x$ such that

III x, a-$(\frac{g_1+g_2}{2})$ III = III x, v_0-$(\frac{g_1+g_2}{2})$ III

III x, a- v III ≥ III x , v_0-vIII for each $v\ \epsilon\ V_x$

Since $(\frac{g_1+g_2}{2})\ \epsilon\ P_G$(a),

It follows that

dist(a, G) = III x, v_0-$(\frac{g_1+g_2}{2})$ III

≤ III x, v_0-g_1 III+ III x, v_0-g_2 III

≤ (III x, a-g_1 III+ III x, a-g_2 III) /2

= d(a,G).

And so III x, v_0-$(\frac{g_1+g_2}{2})$ III = III x, v_0- g_iIII , i=1,2.

Since dim G = 1, we must have $g_1, g_2 \epsilon\ P_G(v_0)$, a contradiction

Hence $g_1 = g_2$ implies P_G(a) is a singleton.

Theorem 3.9 Let X be a 2-normed almost linear space such that V_x is Banach space and the norm of V_x is uniformly kadec-klee (UKK) and let G ⊂ V_x be a W-compact ,convex set. Then for each a ϵR_x(G) the set P_G(a) is compact and convex.

Proof: Clearly G is proximinal in X.

Let now a ϵR_x(G). If P_G(a) is not compact then ther exist a sequence $\{g_n\} \subset P_G$(a) with

Sep $\{g_n\} \geq \varepsilon$ for some $\varepsilon > 0$.

Since P_G(a) is W-compact, may assume that

$g_n \to$ g $\epsilon\ P_G$(a).

Since a ϵR_x(G), for g ϵ G there exist $v_g\ \epsilon\ V_x$ such that III x, a-g III = III x, v_g -g III and III x, a-g_n III and III x, a-g_n III→ III x, v_g-g_n III ,n=1,2,3,….

Here r = III x, g-v_0 III ≥ $sup_{n\epsilon\ N}$ III x, $g_m - v_g$ III

Choose δ such that δ $(\frac{\mathcal{E}}{r})$ >0 then

r^{-1} III x, $g_n - v_g$ III ≤ 1.

$r^{-1}(g_n - v_g) \to r^{-1}$(g$-v_g$) and

Sep { $r^{-1}(g_n - v_g)$ } ≥ $r^{-1}\mathcal{E}$.

Hence by uniformly kadec-klee (UKK) we obtain that r^{-1} III x, g$-v_g$III ≤ 1-δ

a contradiction. Hence P_G(a) is a compact.

Definition 3.10 Let X be a 2- normed almost linear spaces and ϕ≠ G ⊂ V_x, the pair (a, G) is said to have the property (P) if for every r>0 and any Ɛ>0 here is a δ(Ɛ)>0 and a function f: G X G→G such that for every |θ|< δ(Ɛ) we have f(g_1 ,g_2) $\epsilon\ B_X(g_1$, Ɛ) and $B_X(g_1$,r+δ(Ɛ)) ∩ $B_X(g_2\ , r+\theta$)

⊂ B_X(f(g_1 , g_2), $r+\theta$))

Theorem 3.11 Let X be a 2- normed almost linear spaces and G a complete subset of V_x. If pair(a, G) has the property (P), then G is proximinal in X.

Proof:-

For r=rad_G(A), , Ɛ=$\frac{1}{2}$ find the corresponding δ($\frac{1}{2}$)

Then there is a point $g_1\ \epsilon$ G with A⊂ B(g_1,r+ δ($\frac{1}{2}$))

Assume now that for an n ϵ N, the points

g_1 , g_2 , , , $g_n\ \epsilon$ G

And the number δ($\frac{1}{2}$), δ($\frac{1}{4}$), , , , , , δ($\frac{1}{2^n}$) with the property δ($\frac{1}{2^i}$) ≤ $\frac{1}{2^i}$, A⊂ B(g_1,r+ δ($\frac{1}{2^i}$)) i=1,2,3, , , n.

III x, $g_i - g_{i+1}$ III ≤ $\frac{1}{2^i}$, i=1,2, , , , n-1 have already been constructed.

Now for r and $\frac{1}{2^{n+1}}$ find the corresponding δ($\frac{1}{2^{n+1}}$)

It is easy to see that it is possible to choose δ($\frac{1}{2^{n+1}}$) < min(δ($\frac{1}{2^n}$), $\frac{1}{2^{n+1}}$)

There is a point b ϵ G with A⊂ B(b,r+ δ($\frac{1}{2^{n+1}}$))

Using the fact that (a,G) has the property (P)

we obtain A⊂ B(g_n,r+ δ($\frac{1}{2^n}$)) ∩ B(b,r+ δ($\frac{1}{2^{n+1}}$))

⊂ B(g_{n+1}, r+ δ($\frac{1}{2^{n+1}}$)) where g_{n+1}=f(g_n,b).

Then we get III x, $g_n - g_{n+1}$ III ≤ $\frac{1}{2^n}$

By continue the above process we get a cauchy sequence { g_n } in G.

Now let the above sequence has the limit g_0.

This implies $g_0\ \epsilon\ cent_G$(A).

Hence G is proximinal in X .

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