Approximation of function belonging to $W(Lp, \xi(t))$ class by $(E, q)(\bar{N}, p_n)$ means of its conjugate Fourier series

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Abstract

In this paper a theorem on degree of approximation of a function $f \in W(Lp, \xi(t))$ by product summability $(E, q)(\bar{N}, p_n)$ of conjugate series of fourier series associated with f has been proved.
Keywords: Degree of approximation, $W(Lp, \xi(t))$ class (E, q) mean $(\bar{N}, p_n)$ mean, conjugate of Fourier series and Lebesgue integral.

1 Introduction

Let $\sum a_n$ be a given infinite series with the sequence of partial sums $\{s_n\}$. Let $\{p_n\}$ be a sequence of positive real numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \text{ as } n \to \infty \ (p - i = p - i = 0, i \geq 0) \tag{1.1}$$

The sequence to sequence transformation

$$t_n = \frac{1}{p_n} \sum_{v=0}^{n} p_v \ s_v \tag{1.2}$$

defines the sequence $\{t_n\}$ of the $(\bar{N}, p_n)$-mean of the sequence $\{s_n\}$ generated by the sequence of coefficient $\{p_n\}$ if

$$t_n \to s \text{ as } n \to \infty \tag{1.3}$$

Then the series $\sum a_n$ is said to be $(\bar{N}, p_n)$ summable to s. It is clear that $(\bar{N}, p_n)$ method is regular (Hardy [1]).

The sequence to sequence transformation, (Hardy [1])

$$T_n = \frac{1}{(1+q)^n} \sum_{v=0}^{n} \binom{n}{v} q^{n-v} \cdot s_v \tag{1.4}$$

defines the sequences $\{T_n\}$ of the (E,q) means of the sequence $\{s_n\}$ if

$$T_n \to s \ \text{ as } \ n \to \infty \tag{1.5}$$

Then the series $\sum a_n$ is said to be (E,q) summable to s. Clearly (E,q) method is regular (Hardy [1]).

Further, the (E,q) transformation of the $(\bar{N},p_n)$ transform of $\{s_n\}$ is defined by

$$\begin{aligned}\tau_n &= \frac{1}{(1+q)^n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}T_k \\ &= \frac{1}{(1+q)^n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\frac{1}{P_k}\sum_{v=0}^{k}p_v\cdot s_v\right\}\end{aligned} \tag{1.6}$$

If

$$\tau_n \to s \ \text{ as } \ n \to \infty \tag{1.7}$$

then $\sum a_n$ is said to be (E,q) $(\bar{N},p_n)$-summable to s.

Let $f(t)$ be a periodic function with period 2π and L-integrable over $(-\pi,\pi)$. The Fourier series associated with f at any point x is defined by

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx) \equiv \sum_{n=0}^{\infty}A_n(x) \tag{1.8}$$

And the conjugate series of the Fourier series (1.8) is

$$\sum_{n=1}^{\infty}(b_n\cos nx - a_n\sin nx) \equiv \sum_{n=0}^{\infty}B_n(x) \tag{1.9}$$

let $s_n(f:x)$ be the n-th partial sum of (1.9)L_∞-norm of a function $f : R \to R$ is defined by

$$||f|| = \sup\{|f(x)| : x \in R\} \tag{1.10}$$

and the L_v-norm is defined by

$$||f||_v = (\int_0^{2\pi}|f(x)|^v)^{\frac{1}{v}}, \ v \geq 1 \tag{1.11}$$

The degree of approximation of a function $f : R \to R$ by a trigonometric polynomial $P_n(x)$ of degree n under norm $||\cdot||_\infty$ is defined by (Zygmund [4]).

$$||P_n - f||_\infty = \sup\{|p_n(x) - f(x)| : x \in R\} \tag{1.12}$$

and the degree of approximation $E_n(f)$ a function $f \in L_v$ is given by

$$E_n(f) = \min_{P_n} ||P_n - f||_v. \tag{1.13}$$

A function $f \in Lip(\alpha)$ if

$$|f(x+1) - f(x)| = O(|t|^\alpha),\ 0 < \alpha \leq 1,\ t > 0 \tag{1.14}$$

A function $f(x) \in Lip(\alpha, r)$ if

$$\left(\int_0^{2\pi} |f(x+t) - f(x)|^r dx\right)^{1/r} = O(|t|^\alpha),\ 0 < \alpha \leq 1,\ r \geq 1. \tag{1.15}$$

A function $f(x) \in\ Li(\xi|t|, r)$ if

$$\left(\int_0^{2\pi} |f(x+t) - f(x)|^r dx\right)^{1/r} = O(\xi|t|),\ r \geq 1,\ t > 0 \tag{1.16}$$

But $f \in W(Lp, \xi(t))$ if

$$\left(\int_0^{2\pi} |[f(x+t) - f(x)] \sin^\beta x|^p dx\right)^{1/p} = O(\xi|t|),\ \beta \geq 0 \tag{1.17}$$

we use following Notation through out this paper

$$\Psi(t) = \frac{1}{2}\{f(x+t) - f(x-t)\}$$

and

$$\phi(t) = \frac{1}{2}\{f(x+t) - f(x-t) - 2f(x)\}$$

and

$$\bar{k}_n(t) = \frac{1}{\pi(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \left\{ \frac{1}{P_k} \sum_{v=0}^{k} p_v \frac{\cos\frac{1}{2} - \cos(f + \frac{1}{2})t}{\sin t/2} \right\}$$

Further, the method $(E, q)(\bar{N}, p_n)$ is assumed to be regular.
Here we generalize the theorem of Mishra [2].

2 Main Theorem

If $f : R \to R$ is 2π-periodic, Lebesgue integrable $[-\pi, \pi]$ and belonging to the class $W(Lp, \xi(t)),\ p \geq 1$ by $\tau_n(x)$ on its conjugate Fourier series (1.9) is given by

$$||\tau_n - f||_P = O\left((n+1)^{\beta + \frac{1}{p}} \xi\left(\frac{1}{n+1}\right)\right) \tag{2.1}$$

Provided $\xi(t)$ satisfies the following conditions

$$\left\{\int_0^{\frac{1}{n+1}} \left(\frac{t|\phi(t)|}{\xi(t)}\right)^P \sin^{\beta P} t dt\right\}^{1/p} = O\left(\frac{1}{n+1}\right) \tag{2.2}$$

and

$$\left\{\int_0^{\frac{1}{n+1}} \left(\frac{t^{-\delta}|\phi(t)|\sin^\beta t}{\xi(t)}\right)^p dt\right\}^{1/p} = O\{(n+1)^\delta\} \tag{2.3}$$

where δ is an arbitrary number such that $q(1-\delta)-1>0$, (2.2), (2.3) hold uniformly in x and where $\frac{1}{p}+\frac{1}{q}=1$ such that $1 \le p \le \infty$.

3 Lemma

In order to prove it, we shall required the following lemma (Misra [2]).

$$|\bar{k}_n(t)| = \begin{cases} O(n) \text{ for } 0 \le t \le \frac{1}{n+1} \\ O\left(\frac{1}{t}\right) \text{ for } \frac{1}{n+1} \le t \le \pi \end{cases}$$

4 Proof the Theorem

Using Riemann-Lebesgue theorem, we have for the n-th partial sum $\bar{s}_n(f:x)$ of the Conjugate Fourier series (1.9) of $f(x)$ following (Titchmarch [3])

$$\bar{s}_n(f:x) - f(x) = \frac{2}{\pi}\int_0^\pi \phi(t)\bar{k}_n dt$$

the $(\bar{N}, p_n)$ transform of $\bar{s}_n(f:x)$ using 1.2 is given by

$$t_n - f(x) = \frac{2}{\pi P_n}\int_0^\pi \phi(t)\sum_{k=0}^n p_k \frac{\cos\frac{t}{2} - \sin(n+\frac{1}{2})t}{2\sin(t/2)} dt$$

denoting the $(E,q)(\bar{N},p_n)$ transform of $\bar{s}_n(f:x)$ by τ_n, where

$$\begin{aligned} ||\tau_n - f|| &= \frac{1}{\pi(1+1)^n}\int_0^\pi \phi(t)\bar{k}_n(t)dt \\ &= \left\{\int_0^{\frac{1}{n+1}} + \int_{\frac{1}{n+1}}^0\right\}\phi(t)\bar{k}_n(t)dt \\ &= I_1 + I_2 \end{aligned} \tag{4.1}$$

Now

$$I_1 = \int_0^{\frac{1}{n+1}} \phi(t)\bar{k}_n(t)dt$$

Applying Hölder inequality

$$|I_1| \le \left[\int_0^{\frac{1}{n+1}} \left\{\frac{t|\phi(t)|\sin^\beta t}{\xi(t)}\right\}^p dt\right]^{1/p} \cdot \left[\int_0^{\frac{1}{n+1}} \left\{\frac{\xi(t)|\bar{k}_n(t)|}{t\sin^\beta t}\right\}^q dt\right]^{1/q}$$
$$= O\left(\frac{1}{n+1}\right)\left[\int_0^{\frac{1}{n+1}} \left\{\frac{n\xi(t)}{t^{1+\beta}}\right\}^q dt\right]^{1/q}$$
$$= O\left(\frac{1}{n+1}\right)O(n)\left[\int_0^{\frac{1}{n+1}} \left\{\frac{\xi(t)}{t^{1+\beta}}\right\}^q dt\right]^{1/q}$$
$$\le O\left(\frac{1}{n+1}\right)O(n+1)\left[\int_0^{\frac{1}{n+1}} \left\{\frac{\xi(t)}{t^{1+\beta}}\right\}^q dt\right]^{1/q}$$
$$= O\left(\xi\left(\frac{1}{n+1}\right)\right)\left[\int_0^{\frac{1}{n+1}} t^{-(\beta+1)q}dt\right]^{1/q}$$
$$= O(\xi\left(\frac{1}{n+1}\right)\cdot(n+1)^{\beta+1-\frac{1}{q}}$$
$$= O(\xi\left(\frac{1}{n+1}\right)\cdot(n+1)^{\beta+\frac{1}{p}}$$
$$= O\left((n+1)^{\beta+\frac{1}{p}}\cdot\xi\left(\frac{1}{n+1}\right)\right). \tag{4.2}$$

And

$$I_2 = \int_{\frac{1}{n+1}}^0 \phi(t)\ \bar{k}_n(t)dt$$

Using Hölder inequality.

$$|I_2| \le \left[\int_{\frac{1}{n+1}}^\pi \left\{\frac{t^{-\delta}|\phi(t)|\sin^\beta t}{\xi(t)}\right\}^p dt\right]^{1/p} \cdot \left[\int_{\frac{1}{n+1}}^\pi \left\{\frac{\bar{k}_n(t)\cdot\xi(t)}{t^{-\delta}\sin^\beta t}\right\}^{1/q} dt\right]^{1/q}$$
$$= O(n+1)^\delta\left[\int_{\frac{1}{n+1}}^\pi \left\{\frac{\xi(t)}{t^{-\delta+\beta+1}}\right\}^q dt\right]^{1/q}$$
$$= O(n+1)^\delta\left[\int_{1/\pi}^{\frac{1}{n+1}} \left\{\frac{\xi(1/y)}{\bar{y}^{(\beta+1)+\delta}}\right\}^q \frac{dy}{y^2}\right]^{1/q}$$
$$= O\left\{(n+1)^\delta\xi\left(\frac{1}{n+1}\right)\left[(y^{\beta-\delta+1)q-1}\right]^{1/q}\right\}_{1/\pi}^{n+1}$$
$$= O\left\{(n+1)^\delta\xi\left(\frac{1}{n+1}\right)(n+1)^{\beta-\delta+1-\frac{1}{q}}\right\}$$

$$= O\left\{\xi\left(\frac{1}{n+1}\right)(n+1)^{\beta+\frac{1}{p}}\right\}$$
$$= O\left\{(n+1)^{\beta+\frac{1}{p}}\xi\left(\frac{1}{n+1}\right)\right\} \tag{4.3}$$

Now combining (4.1), (4.2), (4.3)

$$||\tau_n - f|| = O\left\{(n+1)^{\beta+\frac{1}{p}}\xi\left(\frac{1}{n+1}\right)\right\}$$

This completes the proof of main theorem.

5 Corollary

Following corollaries can be derived from our main results.

Corollary 5.1. If $\beta = 0$ then the function $f \in W(Lp, \xi(t))$ becomes $f \in Lip(\xi(t), p)$ and degree of approximation is given by

$$||T_n - f|| = O\left((n+1)^{\frac{1}{p}}\xi\left(\frac{1}{n+1}\right)\right),\ p > 0$$

Corollary 5.2. If $\beta = 0$ and $\xi(t) = t^\alpha$ then $f \in W(Lp, \xi(t))$ becomes $f \in Lip(\xi(t), p)$ and degree of approximation is given by

$$||T_n - f|| = O\left(\frac{1}{(n+1)^{\alpha-\frac{1}{p}}}\right),\ p > 0$$

Corollary 5.3. If $\beta = 0,\ \xi(t) = t^\alpha$ and $p \to \infty$ then $f \in W(Lp, \xi(t))$ becomes $Lip\alpha$ and degree of approximation is given by

$$||T_n - f|| = O\left(\frac{1}{(n+1)^\alpha}\right),\ 0 < \alpha < 1$$

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