Average Number of Real Zeros of Random Fractional Polynomial-II

K. Kadambavanam,
PG and Research Department of Mathematics,
Sri Vasavi College,
Erode, Tamilnadu, India.

M. Sudharani,
Department of Mathematics,
Velalar college of Engineering and Technology,
Erode, Tamilnadu, India.

***Abstract* - Let a_0, a_1, ….. be a sequence of independent and identically distributed standard normal random variables. In this paper, the average number of real zeros of the random fractional polynomial $\sum_{k=0}^{n-1} a_k x^{\frac{k}{4}}$ $(0 < x < \infty)$, for large values of n is obtained. Further it is proved that the average number of real zeros EN_n is asymptotic to $\frac{1}{\pi}\log n$.**

Keywords: Random variables, Random fractional polynomial, Normal distribution.

1 INTRODUCTION

Mathematical models are indispensable in many areas of Science and Engineering. For bacterial growth the models usually take the form of differential equations or a system of differential equations.

Differential equations have been extensively studied, both from the analytical and numerical view points. However, there are many situations, where the equations with random coefficients are better suited in describing the behavior of the quantities of interest.

Randomness in the coefficients may arise, because of errors in the observed or measured data, variability in experiment and empirical conditions, uncertainties (variables that cannot be measured, missing data, etc…) or plainly because of lack of knowledge. Differential equations where some or all the coefficients are considered as random variables or that incorporate stochastic effects have been increasingly used in the last few decades to deal with errors and uncertainties and to represent the growing field of great scientific interests. [6, 7, 9].

Fractional calculus is a branch of mathematics that grows out of the traditional definitions of the integral calculus and derivative operators. In the same way fractional exponents is an outgrowth of exponents with integer value [1].

In a letter to L'Hopital in 1695, Leibniz raised the following question [7],

"Can the meaning of derivatives with integer order be generalized to derivatives with non integer orders?" L'Hopital posted the question to Leibniz, what would be the result if the order will be $\frac{1}{2}$ ". Leibniz replied as: 'It will lead to a paradox, from which one day useful consequences will be drawn'.From these words fractional calculus has been initiated. Random fractional algebraic polynomials arise in the study of differential equations with random coefficients.[5,10,11]. Consider the polynomial,

$$a_0 + a_1 x^{\alpha} + a_2 x^{2\alpha} + ...a_{n-1} x^{(n-1)\alpha} \qquad (-\infty < x < \infty). \tag{1.1}$$

If a_k's (k = 0, 1,…, (n-1)) are all independent real random variables, then the above polynomial becomes a random polynomial.

Set, $f(x) = \sum_{k=0}^{n-1} a_k x^{\alpha k}$, $(0 < \alpha < 1)$. The average number of real zeros of this polynomial is denoted by EN_n .In particular if $\alpha = \frac{1}{4}$, the polynomial (1.1) becomes,

$$f(x) = \sum_{k=0}^{n-1} a_k x^{\frac{k}{4}} = a_0 + a_1 x^{\frac{1}{4}} + a_2 x^{\frac{1}{2}} + ... + a_{n-1} x^{\frac{n-1}{4}} \qquad (0 < x < \infty). \tag{1.2}$$

International Journal of Engineering Research & Technology (IJERT)
ISSN: 2278-0181
Vol. 5 Issue 09, September-2016

When $\alpha = 1$, that is for the polynomial $\sum a_k x^k$,where a_k's (k=0,1,…,(n-1)) are independent and standard normal variables, then the number of real zeros, is estimated as $EN_n = \frac{2}{\pi}\log n$ in Kac[3]. This relation is known as Kac's result. When $\alpha = \frac{1}{2}$, the above polynomial (1.1) takes the form $\sum_{k=0}^{n-1} a_k x^{\frac{k}{2}}$.

The average number of real zeros of this fractional polynomial the interval (0,∞) is estimated in [4] as $EN_n = \frac{1}{\pi}\log n$.

Average number of real zeros of various random polynomials is discussed in [2]. In this paper, the fractional polynomial of the form, given in (1.2) has been discussed. The main theorem proved in this article is stated below.

THEOREM (1.1)

If the random variables a_k's (k = 0, 1,…., (n-1)) are all independent standard normal variables ,the average number of real zeros, EN_n of the fractional polynomial (1.2) is given by the following formula,

$$EN_n = \frac{1}{2\pi}\int_0^1 \frac{[1+x^n+2x^{\frac{n}{2}}(n^2-1)-n^2x^{\frac{n+1}{2}}-n^2x^{\frac{n-1}{2}}]^{\frac{1}{2}}}{x^{\frac{3}{4}}(1-x^{\frac{1}{2}})(1-x^{\frac{n}{2}})}\,dx, \tag{1.3}$$

and its asymptotic relation is given by

$$EN_n \sim \frac{1}{\pi}\log n \tag{1.4}$$

and the interval estimation is,

$$\frac{1}{\pi}\log n \le EN_n \le \frac{1}{\pi}\log n + 0.2 \tag{1.5}$$

where EN_n denotes the average number of real zeros of the fractional polynomial (1.2).The following theorem (1.2) is needed to prove the theorem (1.1) completely.

THEOREM (1.2)

For the random fractional polynomial, $\sum_{k=0}^{n-1} a_k x^{\frac{k}{4}} \quad (0 < x < \infty)$, the number of real zeros in (0, 1), and in (1,∞) are equal.

Proof:

Consider the random fractional polynomial,

$$a_0 + a_1x^{\frac{1}{4}} + a_2x^{\frac{1}{2}} + a_3x^{\frac{3}{4}} \ldots + a_{n-1}x^{\frac{n-1}{4}} = 0$$

$$\Rightarrow \quad x^{\frac{n-1}{4}}\left[\frac{a_0}{x^{\frac{n-1}{4}}} + \frac{a_1}{x^{\frac{n-2}{4}}} + \frac{a_2}{x^{\frac{n-3}{4}}} + \ldots + a_{n-1}\right] = 0$$

$$\Rightarrow \frac{1}{y^{\frac{n-1}{4}}}\left[a_0y^{\frac{n-1}{4}} + a_1y^{\frac{n-2}{4}} + a_2y^{\frac{n-3}{4}} + \ldots + a_{n-1}\right] = 0,$$

where $\frac{1}{x} = y$. So, as x varies from 0 to 1, the range of y is from 1 to ∞. Therefore the number of real zeros in the interval (0, 1) is same as that of the interval (1, ∞).

2 PROOF OF THE MAIN THEOREM (1.1)

By Kac's formula [3], for the random polynomial $\sum_{k=0}^{n-1} a_k x^k$,

the number of real zeros in the interval (a,b) is given by the equation,

$$EN_n(a,b) = \frac{1}{\pi}\int_a^b \frac{(AC-B^2)^{\frac{1}{2}}}{A}\, dx\ . \qquad (2.1)$$

In the present case, the fractional polynomial $\sum_{k=0}^{n-1} a_k x^{\frac{k}{4}}$, for $0 < x < \infty$ is considered.

$$\text{Let } \Delta = AC - B^2 \qquad (2.2)$$

where

$$\left.\begin{aligned} A &= \sum_{k=0}^{n-1} x^{\frac{k}{2}} \\ B &= \sum_{k=0}^{n-1} \frac{k}{4}\, x^{\frac{k}{2}-1} \quad \text{and} \\ C &= \sum_{k=0}^{n-1} \frac{k^2}{16}\, x^{\frac{k}{2}-2}. \end{aligned}\right\} \qquad (2.3)$$

Equivalently,

$$\left.\begin{aligned} A &= \frac{1-x^{\frac{n}{2}}}{1-x^{\frac{1}{2}}} \\ B &= \frac{1}{4}\left[\frac{(1-x^{\frac{n}{2}})x^{\frac{1}{2}} - nx^{\frac{n}{2}-1}(1-x^{\frac{1}{2}})}{(1-x^{\frac{1}{2}})^2}\right] \quad \text{and} \\ C &= \frac{1}{16x^{\frac{3}{2}}(1-x^{\frac{1}{2}})^3}\left[(1-x^{\frac{n}{2}})(1+x^{\frac{1}{2}}) - n^2 x^{\frac{n-1}{2}}(1-x^{\frac{1}{2}})^2 - 2nx^{\frac{n}{2}}(1-x^{\frac{1}{2}})\right]. \end{aligned}\right\} \qquad (2.4)$$

Substituting the values of A, B and C, given in the system of equations (2.4) in the equation (2.2) yields,

$$\Delta = \frac{1 + x^n + 2x^{\frac{n}{2}}(n^2-1) - n^2 x^{\frac{n-1}{2}} - n^2 x^{\frac{n+1}{2}}}{16x^{\frac{3}{2}}(1-x^{\frac{1}{2}})^4} \qquad (2.5)$$

and by an analytical computation in the equation (2.1), gives

$$EN_n(0,1) = \frac{1}{4\pi}\int_0^{\infty} \frac{[1 + x^n + 2x^{\frac{n}{2}}(n^2-1) - n^2 x^{\frac{n-1}{2}} - n^2 x^{\frac{n+1}{2}}]^{\frac{1}{2}}}{x^{\frac{3}{4}}(1-x^{\frac{1}{2}})(1-x^{\frac{n}{2}})}\, dx. \qquad (2.6)$$

Employing the theorem (1.2), $EN_n(0,\infty)$ is of the form

$$EN_n(0,\infty) = \frac{1}{2\pi}\int_0^{1} \frac{[1 + x^n + 2x^{\frac{n}{2}}(n^2-1) - n^2 x^{\frac{n-1}{2}} - n^2 x^{\frac{n+1}{2}}]^{\frac{1}{2}}}{x^{\frac{3}{4}}(1-x^{\frac{1}{2}})(1-x^{\frac{n}{2}})}\, dx. \qquad (2.7)$$

3 ASYMPTOTIC VALUE OF EN_n

Consider

$$EN_n(0,\infty) = \frac{1}{2\pi}\int_0^1 \frac{[1-h_n^2(x)]^{\frac{1}{2}}}{x^{\frac{3}{4}}(1-x^{\frac{1}{2}})}\,dx, \tag{3.1}$$

where
$$h_n(x) = \frac{nx^{\frac{n-1}{4}} - nx^{\frac{n+1}{4}}}{1-x^{\frac{n}{2}}}. \tag{3.2}$$

Then
$$h_n(x) > x^{\frac{n-1}{4}},$$

$$1-h_n^2(x) < (1+x^{\frac{n-1}{4}})(1-x^{\frac{n-1}{4}}), \text{ and}$$

$$1-h_n^2(x) \le (1-x^{\frac{n-1}{4}}). \tag{3.3}$$

Suppose f(x) is differentiable on (0, 1), applying the mean value theorem for f(x) on (x,1) gives, $f(1)-f(x) = f'(\theta)(1-x)$ for $x,\theta \in (0,1)$ (3.4)

Set $f(x) = 1 - x^{\frac{n-1}{4}}$,

$$\Rightarrow 1-h_n^2(x) = (1-x)\left[\frac{n-1}{4}\theta^{\frac{n-5}{4}}\right], \quad \text{for } x<\theta<1.$$

$$\Rightarrow 1-h_n^2(x) = (1-x)\left(\frac{n-1}{4}\right), \quad \text{as } \theta \to 1.$$

$$\Rightarrow 1-h_n^2(x) < (1-x^{\frac{1}{2}})\left(\frac{n-1}{4}\right), \quad \text{since } x < x^{\frac{1}{2}}, \text{ for } x \in [0,1].$$

$$\Rightarrow \frac{[1-h_n^2(x)]^{\frac{1}{2}}}{(1-x^{\frac{1}{2}})x^{\frac{3}{4}}} < \frac{\left(\frac{n-1}{4}\right)^{\frac{1}{2}}}{x^{\frac{3}{4}}(1-x^{\frac{1}{2}})^{\frac{1}{2}}}, \quad \text{for } x \in [0,1]. \tag{3.5}$$

On the other hand, $h_n(x) \to 0$, as $n\to\infty$, so $h_n^2(x) \to 0$, as $n\to\infty$.

Then
$$1-h_n^2(x) \le 1$$

$$\Rightarrow [1-h_n^2(x)]^{\frac{1}{2}} \le 1$$

$$\Rightarrow \frac{[1-h_n^2(x)]^{\frac{1}{2}}}{(1-x^{\frac{1}{2}})x^{\frac{3}{4}}} \le \frac{1}{(1-x^{\frac{1}{2}})x^{\frac{3}{4}}}, \quad \text{for } x \in [0, 1], \tag{3.6}$$

and

$$\int_0^1 \frac{[1-h_n^2(x)]^{\frac{1}{2}}}{x^{\frac{3}{4}}(1-x^{\frac{1}{2}})}\,dx < \int_0^{(1-\frac{1}{n})^4} \frac{dx}{x^{\frac{3}{4}}(1-x^{\frac{1}{2}})} + \int_{(1-\frac{1}{n})^4}^1 \frac{\left(\frac{n-1}{4}\right)^{\frac{1}{2}}}{x^{\frac{3}{4}}(1-x^{\frac{1}{2}})^{\frac{1}{2}}}\,dx. \tag{3.7}$$

Evaluation of the first integral in the right hand side of (3.7) yields,

International Journal of Engineering Research & Technology (IJERT)
ISSN: 2278-0181
Vol. 5 Issue 09, September-2016

$$\int_0^{(1-\frac{1}{n})^4} \frac{dx}{x^{\frac{3}{4}}(1-x^{\frac{1}{2}})} = 2[\log(2-\frac{1}{n})+\log n]. \qquad (3.8)$$

The second integral in the right hand side of (3.7) tends to zero, when $n \to \infty$.
Then the inequation (3.7) becomes,

$$\int_0^1 \frac{[1-h_n^2(x)]^{\frac{1}{2}}}{x^{\frac{3}{4}}(1-x^{\frac{1}{2}})} dx < 2[\log(2-\frac{1}{n})+\log n]. \qquad (3.9)$$

From the equation (3.1),

$$EN_n(0,\infty) < \frac{1}{\pi}\log n + \frac{1}{\pi}\log(2-\frac{1}{n}). \qquad (3.10)$$

For large values of n,

$$EN_n(0,\infty) < \frac{1}{\pi}\log n + 0.2. \qquad (3.11)$$

To obtain the lower estimate, let ε and δ be two real quantities with $0 < (\varepsilon, \delta) < 1$,

$h_n^2(x) \to 0$, when $n \to \infty$.

So, for sufficiently large n, $h_n^2(x) < \varepsilon$ (3.12)

$$\int_0^1 \frac{[1-h_n^2(x)]^{\frac{1}{2}}}{x^{\frac{3}{4}}(1-x^{\frac{1}{2}})} dx > \int_0^{(1-n^{\delta-1})^4} \frac{[1-h_n^2(x)]^{\frac{1}{2}}}{x^{\frac{3}{4}}(1-x^{\frac{1}{2}})} dx. \qquad (3.13)$$

$$\Rightarrow h_n(x) < nx^{\frac{n-1}{4}} \le n[1-n^{\delta-1}]^{\frac{n-1}{4}}, \quad \text{for } x \in [0,(1-n^{\delta-1})^4].$$

Using the relation (3.12),

$$\int_0^1 \frac{[1-h_n^2(x)]^{\frac{1}{2}}}{x^{\frac{3}{4}}(1-x^{\frac{1}{2}})} dx > \int_0^{(1-n^{\delta-1})^4} \frac{[1-h_n^2(x)]^{\frac{1}{2}}}{x^{\frac{3}{4}}(1-x^{\frac{1}{2}})} dx > \int_0^{(1-n^{\delta-1})^4} \frac{(1-\varepsilon)^{\frac{1}{2}}}{(1-x^{\frac{1}{2}})x^{\frac{3}{4}}} dx. \qquad (3.14)$$

Making suitable transformation,

$$\int_0^1 \frac{[1-h_n^2(x)]^{\frac{1}{2}}}{x^{\frac{3}{4}}(1-x^{\frac{1}{2}})} dx > 2(1-\varepsilon)^{\frac{1}{2}}(1-\delta)\log n. \qquad (3.15)$$

That is,

$$EN_n(0,\infty) > \frac{1}{\pi}(1-\varepsilon)^{\frac{1}{2}}(1-\delta)\log n. \qquad (3.16)$$

From the inequations (3.11) and (3.16) the asymptotic formula

$$EN_n(0,\infty) \sim \frac{1}{\pi}\log n \qquad (3.17)$$

is obtained.

Equation (3.17) represents the asymptotic value of the real zeros of the random fractional polynomial for $0 < x < \infty$. This proves the theorem 1.1

4 A GENERALISATION OF THE RESULTS ON RANDOM AND FRACTIONAL POLYNOMIALS

In this section, a generalisation of the Kac's formula [3], and the theorem (1.1) on random fractional polynomial is proved.

Consider the algebraic equation,

$$a_0 + a_1 x^{\alpha} + a_2 x^{2\alpha} + \dots a_{n-1} x^{(n-1)\alpha} = 0, \qquad (-\infty < x < \infty) \tag{4.1}$$

where the a_k's (k = 0,1,…(n-1)) are independent random variables assuming real values only.

Then $$f(x) = \sum_{k=0}^{n-1} a_k x^{\alpha k} , \qquad (0 < \alpha < 1).$$

By Kac's formula [3],

$$EN_n(a,b) = \frac{1}{\pi} \int_a^b \frac{(AC - B^2)^{\frac{1}{2}}}{A} dx \tag{4.2}$$

Let $$\Delta = AC - B^2 \tag{4.3}$$

where,

$$A = \mathrm{var}[f(x)] \tag{4.4}$$

$$B = \mathrm{cov}[f(x) f'(x)] \tag{4.5}$$

$$C = Var[f'(x)]. \tag{4.6}$$

Then from [3],

$$\left.\begin{aligned} A &= \sum_{k=0}^{n-1} x^{2k\alpha} \\ B &= \sum_{k=0}^{n-1} k\alpha x^{2k\alpha-1} \quad \text{and} \\ C &= \sum_{k=0}^{n-1} k^2 \alpha^2 x^{2k\alpha-2} \end{aligned}\right\} \tag{4.7}$$

Equivalently,

$$\left.\begin{aligned} A &= \frac{1 - x^{2n\alpha}}{1 - x^{2\alpha}} \\ B &= \left[\frac{\alpha x^{2\alpha-1} - \alpha x^{2\alpha(n+1)-1} - n\alpha x^{2n\alpha-1}(1 - x^{2\alpha})}{(1 - x^{2\alpha})^2} \right] \quad \text{and} \\ C &= \frac{\left[\alpha^2 x^{2\alpha-2}(1 - x^{2n\alpha})(1 + x^{2\alpha}) - n^2\alpha^2 x^{2n\alpha-2}(1 - x^{2\alpha})^2 - 2n\alpha^2 x^{2n\alpha+2\alpha-2}(1 - x^{2\alpha}) \right]}{(1 - x^{2\alpha})^3}. \end{aligned}\right\} \tag{4.8}$$

Substituting the values of A, B and C, given in the system of equations (4.8) in the equation (4.3) yields,

$$\Delta = \frac{\alpha^2 x^{2\alpha-2} - n^2\alpha^2 x^{2n\alpha-2} + 2\alpha^2 x^{2n\alpha+2\alpha-2} - n^2\alpha^2 x^{2n\alpha+4\alpha-2} + \alpha^2 x^{4n\alpha+2\alpha-2}}{(1 - x^{2\alpha})^4}. \tag{4.9}$$

Substituting the value of Δ obtained in the equation (4.9) to the equation (4.2) yields,

$$EN_n(-\infty, \infty) =$$

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(\alpha^2 x^{2\alpha-2} - n^2\alpha^2 x^{2n\alpha-2} + 2\alpha^2 x^{2n\alpha+2\alpha-2}(n^2 - 1) - n^2\alpha^2 x^{2n\alpha+4\alpha-2} + \alpha^2 x^{4n\alpha+2\alpha-2})^{\frac{1}{2}}}{(1 - x^{2\alpha})(1 - x^{2n\alpha})} \, dx. \tag{4.10}$$

When α =1, equation (4.10) coincides the relation derived in Kac's formula [3].

$$EN_n = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(1 + x^{4n} + 2x^{2n}(n^2 - 1) - n^2 x^{2n-2} - n^2 x^{2n+2})^{\frac{1}{2}}}{(1 - x^2)(1 - x^{2n})} \, dx. \tag{4.11}$$

That is for the polynomial $\sum a_k x^k$,where a_k's (k = 0,1,…,(n-1)) are independent and normally distributed random variables with mean 0 and variance 1, Kac[3] estimated the average number of real zeros in $(-\infty,\infty)$ as $\frac{2}{\pi}\log n.$

But when $\alpha = \frac{1}{2}$, the equation (4.10) becomes,

$$EN_n(-\infty,\infty) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{(1+x^{2n}+2x^n(n^2-1)-n^2x^{n-1}-n^2x^{n+1})^{\frac{1}{2}}}{x^{\frac{1}{2}}(1-x)(1-x^n)}$$

This result coincides with the result derived in [4].

REFERENCES

[1] Adam Loverro, 2004, 'Fractional Calculus: History, Definitions and Applications For the Engineer'.
[2] Bharucha-Reid A.T. and Sambandham M, 1986, 'Random Polynomials', Academic Press, Orlando,
[3] Kac.M ,1943, 'On the Average Number Of Real Zeros Of a Random Algebraic Equation', Bull.Ame.math.Soc, 49,(314-320).
[4] Kadambavanam K, Sudharani M, Oct 2014, 'Average Number Of Real Zeros Of Random Fractional Polynomial', Journal of Global Research in Mathematical Archives,
[5] Vol 2,No 10,(67-75)
[6] A.A.Kilbas, H.M.Srivastava, and J.J. Trujillo Theory and Applications of Fractional Differential Equations, North-Holland Mathematics Studies, 2006
[7] Neckel Tobias and Florian Rupp,2013,'Random Differential Equations in Scientific Computing',De Gruyter Open,Warsaw,Poland.
[8] Oksendal B, 2003, 'Stochastic Differential Equations',Sixth Ed.,
[9] Springer -Verlag, Heidelberg.
[10] Podlubny I, 1999, 'Fractional Differential Equations', 'Mathematics in Science And Engineering V198',Academic Press.
[11] Soong T, 1992. 'Probabilistic Modeling and Analysis in Science And Engineering',Wiley Newyork.
[12] Tomas Kisela, Fractional Differential Equations and Their Applications, Diploma Thesis,
[13] Brno University of Technology, Brno 2008
[14] Yong Zhou, Basic Theory of Fractional Differential Equations, World Science Publications, 2014