Argument Estimates Of Strongly Close-to-star Functions In A Sector

†T.N.Shanmugam, ‡C.Ramachandran, †R.Ambrose Prabhu

†Department of Mathematics,
College of Engineering Guindy, Anna University, Chennai - 600 025,Tamilnadu,India
‡Department of Mathematics,
University College of Engineering Villupuram, Villupuram - 605 602,Tamilnadu,India

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Abstract

In the present investigation, we obtain some sufficient condition for a normalized strongly close-to-star functions in the open disk \( U = \{ z \in \mathbb{C} : |z| < 1 \} \) to satisfy the condition

\[-\frac{\pi}{2} \beta \leq \arg \left\{ \frac{f(z)}{g(z)} \right\} \leq \frac{\pi}{2} \alpha, \quad 0 \leq \alpha, \beta \leq 1.\]

The aim of this paper is to generalize a result obtained by N.E.Cho and S.Owa.

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1 Introduction

Let \( A \) denote the class of functions of the form :

\[ f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in U, \tag{1.1} \]

which are analytic in the open unit disk \( U = \{ z \in \mathbb{C} : |z| < 1 \} \). Let \( S \) be the subclass of \( A \) consisting of all univalent functions. Let us denote \( S^*, K \) and \( C \) be the subclasses of \( A \), consisting of functions which are respectively starlike,convex and close-to-convex in \( U \).

Let \( f(z) \) and \( g(z) \) be analytic functions in \( U \). We say that \( f(z) \) is subordinate to \( g(z) \) if there exist analytic function \( w(z) \) such that \( w(0) = 0, |w(z)| < 1 \) with \( f(z) = g(w(z)) \) and is denoted by \( f \prec g \).

Let \( S^*[A,B] = \left\{ f \in A : \frac{zf'(z)}{f(z)} \prec \frac{1 + Az}{1 + Bz}, \quad z \in U, \quad -1 \leq B < A \leq 1 \right\} \)

and

\[ K[A,B] = \left\{ f \in A : 1 + \frac{zf''(z)}{f'(z)} \prec \frac{1 + Az}{1 + Bz}, \quad z \in U, \quad -1 \leq B < A \leq 1 \right\} \]

The class \( S^*[A,B] \) and related classes were studied by Janowski[1] and Silverman and Silvia [4] proved
that a function \( f(z) \) is in \( S^* [A, B] \) iff

\[
\left| \frac{zf'(z)}{f(z)} \right| = \frac{1 - AB}{1 - B^2} < \frac{A - B}{1 - B^2} \quad (z \in \mathbb{U}; B \neq -1) \tag{1.2}
\]

and

\[
\text{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \frac{1 - A}{A - B} \quad (z \in \mathbb{U}; B = -1) \tag{1.3}
\]

Lemma 1.1. [3] Let \( p(z) \) be analytic in \( \mathbb{U} \) with \( p(0) = 1 \) and \( p(z) \neq 0 \). If there exists two points \( z_1, z_2 \in \mathbb{U} \) such that

\[
-\frac{\pi}{2} \beta = \arg p(z_1) < \arg p(z) < \arg p(z_2) = \frac{\pi}{2} \alpha, \quad \alpha, \beta > 0 \quad \text{and, for} \ |z| < |z_1| = |z_2|,
\]

then we have

\[
z_1 p'(z_1) = i \left( \frac{\alpha + \beta}{2} \right) m
\]

and

\[
z_2 p'(z_2) = i \left( \frac{\alpha + \beta}{2} \right) m
\]

where \( m \geq 1 - |\delta| \frac{1 + |\delta|}{1 + |\delta|} \) and \( \delta = \tan \left( \frac{\alpha - \beta}{\alpha + \beta} \right) \).

Theorem 1.1. Let \( f \in \mathcal{A} \). If

\[
\left| \arg \left\{ \left( \frac{f'(z)}{g'(z)} \right)^a \left( \frac{f(z)}{g(z)} \right)^b \right\} \right| \leq \frac{\pi}{2} \delta
\]

for some

\[
g(z) \in \mathcal{K} [A, B],
\]

then

\[
\left| \arg \left( \frac{f(z)}{g(z)} \right) \right| < \frac{\pi}{2} \alpha
\]

where \( 0 < \alpha \leq 1 \) is the solution of the equation

\[
\delta = \left\{ \begin{array}{ll}
(a + b)\alpha + \frac{\pi}{2} \arctan \left( \frac{1}{1 + t(A, B)} \right) & , B \neq -1 \\
\frac{\tan \left( \frac{1}{2} \arctan \left( \frac{1}{1 + t(A, B)} \right) \right)}{1 + t(A, B)} & , B = -1
\end{array} \right.
\]

where \( t(A, B) = \frac{2}{\pi} \arcsin \left( \frac{A - B}{1 - AB} \right) \).

Proof. Let \( p(z) = \frac{f(z)}{g(z)} \), \( q(z) = \frac{zg'(z)}{g(z)} \)

by differentiating logarithmically, we have

\[
\frac{p'(z)}{p(z)} = \frac{f'(z)}{f(z)} - \frac{g'(z)}{g(z)}
\]

A simple computation shows that

\[
\left( \frac{f'(z)}{g'(z)} \right)^a \left( \frac{f(z)}{g(z)} \right)^b = (p(z))^a + b \left( 1 + \frac{1}{q(z)} \right)^a
\]

Since \( g(z) \in \mathcal{K} [A, B], \ g(z) \in S^* [A, B] \).
If we take $q(z) = \rho e^{i \frac{\pi}{2} \phi}$, $z \in U$, then it follows from (1.2) and (1.3) that

$$\frac{1 - A}{1 - B} < \rho < \frac{1 + A}{1 + B}, \quad -t(A, B) < \phi < t(A, B), \text{if } B \neq -1,$$

and

$$\frac{1 - A}{2} < \rho < \infty, \quad -1 < \phi < \infty, \text{if } B = -1,$$

where $t(A, B) = \frac{2}{\pi} \sin^{-1}\left(\frac{A - B}{1 - AB}\right)$.

Let $p(z) = \frac{f(z)}{g(z)}$, $f \in A$ and $g \in A$. If there exists two points $z_1, z_2 \in U$ such that

$$\frac{\pi}{2} \beta = \arg p(z_1) < \arg p(z) < \arg p(z_2) = \frac{\pi}{2} \alpha, \quad \alpha, \beta > 0 \text{ and, for } |z| < |z_1| = |z_2|,$$

then by lemma (1.1), we have

$$\frac{z_1 p'(z_1)}{p(z_1)} = -i \left(\frac{\alpha + \beta}{4}\right) \left(\frac{1 + t_1^2}{t_1}\right) m$$

and

$$\frac{z_2 p'(z_2)}{p(z_2)} = i \left(\frac{\alpha + \beta}{4}\right) \left(\frac{1 + t_2^2}{t_2}\right) m. \quad (1.4)$$

where

$$e^{-i \frac{\pi}{2} \left(\frac{\alpha - \beta}{\alpha + \beta}\right) (p(z_1)) \left(\frac{2}{\alpha + \beta}\right)} = -it_1$$

and

$$e^{-i \frac{\pi}{2} \left(\frac{\alpha - \beta}{\alpha + \beta}\right) (p(z_2)) \left(\frac{2}{\alpha + \beta}\right)} = it_2, \quad t_1, t_2 > 0. \quad (1.5)$$

and

$$m \geq \frac{1 - |\delta|}{1 + |\delta|} \quad (1.6)$$

Let us put $z = z_2$. Then from (1.4),(1.5)and (1.6), we have

\[
\arg \left\{ \left( \frac{f'(z_2)}{g'(z_2)} \right)^a \left( \frac{f(z_2)}{g(z_2)} \right)^b \right\} = (a + b)\arg p(z_2) + a\arg \left\{ 1 + \frac{1}{q(z_2)} \frac{z_2 p'(z_2)}{p(z_2)} \right\}
\]

\[
= (a + b)\frac{\pi}{2} \alpha + a \arg \left( 1 + \frac{e^{-\frac{i \pi}{2} \phi}}{\rho} \left(\frac{\alpha + \beta}{4}\right) \left(\frac{1 + t_2}{t_2}\right) m \right)
\]

\[
= \frac{\pi}{2} \alpha (a + b) + a \arg \left( \rho + me^{i \frac{\pi}{4}(1 - \phi)} \left(\frac{\alpha + \beta}{4}\right) \left(\frac{t_2 + 1}{t_2}\right) \cos \frac{\pi}{2} (1 - \phi) + isi \frac{\pi}{2} (1 - \phi) \right)
\]

\[
\geq \frac{\pi}{2} \alpha (a + b) + a \tan^{-1} \left\{ \frac{m \left(\frac{\alpha + \beta}{4}\right) \left(\frac{t_2 + 1}{t_2}\right) \sin \frac{\pi}{4} (1 - \phi)}{\rho + m \left(\frac{\alpha + \beta}{4}\right) \left(\frac{t_2 + 1}{t_2}\right) \cos \frac{\pi}{4} (1 - \phi)} \right\}
\]
Let us take \( g(x) = x + \frac{1}{x}, \ x > 0 \). Then attains the minimum value at \( x = 1 \). Therefore, we have

\[
\arg \left\{ \left( \frac{f'(z_2)}{g'(z_2)} \right)^a \left( \frac{f(z_2)}{g(z_2)} \right)^b \right\} \geq \frac{\pi}{2} \alpha(a + b) + \frac{2}{\pi} \alpha(b - b) + \frac{2}{\pi} \alpha(a + b) + \frac{2}{\pi} \alpha(b - b) - \alpha \tan^{-1} \left\{ \frac{m(\alpha + \beta)}{\pi(1 - \alpha)} \right\}
\]

where

\[
\delta = \frac{1}{\alpha + \beta}, \ \text{and} \ \tan \left( \frac{\alpha + \beta}{\alpha + \beta} \right)
\]

This contradicts the assumption of the theorem. For the case \( z = z_1 \), applying the same method as above, we have

\[
\arg \left\{ \left( \frac{f'(z_1)}{g'(z_1)} \right)^a \left( \frac{f(z_1)}{g(z_1)} \right)^b \right\} \leq -\frac{\pi}{2} \beta(a + b) - \beta \tan^{-1} \left\{ \frac{m(\alpha + \beta)}{\pi(1 - \alpha)} \right\}
\]

This contradiction completes the proof of the theorem. \( \square \)

Taking \( \alpha = \beta = 1 \) in theorem (1.1), we have the result obtained by NAK Euncho and Shigeyoshi owa [2]

By setting \( a = 1, b = 0, \delta = 1, A = 1 \) and \( B = -1 \) in theorem (1.1), we have

**Corollary 1.1.** Every close-to-convex function is close-to-star in \( U \). ie,

\[
\left| \arg \left( \frac{f'(z)}{g'(z)} \right) \right| < \frac{\pi}{2}
\]

ie,

\[
Re \left( \frac{f'(z)}{g'(z)} \right) \geq 0 \quad \text{or} \quad Re \left( \frac{f'(z)}{g'(z)} \right) \leq \frac{1 + z}{1 - z}.
\]

If we put \( g(z) = z \) in theorem (1.1), then by letting \( B \rightarrow A(A < 1) \), we obtain

**Corollary 1.2.** If \( f \in A \) and

\[
\left| \arg \left( \left( \frac{f'(z)}{z} \right)^a \left( \frac{f(z)}{z} \right)^b \right) \right| < \frac{\pi}{2} \delta \ (a > 0, b \in \mathbb{R}, 0 < \delta \leq 1)
\]

then

\[
|\arg f'(z)| < \frac{\pi}{2} \delta
\]
where $\alpha(0 < \alpha \leq 1)$ is the solution of the equation:

$$\delta = (a + b)\alpha + \frac{2}{\pi}a \tan^{-1}(\alpha).$$

References


