Asymptotic Solutions and Range of Validity for Heat Transfer in Packed-Bed and Capillary Tubes

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Abstract:- In this paper, an asymptotic analytical solution was used to evaluate convective-diffusive heat transfer in packed-bed and capillary tubes. Asymptotic solution to the heat transfer problems found in packed-bed systems with axial convective, radial diffusive with constant heat flux at the wall, and in capillary tubes were studied and analyzed from the results of the analysis reported by Bird et al. [1] and Arce and Trigatti, [2]. The role of asymptotic solution and its ranges of the validity of both for the packed-bed and capillary tubes were assessed and ranges of validity reported.

Keywords: Asymptotic solution; heat transfer; convection; capillary tube; packed-bed tube.

I INTRODUCTION

Capillary and packed-bed tubes are used in various industrial applications and are important to the practical aspects of engineering processes such as reactors, heaters, evaporators, thermal, and energy storage units [1, 2]. Analysis of capillary tubes with laminar flow regime and packed-bed tubes for Newtonian fluids under incompressible conditions was presented with focus on assessing the range of validity of an asymptotic solution for the heat transfer problem. A restriction on the asymptotic solutions is based on the fact that convective-diffusive transfer within the domain is controlled by *a constant rate* at the capillary walls. The assessment of the asymptotic solution is very important in the analysis of the design of reactors or heat exchangers [3]. Several studies have been done to correlate heat transfer by forced convection from circular tubes [4]. However, it appears that the validity ranges are not reported in the literature.

II MODEL FORMULATION

A General Assumptions for the Mathematical Model

The geometry of the capillary and packed-bed tubes are assumed to be cylindrical with a length that satisfies the long channel assumption (R/L<< 1), see figures 1 and 2. The energy equation for capillary cylindrical coordinates is given by Bird et al. [1] and was described in section B. The fluid is assumed to be a Newtonian fluid and the flow is assumed to be pressure-driven, fully developed flow, and incompressible. Convection is assumed to dominate in the axial direction while diffusion dominates the transport in the radial direction of the flow [5]. The fluid is assumed to exchange heat at a constant rate without a volumetric source present within the bulk domain.

B Mathematical Model Equation for Capillary Tube

The basic equations for the system under study and the boundary conditions were properly derived from the general heat transfer equation given by Bird et al. [1]. For the capillary tube, the mathematical model for the steady state convective diffusion equation under the assumptions stated is given by

$$pC_pV_z(r)\frac{\partial T}{\partial z} = k\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)\right] \quad (1)$$

Where

$$V_z(r) = V_{\max}\left[1-\left(\frac{r}{R}\right)^2\right] \quad (2)$$

The boundary conditions for equation (2) and the parameters are introduced as follows:

at r = 0, T = finite or, alternatively, $\frac{\partial \mathrm{T}}{\partial r} = 0$ (symmetry) (3)

at r = R, $k\frac{\partial T}{\partial r} = q_0$ (constant) (4)

at z = 0, T = T_1 (5)

By defining the following dimensionless variables [1]

$$\Theta = \frac{T-T_1}{q_0R/k}\;;\quad \rho = \frac{r}{R}\;;\quad Z = \frac{z}{\rho c_p v_{z,\max}R^2/k} \quad (6)$$

The capillary equation of heat transfer then reduces to

$$(1-\rho^2)\frac{\partial\Theta}{\partial Z}=\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Theta}{\partial\rho}\right) \qquad (7)$$

The boundary conditions in equations (1) - (5) in non-dimensional terms are

$$\text{at } \rho=0,\ \Theta=\ finite \qquad (8)$$

$$\text{at } \rho=1,\ \frac{\partial\Theta}{\partial\rho}=1 \qquad (9)$$

$$\text{at } \rho=0,\ \Theta=0 \qquad (10)$$

An asymptotic solution for this model was explored in the sections below.

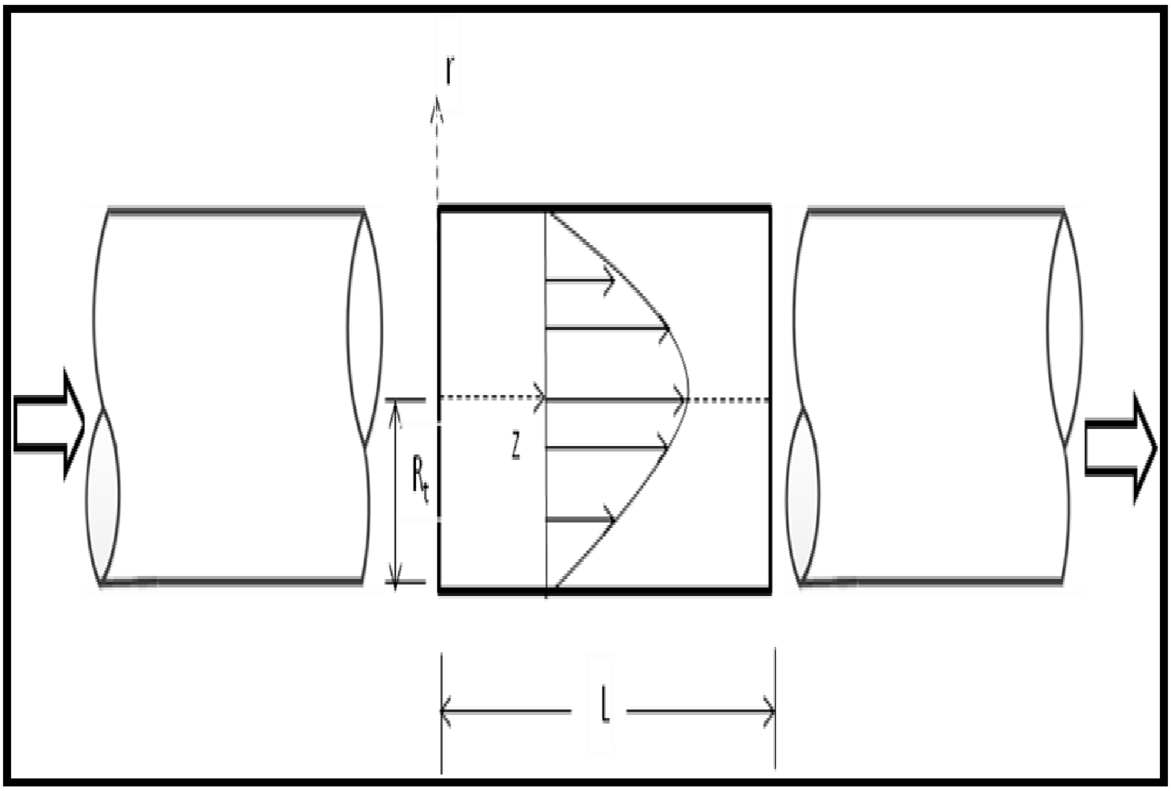

Figure 1: System description of a capillary tube.

C Mathematical Model Equation for Packed-Bed Tube

The convective-diffusive transport equation for the case of a Poiseuille flow in a packed-bed tube as reported by Arce and Trigatti [2] is given by

$$p_m c_{pm} v\frac{\partial T}{\partial z}=\frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) \qquad (11)$$

Where *v* is the area-averaged velocity of the fluid moving through the packed-bed, and with the same assumptions stated above, the wall of the packed-bed tube at R_t shows that the system (see figure 1) either gains or losses heat at a constant rate q_0 and at the center of the tube, symmetry is assumed by Arce and Trigatti [2]. The boundary conditions for equation 11 can be stated as

$$\left.\frac{\partial T}{\partial r}\right|_{r=0} = 0\ ; \quad -k\left.\frac{\partial T}{\partial r}\right|_{r=R_c} = q_0\ ; \text{ and } T(x=0) = T_0 \tag{12}$$

And if the dimensionless variables are given as

$$\rho = \frac{r}{R};\ Z = \frac{z}{\rho_m c_{pm} v R^2 / k}\ ; \text{ and } \ \theta = \frac{(T - T_0)}{\left(q_0 \dfrac{R}{k}\right)} \tag{13}$$

Then, by using eq. (12) and eq. (13), the following non-dimensionless forms are as follows

$$\frac{\partial \theta}{\partial Z} = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial \theta}{\partial \rho}\right) \tag{14}$$

And the boundaries and entrance conditions are given by

$$\left.\frac{\partial \theta}{\partial \rho}\right|_{\rho=0} = 0; \quad \left.\frac{\partial \theta}{\partial \rho}\right|_{\rho=1} = -1; \quad \text{and} \quad \theta(Z=0) = 0 \tag{15}$$

An asymptotic solution for this differential model was obtained below.

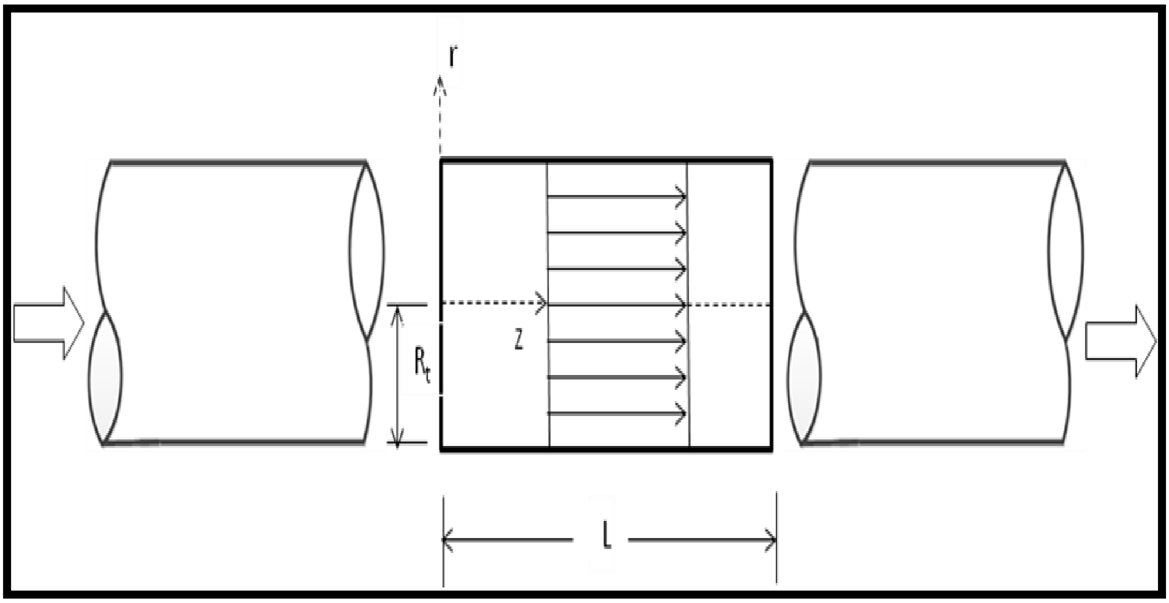

Figure 2: System description of a packed-bed tube.

III ASYMPTOTIC SOLUTIONS

A Capillary Tube Model Equation

For distances that are away (z > 0) from the entrance of the capillary tube, the temperature profile can be approximated by a linear relation in the z-direction as reported by Bird et al. [1]. This assumption yields the mathematical formulation of the asymptotic solution and in terms of the nomenclature used above as follows

$$\Theta^{\infty}(\rho, Z) = C_0 Z + \psi(\rho) \tag{16}$$

The fact that this solution is valid for case z > 0 implies that the condition at z = 0 needs to be replaced by a "global balance" as stated by Bird et al. [1]. This is the reason to use $\Theta^{\infty}(\rho, Z)$ instead of simply $\Theta(\rho, Z)$.

$$2\pi R z q_0 = \int_0^{2\pi}\int_0^{R} pC_p (T - T_1) v_z\, r dr d\theta \tag{17}$$

Now, in dimensionless form, this equation leads to

$$Z = \int_0^1 \Theta^{\infty}(\rho, Z)(1 - \rho^2)\rho d\rho \tag{18}$$

As stated by the Bird et al. [1] function $\psi(\rho)$ in equation (16) must satisfy

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{d\psi}{d\rho}\right)=C_o\left(1-\rho^2\right) \quad (19)$$

And by applying the boundary conditions

at $\rho=0$, ψ = finite and (20)

at $\rho=1$, $\frac{d\psi}{dr}=1$ (21)

By solving equation (19), the following result is obtained

$$\psi(\rho)=C_0\left(\frac{\rho^2}{4}-\frac{\rho^4}{16}\right)+C_1\ln\rho+C_2 \quad (22)$$

Where C_1 and C_2 are constants of integration that by applying the boundary condition given by equations (20) and (21) can be determined as

$$C_1=0 \quad (23)$$

$$C_2=-\frac{7}{24} \quad (24)$$

The equation (22) leads to the following general solution for $\psi(\rho)$

$$\psi(\rho)=C_0\left(\frac{\rho^2}{4}-\frac{\rho^4}{16}\right)-\frac{7}{24} \quad (25)$$

Substituting equation (25) into Equation (16) and by applying the asymptotic solution, then the temperature profile is given by

$$\Theta^\infty(\rho,Z)=C_0Z+C_0\left(\frac{\rho^2}{4}-\frac{\rho^4}{16}\right)+C_1\ln\rho+C_2 \quad (26)$$

Or, alternatively,

$$\Theta^\infty(\rho,Z)=C_0Z+C_0\left(\frac{\rho^2}{4}-\frac{\rho^4}{16}\right)-\frac{7}{24} \quad (27)$$

Constant C_0 can now be obtained to yield

$$C_0=4 \quad (28)$$

Therefore, the final solution for the asymptotic solution for the cylindrical tube is obtained (see Figure 3) and given by

$$\Theta^\infty(\rho,Z)=4Z+\rho^2-\frac{1}{4}\rho^4-\frac{7}{24} \quad (29)$$

B Packed-Bed Tube Model Equation

Similarly to the case of a capillary tube, the asymptotic solution for the Poiseuille flow with convective-diffusive heat transfer in a packed-bed tube can be written as

$$\theta^\infty(\rho,Z)=C_0Z+\psi(\rho) \quad (30)$$

In order to determine function $\psi(\rho)$, this must satisfy the following differential equation

$$\frac{d}{d\rho}\left(\rho\frac{d\psi}{d\rho}\right)=c_0\rho \quad (31)$$

As well as the boundary conditions

$$\left.\frac{d\psi}{d\rho}\right|_{\rho=0}=0; \quad \text{and} \quad \left.\frac{d\psi}{d\rho}\right|_{\rho=1}=-1 \quad (32)$$

The solution is reported by Arce and Trigatti [2] and only a summary of the key steps for the convenience of the reader was done. As before, see Equation (17), the condition at the inlet of the packed-bed must be replaced by an integral condition valid *also* for the function $\theta^\infty(\rho,Z)$. This equation can be derived using an integral energy balance in the system.

[Energy due to transport by convection]

$$\equiv\int_0^{2\Pi}\int_0^R rdrd\theta p_m c_{pm}V(T-T_0) \quad (33)$$

[Energy loss at the wall of the tube] $\equiv -w\pi Rxq_0$ (34)

Now, by using Equation (33) and Equation (34) and invoking the energy balance in the system gives

$$-Z=\int_0^1 d\rho'\rho'\theta(\rho',Z) \quad (35)$$

After using the non-dimensional variables, the solution for $\psi(\rho)$ leads to

$$\psi(\rho)=\left(\frac{C_o}{4}\right)\rho^2+C_1\ln\rho+C_2 \quad (36)$$

And after using the boundary conditions in Equation (32), function $\psi(\rho)$ is given by

$$\psi(\rho)=\frac{1}{4}\left(1-2\rho^2\right) \quad (37)$$

The constant C_0 can be obtained from Equation (35) to give

$$C_0 = 2 \quad (38)$$

Finally, the asymptotic function for $\theta^{\infty} = \theta^{\infty}(\rho, z)$ is given by

$$\theta^{\infty}(\rho, \mathrm{Z}) = -2\mathrm{Z} + \frac{1}{4}\left(1-2\rho^2\right) \quad (39)$$

For the case of the packed-bed tube in cylindrical coordinates is shown in Figure 4.

IV RANGE OF VALIDITY OF ASYMPTOTIC SOLUTION

A Case 1- Capillary Tube

The ranges of validity of the asymptotic solution in a capillary tube can be obtained using Equation (29) and after imposing the condition $\Theta^{\infty}(\rho, Z) = 0$, the following equation is obtained

$$Z^* = \left(-\frac{\rho^{*2}}{4} + \frac{1}{16}\rho^{*4} + \frac{7}{96}\right) \text{ for } 0 \le \rho \le 1 \quad (40)$$

From the physical point of view, this system describes a heating fluid under Poiseuille flow conditions. This implies that according to the definition of Θ^{∞} used in here, the proper domain for this solution should be $\Theta^{\infty}>0$. Now, values that are located within the domain of $\Theta^{\infty}<0$ (see Figure 3) are not physically meaningful. Therefore, the boundary between the two domains for Θ^{∞} can be calculated from the condition $\Theta^{\infty} = 0$ or alternatively

$$Z^* = \frac{1}{4}\left(-\rho^{*2} + \frac{1}{16}\rho^{*4} + \frac{7}{96}\right) \quad (41)$$

The plot of this boundary is shown in Figure 3 and it is observed that the upper region for the boundary between the two domains for Θ^{∞} is the physical meaningful domain. From Figure 3, this is clearly when $Z \ge 0.073$ and $\rho \in (0 \le \rho \le 1)$. In other words, for all values $\rho \in (0 \le \rho \le 1)$, the smallest value of Z, when the solution Θ^{∞}, is valid is $Z = 0.073$. Moreover, if values of Z are such that $Z \in 0 \le Z \le Z = 0.073$, the solution is only valid for $\rho \in (\rho^* \le \rho \le 1)$; *where* ρ^* = 0.29. The reason behind this condition as indicated above is the fact that $\Theta^{\infty}(\rho, Z)$ changes sign within the domain of validity. This change of sign is a result that the asymptotic solution is no longer valid; therefore, the condition determines the values from which $\Theta^{\infty}(\rho, Z)$ yields physically meaningful results.

B Case 2- Packed-Bed Tube

For the packed-bed tube, the system is assumed to be cooling. From the physical point of view, under Poiseuille flow conditions and in order to establish the ranges of validity for the packed-bed tube, the function $\theta^{\infty}(\rho, \mathrm{Z})$, it is seen that when Z takes smaller values, the function $\theta^{\infty}(\rho, \mathrm{Z})$ tends to cross the axis $\theta^{\infty}(\rho, \mathrm{Z}) = 0$ as it should be since the validity of $\theta^{\infty}(\rho, \mathrm{Z})$ is for $\mathrm{Z} \rightarrow$ large. A condition for the validity can be obtained by simply stating that

$$\theta^{\infty}(\rho, \mathrm{Z}) = 0 \quad (42)$$

With this, Equation (39) can be written as

$$Z^* = \frac{1}{8}\left(1-2\rho^{*2}\right) \quad (43)$$

This equation is used to plot the graph as seen in Figure 4. From this figure 4, it is possible to see the following: first for $\rho \in 0 \le \rho \le 1$ and $Z \ge 0.125$, the solution $\theta^{\infty} < 0$ and belongs to a physically meaningful domain. However, for values of Z < 0.125, the asymptotic solution is still valid (partially) for values of $\rho \in 0.5 \le \rho \le 1$. In other words, if the full domain of ρ $(0 \le \rho \le 1)$ is desired to be considered, the Z is restricted to Z > 0.125 in order for θ^{∞} to be within the range of validity. On the other hand, if the full range of Z $(0 \le Z \le \infty)$ is considered, then the partial values of $\rho(0.5 \le \rho \le 1)$ must be used in order for θ^{∞} to be within the region of validity.

V NUMERICAL ILLUSTRATIONS

Based on the prediction suggested by the validity of the capillary tube, Figure 5 is presented. All values of Z>0.073 (i.e., Z=1, Z= 0.075, Z=0.5) comply with the solution $\theta^{\infty}(\rho, \mathrm{Z})$ being within the physically meaningful domain. For example, for Z= 0.25, Z= 0.1, Z= 0.05 and Z= 0.01, the values of $\theta^{\infty}(\rho, Z)$ fall outside the region of physically meaningful domain. Similarly, Figure 6 illustrates the results for the case of the packed-bed tube- for example, Z=0.25, Z=0.5, Z=0.75, and Z=1 all have values of $\theta^{\infty}(\rho, Z)$ being inside the physically meaningful domain for this case. However, the values for

Z=0.1, Z=0.05, and Z=0.01 do not belong to this region. As a final remark, the results of these figures effectively illustrate the useful predictions of the ranges of validity.

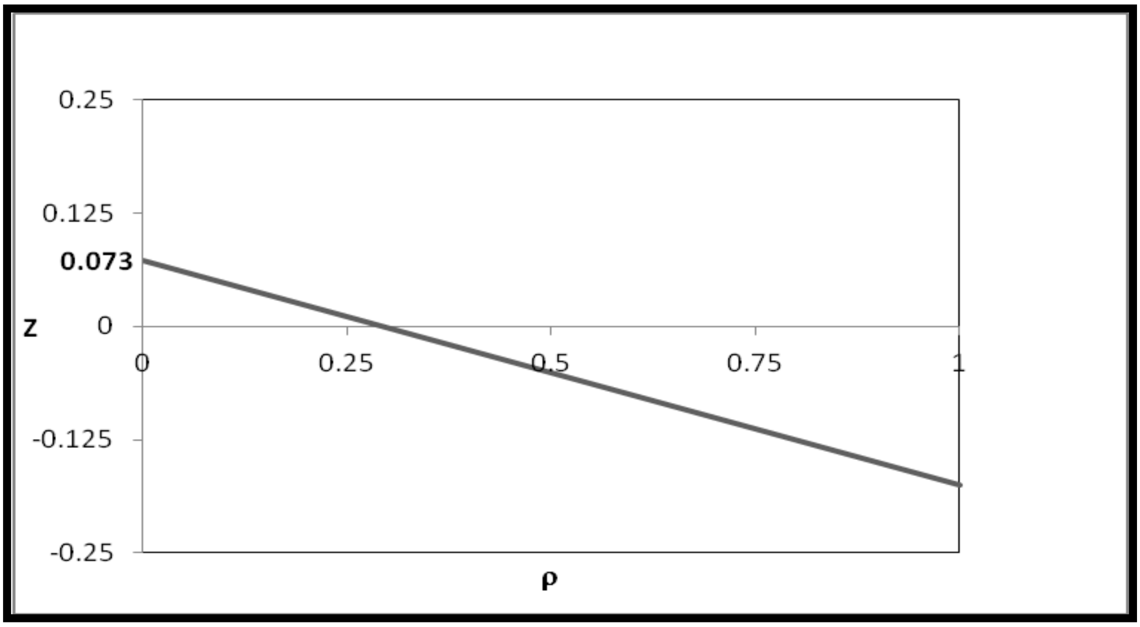

Figure 3: Sketch of the range of validity of asymptotic solution in a cylindrical capillary tube.

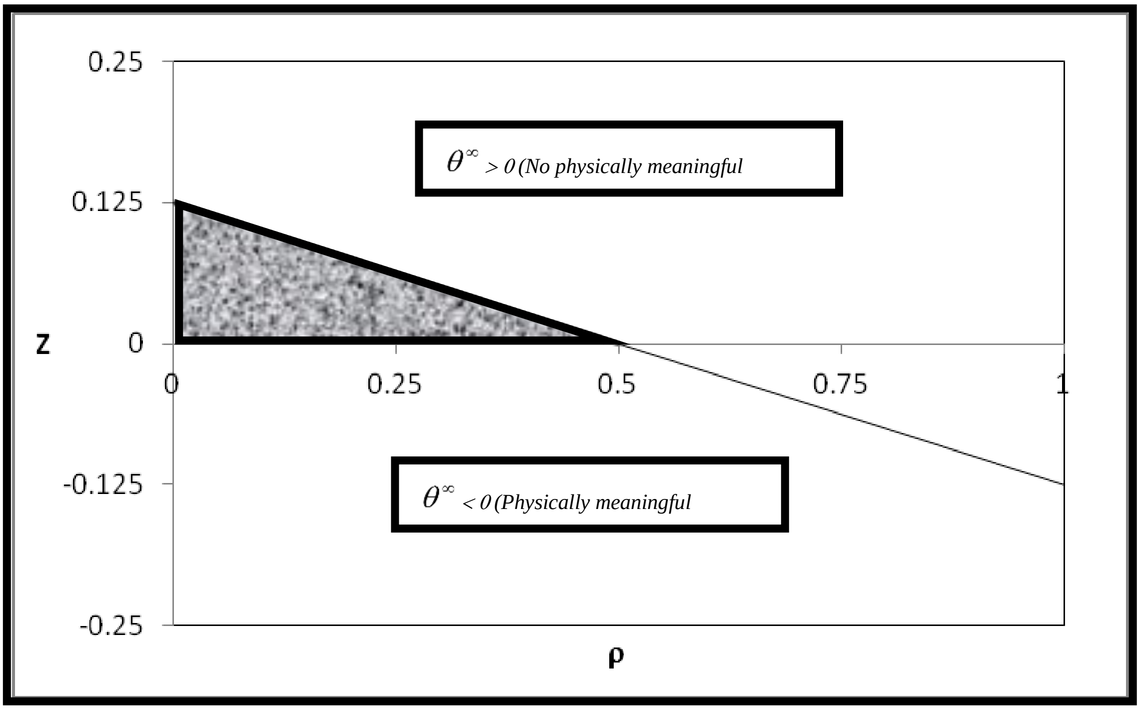

Figure 4: Sketch of the range of validity of asymptotic solution in a packed-bed tube.

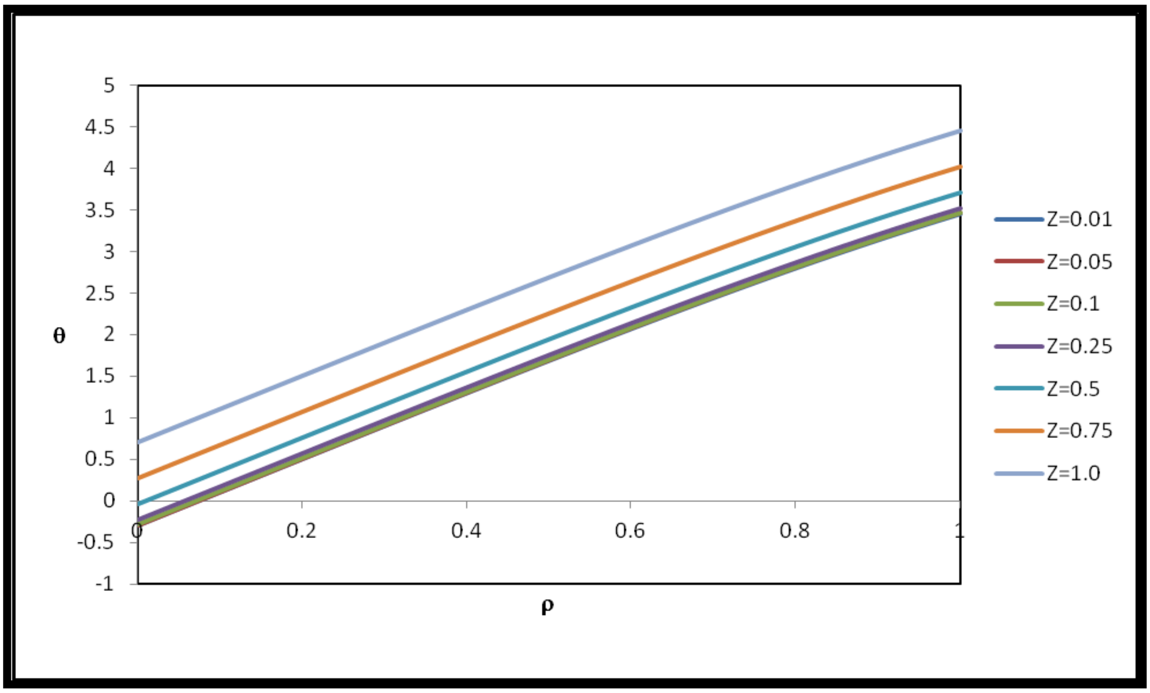

Figure 5: Heating process in a capillary tube.

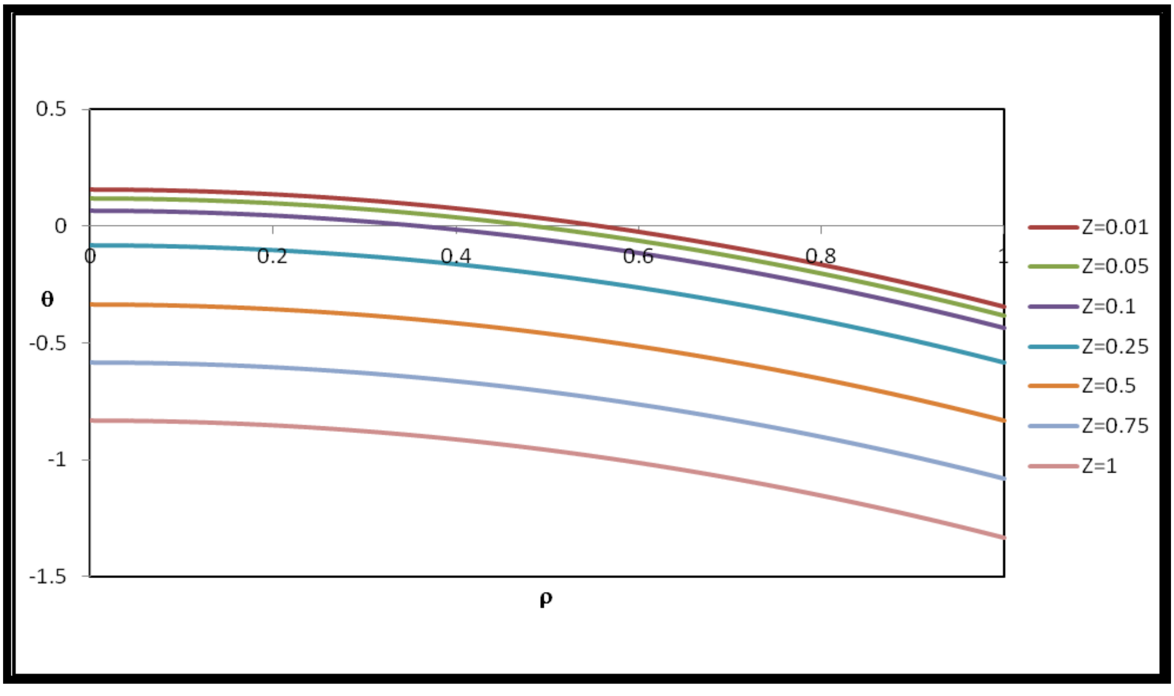

Figure 6: Cooling process in a packed-bed tube.

VI SUMMARY AND CONCLUDING REMARKS

Differential models for two cases of heat transfer (i.e. a capillary system under *heating conditions* and a packed-bed system under *cooling conditions*) were developed and their boundary conditioned formulated. Asymptotic solutions for both models were proposed by following the suggestions of Bird et al. [1] that are consistent with a case of constant rate of cooling (or heating) at the external walls [6]. Moreover, to the best of the researcher's knowledge, the ranges of validity of both asymptotic solutions were determined and illustrations of the behavior of the systems (within the ranges) were presented. The analysis suggests that for these cases, the asymptotic solutions present limitations for *both independent variables* as opposed to only one as suggested before. Finally, these solutions and the range of validity are useful to improve the understanding of the behavior of the systems described.

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REFERENCES

[1] R. B., Bird, W. E. Stewart, and E. N. Lightfoot, Transport Phenomena, 2nd ed., Wiley, New York, 2002, pp. 311-316.
[2] P. E. Arce and I. M. B. Trigatti, "Convective-diffusive heat transfer in packed-bed with constant heat flux at the wall: An asymptotic solution for $\xi \rightarrow$ large," *International Communications in Heat and Mass Transfer*. 21(3), 1994: 435-446.
[3] M. Mansour, A. Sheboul, and A. N. Hussein, "An analytical solution for diffusion and reaction in a laminar flow tubular reactor," *International Communications in Heat and Mass Transfer*. 16, 1989: 603-608.
[4] S. M. Ross, "A mathematical model of mass transport in a long permeable tube with radial convection," *Journal of Fluid Mechanics*. 63, 1974: 157-175.
[5] G. R. Ahmed and M. M. Yovanovich, "Analytical method for forced convection from flat plates, circular cylinders, and spheres," *Journal of Thermophysics and Heat Transfer*. 9 (3), 1995: 516-523.
[6] P. E. Arce, A. E. Cassano, and H.A. Irazoqui, "The tubular reactor with laminar flow regime: An integral equation approach homogeneous reaction with arbitrary kinetics. *Computers and Chemical Engineering*. 12, 1988: 1103-1113.

NOMENCLATURE

c_{pm}	Heat capacity of the cooling/heating fluid
c_0	Proportionality constant
c_1, c_2	Integration constants
D_H	Diameter
h	Convective heat transfer coefficient
H	Width of the packed-bed
K	Thermal diffusivity
kg	Global mass constant
L	Length
ρ, ρ_m	Density $\left(kg/m^3\right)$ Convective heat transfer coefficient
q_0	Heat flux at the wall of the system
r	Radial position of the packed-bed tube
R	Radius of the packed-bed tube
T	Temperature profile
T_0	Non-dimensional temperature profile
v, V	Area Averaged Velocity of the fluid
$v_{z,\max}$	Maximum Velocity in the z-direction
$v_z(r)$	Average velocity in the z- direction

Greek symbols

μ	Dynamic viscosity (Pa s)
ψ	Psi
Π	Pi
χ	Chi
Θ	Theta
$\vec{\nabla}$	Gradient
ζ	Radial component
ξ	Non-dimensional axial position