Applications on the Conditions of the General Subordination Theorem

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***Abstract* - In this work, we are choosing the distinguished dominant function q which satisfy the general subordination theorem's conditions. As an impact we will get some fascinating specification about starlikeness of an analytic function.**

Keywords - First order differential subordination; dominant function; analytic functions; unit disc

I. INTRODUCTION

The general subordination theorem's conditions is defined as follows:

Let α,λ and q be a complex number with Re $\alpha>0.$ If $f\in A,\dfrac{f(z)}{z}\neq 0$ be a function in U satisfies the following,

$$\phi(\alpha,\lambda;\mathrm{p}(z))\prec\phi(\alpha,\lambda;q(z)),\ z\in U.$$

Setting $p(z)=\dfrac{zf'(z)}{f(z)}$ in above first order differential subordination equation, we get

$$\phi\left(\alpha,\lambda;\frac{zf'(z)}{f(z)}\right)\prec\phi(\alpha,\lambda;q(z)),\ z\in U,$$

so

$$\frac{zf'(z)}{f(z)}\prec q(z),\forall\,\mathrm{z}\in\mathrm{U}.$$

where U is an unit disc.

The unit disc is defined by the function $U=\{z:|z|<1\}$ and these functions are normalized by the conditions $f(0)=0$ and $f'(0)=1.$

Let us take $q(z)=\dfrac{1+az}{1-z},-1<a\leq 1.$ obviously, q is convex univalent in $U.$

II. APPLICATIONS

THEOREM 2.1

Let the positive real numbers be α and λ. Assume that $a,\ -1<a\leq 1,$ is a real number such that $a\leq\dfrac{1}{\alpha}$ whenever $\alpha>1.$ Let $f\in A,\dfrac{f(z)}{z}\neq 0$ in $U,$ which satisfy

$$\frac{zf'(z)}{f(z)}\left(1-\alpha+\alpha(1-\lambda)\frac{zf'(z)}{f(z)}+\alpha\lambda\left(1+\frac{zf''(z)}{f'(z)}\right)\right)\prec h(z),z\in U,$$

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where

$$h(z)=(1-\alpha)\left(\frac{1+az}{1-z}\right)+\alpha\left(\frac{1+az}{1-z}\right)^2+\alpha\lambda\left(\frac{(a+1)z}{(1-z)^2}\right),$$

then,

$$\frac{zf'(z)}{f(z)}\prec\frac{1+az}{1-z},\forall z\in U.$$

PROOF

Let h be defined by,

$$h(z)=a(\alpha(a+1)-1)+(a+1)\left(\frac{1-a\alpha+(2\alpha a+\alpha\lambda+\alpha-1)(z)}{(1-z)^2}\right),$$

we put $h(0)=1$ in the above equation, we get

$$h(z)=a(\alpha(a+1)-1)+(a+1)\left(\frac{1-a\alpha+(2\alpha a+\alpha\lambda+\alpha-1)(z)}{(1-z)^2}\right),$$

$$h(0)=a(\alpha(a+1)-1)+(a+1)\left(\frac{1-a\alpha+(2\alpha a+\alpha\lambda+\alpha-1)(0)}{(1-0)^2}\right),$$

$$h(0)=a(\alpha(a+1)-1)+(a+1)\left(\frac{1-a\alpha+0}{1}\right),$$

$$h(0)=a(\alpha a+\alpha-1)+(a+1)(1-a\alpha),$$

$$h(0)=a^2\alpha+\alpha a-a+a-a^2\alpha+1-a\alpha,$$

$$h(0)=1.$$

Then we have to put $h(-1)=\dfrac{\alpha(a+1)(a-1-\lambda)+2(1-a)}{4}$

$$h(-1)=a(\alpha(a+1)-1)+(a+1)\left(\frac{1-a\alpha+(2\alpha a+\alpha\lambda+\alpha-1)(-1)}{(1-(-1))^2}\right),$$

$$h(-1)=a(\alpha(a+1)-1)+(a+1)\left(\frac{1-a\alpha-2a\alpha-\alpha\lambda-\alpha+1}{4}\right),$$

$$h(-1)=a(\alpha a+\alpha-1)+\left(\frac{a-a^2\alpha-2a^2\alpha-a\alpha\lambda-a\alpha+a+1-a\alpha-2a\alpha-\alpha\lambda-\alpha+1}{4}\right),$$

$$h(-1)=a^2\alpha+a\alpha-a+\left(\frac{2a-3a^2\alpha-3a\alpha-a\alpha\lambda-a\alpha+2-\alpha\lambda-\alpha}{4}\right),$$

$$h(-1)=\left(\frac{4\alpha a^2+4a\alpha-4a+2a-3a^2\alpha-3a\alpha-a\alpha\lambda-a\alpha+2-\alpha\lambda-\alpha}{4}\right),$$

$$h(-1)=\left(\frac{\alpha a^2-2a-a\alpha\lambda+2-\alpha\lambda-\alpha}{4}\right),$$

$$h(-1)=\left(\frac{\alpha a^2-2a-a\alpha\lambda+2-\alpha\lambda-\alpha+a\alpha-a\alpha}{4}\right),$$

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$$h(-1)=\frac{\alpha(a+1)(a-1-\lambda)+2(1-a)}{4}.$$

So it is clear that the function h is close-to-convex in U. The curve is symmetrical about the real axis and it intersects the real axis at one point only.

The boundary of the curve is given by

$$h(e^{i\theta})=u(\theta)+iv(\theta),\theta\in(-\pi,\pi).$$

Already we have,

$$h(z)=a(\alpha(a+1)-1)+(a+1)\left(\frac{1-a\alpha+(2\alpha a+\alpha\lambda+\alpha-1)(z)}{(1-z)^2}\right),$$

$$h(e^{i\theta})=a(\alpha(a+1)-1)+(a+1)\left(\frac{1-a\alpha+(2\alpha a+\alpha\lambda+\alpha-1)(e^{i\theta})}{(1-e^{i\theta})^2}\right),$$

$$h(e^{i\theta})=a(\alpha(a+1)-1)+(a+1)\left(\frac{1-a\alpha+(2\alpha a+\alpha\lambda+\alpha-1)(\cos\theta+i\sin\theta)}{(1-(\cos\theta+i\sin\theta))^2}\right),$$

$$h(e^{i\theta})=a(\alpha(a+1)-1)+(a+1)\left(\frac{1-a\alpha+(2\alpha a+\alpha\lambda+\alpha-1)(\cos\theta+i\sin\theta)}{((1-\cos\theta)+i\sin\theta)^2}\right),$$

$$h(e^{i\theta})=a(\alpha(a+1)-1)+(a+1)\left(\frac{1-a\alpha+(2\alpha a+\alpha\lambda+\alpha-1)(\cos\theta+i\sin\theta)}{(-2(1-\cos\theta)(\cos\theta+i\sin\theta))}\right),$$

$$h(e^{i\theta})=a(\alpha(a+1)-1)-\frac{(a+1)}{2(1-\cos\theta)}\left(\frac{1-a\alpha+(2\alpha a+\alpha\lambda+\alpha-1)(\cos\theta+i\sin\theta)}{(\cos\theta+i\sin\theta)}\right)\times\frac{\cos\theta-i\sin\theta}{\cos\theta-i\sin\theta}$$

$$h(e^{i\theta})=a(\alpha(a+1)-1)-\frac{(a+1)}{2(1-\cos\theta)}\left(\frac{(1-a\alpha)(\cos\theta-i\sin\theta)+(2\alpha a+\alpha\lambda+\alpha-1)}{1}\right),$$

$$h(e^{i\theta})=a(\alpha(a+1)-1)-\frac{(a+1)}{2(1-\cos\theta)}((1-a\alpha)(\cos\theta)+(2\alpha a+\alpha\lambda+\alpha-1))+\frac{(a+1)}{2(1-\cos\theta)}((1-a\alpha)i\sin\theta),$$

Taking the real and imaginary parts,we get

$$u(\theta)=\frac{(a\alpha(a+1)-a)2(1-\cos\theta)-(a+1)(1-a\alpha)(\cos\theta)-(a+1)(2\alpha a+\alpha\lambda+\alpha-1)}{2(1-\cos\theta)},$$

$$u(\theta)=\frac{2(a^2\alpha+a\alpha-a)(1-\cos\theta)-(a+1)(\cos\theta-a\alpha\cos\theta+2\alpha a+\alpha\lambda+\alpha-1)}{2(1-\cos\theta)},$$

$$u(\theta)=\frac{2(a^2\alpha+a\alpha-a-a^2\alpha\cos\theta-a\alpha\cos\theta+a\cos\theta)-(a+1)\cos\theta+(a+1)a\alpha\cos\theta-(a+1)(2\alpha a+\alpha\lambda+\alpha-1)}{2(1-\cos\theta)}$$

$$u(\theta)=\frac{(1-a)+\cos\theta((a-1)-a\alpha(a+1))-\alpha(\lambda+1)(1+a)}{2(1-\cos\theta)}.$$

Then imaginary part is,

$$v(\theta)=\frac{(a+1)}{2(1-\cos\theta)}((1-a\alpha)\sin\theta).$$

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Eliminating θ, we get the equation of the boundary curve as

$$v^2 = -\frac{(1-a\alpha)^2(1+a)}{\alpha(\lambda+1+a)}\left(u - \frac{\alpha\left(a^2-\lambda-a\lambda-1\right)+2(1-a)}{4}\right),$$

which is a parabola opening towards the left, with its vertex at the point $\left(\frac{\alpha\left(a^2-\lambda-a\lambda-1\right)+2(1-a)}{4}, 0\right)$ and the negative real axis as the axis of parabola.

Hence, $h(U)$ is the exterior of this parabola and includes the right plane

$$u \geq \frac{\alpha\left(a^2-\lambda-a\lambda-1\right)+2(1-a)}{4}.$$

THEOREM 2.2

Let the real number be $\alpha, \alpha > 0,$ Assume the real number be $\beta, 0 \leq \beta < 1,$ such that $\beta \geq \frac{1}{2} - \frac{1}{2\alpha}$ where $\alpha > 1.$

For all $z \in U$, let $f \in A, \frac{f(z)}{z} \neq 0$ in U, which satisfy the differential subordination

$$\frac{zf'(z)}{f(z)} + \alpha\frac{z^2 f''(z)}{f(z)} \prec h(z),$$

where

$$h(z) = (1-\alpha)\left(\frac{1+(1-2\beta)z}{1-z}\right) + \alpha\left(\frac{1+(1-2\beta)z}{1-z}\right)^2 + \alpha\left(\frac{2(1-\beta)z}{(1-z)^2}\right),$$

then $f \in S^*(\beta)$.

PROOF

We know that,

$$h(z) = (1-\alpha)\left(\frac{1+az}{1-z}\right) + \alpha\left(\frac{1+az}{1-z}\right)^2 + \alpha\lambda\left(\frac{(a+1)z}{(1-z)^2}\right),$$

setting $a = 1-2\beta, 0 \leq \beta < 1$ and $\lambda = 1$ in above equation, we get

$$h(z) = (1-\alpha)\left(\frac{1+(1-2\beta)z}{1-z}\right) + \alpha\left(\frac{1+(1-2\beta)z}{1-z}\right)^2 + \alpha\lambda\left(\frac{((1-2\beta)+1)z}{(1-z)^2}\right),$$

$$h(z) = (1-\alpha)\left(\frac{1+(1-2\beta)z}{1-z}\right) + \alpha\left(\frac{1+(1-2\beta)z}{1-z}\right)^2 + \alpha(1)\left(\frac{((1-2\beta)+1)z}{(1-z)^2}\right),$$

$$h(z) = (1-\alpha)\left(\frac{1+(1-2\beta)z}{1-z}\right) + \alpha\left(\frac{1+(1-2\beta)z}{1-z}\right)^2 + \alpha\left(\frac{(2-2\beta)z}{(1-z)^2}\right),$$

$$h(z) = (1-\alpha)\left(\frac{1+(1-2\beta)z}{1-z}\right) + \alpha\left(\frac{1+(1-2\beta)z}{1-z}\right)^2 + \alpha\left(\frac{2(1-\beta)z}{(1-z)^2}\right),$$

then $f \in S^*(\beta)$.

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LEMMA 2.1

If $h(z)=u+iv,$ then $h(U)$ is the exterior of the parabola given by

$$v^2=-\frac{(1-\alpha(1-2\beta))^2(2-2\beta)}{\alpha(3-2\beta)}\left(u-\left(\alpha\beta\left(\beta-\frac{1}{2}\right)+\beta-\frac{\alpha}{2}\right)\right),$$

with its vertex at $\left(\alpha\beta\left(\beta-\frac{1}{2}\right)+\beta-\frac{\alpha}{2},0\right).$

PROOF

By Theorem 2.1, we have the equation of the boundary curve as

$$v^2=-\frac{(1-a\alpha)^2(1+a)}{\alpha(\lambda+1+a)}\left(u-\frac{\alpha(a^2-\lambda-a\lambda-1)+2(1-a)}{4}\right),$$

which is a parabola opening towards the left, with its vertex at the point $\left(\frac{\alpha(a^2-\lambda-a\lambda-1)+2(1-a)}{4},0\right).$

Here setting $a=1-2\beta, 0\le\beta<1$ and $\lambda=1,$

$$v^2=-\frac{(1-(1-2\beta)\alpha)^2(1+(1-2\beta))}{\alpha(1+1+(1-2\beta))}\left(u-\frac{\alpha((1-2\beta)^2-1-(1-2\beta)(1)-1)+2(1-(1-2\beta))}{4}\right),$$

$$v^2=-\frac{(1-(1-2\beta)\alpha)^2(1+1-2\beta)}{\alpha(2+1-2\beta)}\left(u-\frac{\alpha((1-2\beta)^2-1-1+2\beta-1)+2(1-1+2\beta)}{4}\right),$$

$$v^2=-\frac{(1-(1-2\beta)\alpha)^2(2-2\beta)}{\alpha(3-2\beta)}\left(u-\frac{\alpha(1-4\beta+4\beta^2-1-1+2\beta-1)+2-2+4\beta}{4}\right),$$

$$v^2=-\frac{(1-(1-2\beta)\alpha)^2(2-2\beta)}{\alpha(3-2\beta)}\left(u-\frac{\alpha(4\beta^2-2\beta-2)+4\beta}{4}\right),$$

$$v^2=-\frac{(1-(1-2\beta)\alpha)^2(2-2\beta)}{\alpha(3-2\beta)}\left(u-\frac{(4\alpha\beta^2-2\alpha\beta-2\alpha)+4\beta}{4}\right),$$

$$v^2=-\frac{(1-(1-2\beta)\alpha)^2(2-2\beta)}{\alpha(3-2\beta)}\left(u-\left(\alpha\beta^2-\frac{\alpha\beta}{2}-\frac{\alpha}{2}+\beta\right)\right),$$

$$v^2=-\frac{(1-(1-2\beta)\alpha)^2(2-2\beta)}{\alpha(3-2\beta)}\left(u-\left(\alpha\beta\left(\beta-\frac{1}{2}\right)-\frac{\alpha}{2}+\beta\right)\right),$$

with its vertex at $\left(\alpha\beta\left(\beta-\frac{1}{2}\right)+\beta-\frac{\alpha}{2},0\right).$

LEMMA 2.2

Let the real number be α , $0<\alpha<2,$. If $f\in A, \frac{f(z)}{z}\ne 0$ in $U,$ which satisfies the differential subordination

$$\frac{zf'(z)}{f(z)}+\alpha\frac{z^2f''(z)}{f(z)}\prec h(z), z\in U,$$

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then $f \in S^*\left(\frac{\alpha}{2}\right)$.

PROOF

Putting $a = 1-\alpha$ and $\lambda = 1$ in Theorem 2.1, we get the following one:
we know that,

$$h(z) = (1-\alpha)\left(\frac{1+az}{1-z}\right) + \alpha\left(\frac{1+az}{1-z}\right)^2 + \alpha\lambda\left(\frac{(a+1)z}{(1-z)^2}\right),$$

Substituting the corresponding values in above equation, we get

$$h(z) = (1-\alpha)\left(\frac{1+(1-\alpha)z}{1-z}\right) + \alpha\left(\frac{1+(1-\alpha)z}{1-z}\right)^2 + \alpha(1)\left(\frac{((1-\alpha)+1)z}{(1-z)^2}\right),$$

$$h(z) = (1-\alpha)\left(\frac{1+(1-\alpha)z}{1-z}\right) + \alpha\left(\frac{1+(1-\alpha)z}{1-z}\right)^2 + \alpha\left(\frac{(2-\alpha)z}{(1-z)^2}\right),$$

and $h(U)$ is the exterior of the parabola given by the equation,

$$v^2 = -\frac{(1-a\alpha)^2(1+a)}{\alpha(\lambda+1+a)}\left(u - \frac{\alpha(a^2-\lambda-a\lambda-1)+2(1-a)}{4}\right)$$

Substituting $a = 1-\alpha$ and $\lambda = 1$ in above equation, we get

$$v^2 = -\frac{(1-(1-\alpha)\alpha)^2(1+1-\alpha)}{\alpha(1+1+1-\alpha)}\left(u - \frac{\alpha\left((1-\alpha)^2-1-(1-\alpha)1-1\right)+2(1-(1-\alpha))}{4}\right),$$

$$v^2 = -\frac{(1-(1-\alpha)\alpha)^2(1+1-\alpha)}{\alpha(1+1+1-\alpha)}\left(u - \frac{\alpha\left((1-\alpha)^2-1-(1-\alpha)1-1\right)+2(1-1+\alpha)}{4}\right),$$

$$v^2 = -\frac{(1-\alpha+\alpha^2)^2(2-\alpha)}{\alpha(3-\alpha)}\left(u - \frac{\alpha(1-2\alpha+\alpha^2-1-1+\alpha-1)+2\alpha}{4}\right),$$

$$v^2 = -\frac{(1-\alpha+\alpha^2)^2(2-\alpha)}{\alpha(3-\alpha)}\left(u - \frac{\alpha(\alpha^2-\alpha-2)+2\alpha}{4}\right),$$

$$v^2 = -\frac{(1-\alpha+\alpha^2)^2(2-\alpha)}{\alpha(3-\alpha)}\left(u - \left(\frac{\alpha\alpha^2-\alpha^2-2\alpha+2\alpha}{4}\right)\right),$$

$$v^2 = -\frac{(1-\alpha+\alpha^2)^2(2-\alpha)}{\alpha(3-\alpha)}\left(u - \left(\frac{\alpha^2(\alpha-1)}{4}\right)\right),$$

with its vertex at the point $\left(-\frac{\alpha^2}{4}(1-\alpha), 0\right)$.

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LEMMA 2.3

For $0<\alpha\le 1$ if $f\in A, \dfrac{f(z)}{z}\neq 0$ in U, which satisfies

$$\frac{zf'(z)}{f(z)}+\alpha\frac{z^2f''(z)}{f(z)}\prec(1-\alpha)\left(\frac{1+z}{1-z}\right)+\alpha\left(\frac{1+z}{1-z}\right)^2+\alpha\frac{2z}{(1-z)^2}=h(z),(\text{say})$$

then $f\in S^*\left(\dfrac{\alpha}{2}\right)$.

PROOF

The differential subordination equation is

$$\frac{zf'(z)}{f(z)}+\alpha\frac{z^2f''(z)}{f(z)}\prec h(z), z\in U,$$

where

$$h(z)=(1-\alpha)\left(\frac{1+az}{1-z}\right)+\alpha\left(\frac{1+az}{1-z}\right)^2+\alpha\lambda\left(\frac{(a+1)z}{(1-z)^2}\right),$$

Taking $a=\lambda=1$,

$$h(z)=(1-\alpha)\left(\frac{1+(1)z}{1-z}\right)+\alpha\left(\frac{1+(1)z}{1-z}\right)^2+\alpha(1)\left(\frac{(1+1)z}{(1-z)^2}\right),$$

$$h(z)=(1-\alpha)\left(\frac{1+z}{1-z}\right)+\alpha\left(\frac{1+z}{1-z}\right)^2+\alpha\left(\frac{2z}{(1-z)^2}\right),$$

$h(U)$ is the exterior of the parabola given by the equation,

$$v^2=-\frac{(1-a\alpha)^2(1+a)}{\alpha(\lambda+1+a)}\left(u-\frac{\alpha(a^2-\lambda-a\lambda-1)+2(1-a)}{4}\right),$$

Taking $a=\lambda=1$,

$$v^2=-\frac{(1-(1)\alpha)^2(1+1)}{\alpha(1+1+1)}\left(u-\frac{\alpha(1-1-1-1)+2(1-1)}{4}\right),$$

$$v^2=-\frac{(1-\alpha)^2(2)}{3\alpha}\left(u-\frac{\alpha(-2)+0}{4}\right),$$

$$v^2=-\frac{2(1-\alpha)^2}{3\alpha}\left(u+\frac{2\alpha}{4}\right),$$

$$v^2=-\frac{2(1-\alpha)^2}{3\alpha}\left(u+\frac{\alpha}{2}\right),$$

with its vertex at $\left(-\dfrac{\alpha}{2},0\right)$.

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LEMMA 2.4

Let the real number be $\lambda, \lambda > 0$. For $-1 < \alpha \le 1$ if $f \in A, \frac{f(z)}{z} \neq 0$ in U, which satisfies the differential subordination

$$\frac{zf'(z)}{f(z)}\left(1+(1-\lambda)\frac{zf'(z)}{f(z)}+\lambda\left(1+\frac{zf''(z)}{f'(z)}\right)\right) \prec \left(\frac{1+az}{1-z}\right)+\left(\frac{1+az}{1-z}\right)^2+\lambda\left(\frac{(a+1)z}{(1-z)^2}\right),$$

in U, then

$$\frac{zf'(z)}{f(z)} \prec \left(\frac{1+az}{1-z}\right), \forall z \in U.$$

PROOF

The equation,

$$\frac{zf'(z)}{f(z)}\left(1-\alpha+\alpha(1-\lambda)\frac{zf'(z)}{f(z)}+\alpha\lambda\left(1+\frac{zf''(z)}{f'(z)}\right)\right) \prec h(z),$$

where

$$h(z)=(1-\alpha)\left(\frac{1+az}{1-z}\right)+\alpha\left(\frac{1+az}{1-z}\right)^2+\alpha\lambda\left(\frac{(a+1)z}{(1-z)^2}\right),$$

put $\alpha = \frac{1}{2}$ in above equation, we get

$$\frac{zf'(z)}{f(z)}\left(1-\frac{1}{2}+\frac{1}{2}(1-\lambda)\frac{zf'(z)}{f(z)}+\frac{1}{2}\lambda\left(1+\frac{zf''(z)}{f'(z)}\right)\right) \prec h(z),$$

$$\frac{zf'(z)}{f(z)}\left(\frac{2-1}{2}+\frac{1}{2}(1-\lambda)\frac{zf'(z)}{f(z)}+\frac{1}{2}\lambda\left(1+\frac{zf''(z)}{f'(z)}\right)\right) \prec h(z),$$

$$\frac{zf'(z)}{f(z)}\left(\frac{1}{2}+\frac{1}{2}(1-\lambda)\frac{zf'(z)}{f(z)}+\frac{1}{2}\lambda\left(1+\frac{zf''(z)}{f'(z)}\right)\right) \prec h(z),$$

$$\frac{1}{2}\frac{zf'(z)}{f(z)}\left(1+(1-\lambda)\frac{zf'(z)}{f(z)}+\lambda\left(1+\frac{zf''(z)}{f'(z)}\right)\right) \prec h(z), \tag{1}$$

Next we have to find $h(z)$,

$$h(z)=(1-\alpha)\left(\frac{1+az}{1-z}\right)+\alpha\left(\frac{1+az}{1-z}\right)^2+\alpha\lambda\left(\frac{(a+1)z}{(1-z)^2}\right),$$

put $\alpha = \frac{1}{2}$ in $h(z)$,

$$h(z)=\left(1-\frac{1}{2}\right)\left(\frac{1+az}{1-z}\right)+\frac{1}{2}\left(\frac{1+az}{1-z}\right)^2+\frac{1}{2}\lambda\left(\frac{(a+1)z}{(1-z)^2}\right),$$

$$h(z)=\left(\frac{2-1}{2}\right)\left(\frac{1+az}{1-z}\right)+\frac{1}{2}\left(\frac{1+az}{1-z}\right)^2+\frac{1}{2}\lambda\left(\frac{(a+1)z}{(1-z)^2}\right),$$

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$$h(z)=\left(\frac{1}{2}\right)\left(\frac{1+az}{1-z}\right)+\frac{1}{2}\left(\frac{1+az}{1-z}\right)^{2}+\frac{1}{2}\lambda\left(\frac{(a+1)z}{(1-z)^{2}}\right),$$

$$h(z)=\frac{1}{2}\left\{\left(\frac{1+az}{1-z}\right)+\left(\frac{1+az}{1-z}\right)^{2}+\lambda\left(\frac{(a+1)z}{(1-z)^{2}}\right)\right\}, \qquad (2)$$

substituting the value of $h(z)$ from “(2)” in “(1)” equation, we get

$$\frac{1}{2}\frac{zf'(z)}{f(z)}\left(1+(1-\lambda)\frac{zf'(z)}{f(z)}+\lambda\left(1+\frac{zf''(z)}{f'(z)}\right)\right)\prec\frac{1}{2}\left\{\left(\frac{1+az}{1-z}\right)+\left(\frac{1+az}{1-z}\right)^{2}+\lambda\left(\frac{(a+1)z}{(1-z)^{2}}\right)\right\},$$

$$\frac{zf'(z)}{f(z)}\left(1+(1-\lambda)\frac{zf'(z)}{f(z)}+\lambda\left(1+\frac{zf''(z)}{f'(z)}\right)\right)\prec\left\{\left(\frac{1+az}{1-z}\right)+\left(\frac{1+az}{1-z}\right)^{2}+\lambda\left(\frac{(a+1)z}{(1-z)^{2}}\right)\right\},$$

in U, then

$$\frac{zf'(z)}{f(z)}\prec\left(\frac{1+az}{1-z}\right),\forall \mathrm{z}\in\mathrm{U}.$$

Here, the image of the unit disc U under the function $h(z)=u+iv$ is the exterior of the parabola,

$$v^{2}=-\frac{(1-a\alpha)^{2}(1+a)}{\alpha(\lambda+1+a)}\left(u-\frac{\alpha\left(a^{2}-\lambda-a\lambda-1\right)+2(1-a)}{4}\right).$$

Put $\alpha=\frac{1}{2}$ in above equation, we get

$$v^{2}=-\frac{\left(1-a\left(\frac{1}{2}\right)\right)^{2}(1+a)}{\frac{1}{2}(\lambda+1+a)}\left(u-\frac{\frac{1}{2}\left(a^{2}-\lambda-a\lambda-1\right)+2(1-a)}{4}\right),$$

$$v^{2}=-\frac{\left(\frac{2-a}{2}\right)^{2}(1+a)}{\frac{1}{2}(\lambda+1+a)}\left(u-\frac{\frac{2}{2}\left(a^{2}-\lambda-a\lambda-1\right)+2(2)(1-a)}{4}\right),$$

$$v^{2}=-\frac{2}{1}\times\frac{1}{2}\frac{(2-a)^{2}(1+a)}{(\lambda+1+a)}\left(u-\frac{\left(a^{2}-\lambda-a\lambda-1\right)+4(1-a)}{4}\right),$$

$$v^{2}=-\frac{(2-a)^{2}(1+a)}{(\lambda+1+a)}\left(u-\frac{\left(a^{2}-\lambda-a\lambda-1\right)+4(1-a)}{4}\right),$$

which has its vertex at the point $\left(\frac{\left(a^{2}-\lambda-a\lambda-1\right)+4(1-a)}{4},0\right)$.

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LEMMA 2.5

Let the given real number be $\lambda, \lambda > 0$. For $-1 < \alpha \le 1$ if $f \in A, \frac{f(z)}{z} \ne 0$ in U, which satisfies the differential subordination

$$\frac{zf'(z)}{f(z)}\left((1-\lambda)\frac{zf'(z)}{f(z)}+\lambda\left(1+\frac{zf''(z)}{f'(z)}\right)\right)\prec\left(\frac{1+az}{1-z}\right)^2+\lambda\left(\frac{(a+1)z}{(1-z)^2}\right)=h(z),(say)$$

in U, then

$$\frac{zf'(z)}{f(z)}\prec\left(\frac{1+az}{1-z}\right),\forall z\in \mathrm{U}.$$

PROOF

The differential subordination equation is,

$$\frac{zf'(z)}{f(z)}\left(1-\alpha+\alpha(1-\lambda)\frac{zf'(z)}{f(z)}+\alpha\lambda\left(1+\frac{zf''(z)}{f'(z)}\right)\right)\prec h(z),$$

where

$$h(z)=(1-\alpha)\left(\frac{1+az}{1-z}\right)+\alpha\left(\frac{1+az}{1-z}\right)^2+\alpha\lambda\left(\frac{(a+1)z}{(1-z)^2}\right),$$

$$\frac{zf'(z)}{f(z)}\left(1-\alpha+\alpha(1-\lambda)\frac{zf'(z)}{f(z)}+\alpha\lambda\left(1+\frac{zf''(z)}{f'(z)}\right)\right)\prec(1-\alpha)\left(\frac{1+az}{1-z}\right)+\alpha\left(\frac{1+az}{1-z}\right)^2+\alpha\lambda\left(\frac{(a+1)z}{(1-z)^2}\right),$$

Put $\alpha = 1$ in above equation, we get

$$\frac{zf'(z)}{f(z)}\left(1-1+1(1-\lambda)\frac{zf'(z)}{f(z)}+(1)\lambda\left(1+\frac{zf''(z)}{f'(z)}\right)\right)\prec(1-1)\left(\frac{1+az}{1-z}\right)+1\left(\frac{1+az}{1-z}\right)^2+(1)\lambda\left(\frac{(a+1)z}{(1-z)^2}\right)$$

$$\frac{zf'(z)}{f(z)}\left((1-\lambda)\frac{zf'(z)}{f(z)}+\lambda\left(1+\frac{zf''(z)}{f'(z)}\right)\right)\prec\left(\frac{1+az}{1-z}\right)^2+\lambda\left(\frac{(a+1)z}{(1-z)^2}\right),$$

in U, then

$$\frac{zf'(z)}{f(z)}\prec\left(\frac{1+az}{1-z}\right),\forall z\in \mathrm{U}.$$

Then we can verified that the image of the open unit disc U under the function h (i.e) $h(U)$ is the exterior of the parabola given by

$$v^2=-\frac{(1-a\alpha)^2(1+a)}{\alpha(\lambda+1+a)}\left(u-\frac{\alpha\left(a^2-\lambda-a\lambda-1\right)+2(1-a)}{4}\right).$$

Put $\alpha = 1$ in above equation, we get

$$v^2=-\frac{(1-a(1))^2(1+a)}{1(\lambda+1+a)}\left(u-\frac{1\left(a^2-\lambda-a\lambda-1\right)+2(1-a)}{4}\right),$$

$$v^2=-\frac{(1-a)^2(1+a)}{(\lambda+1+a)}\left(u-\frac{\left(a^2-\lambda-a\lambda-1+2-2a\right)}{4}\right),$$

$$v^2=-\frac{(1-a)^2(1+a)}{(\lambda+1+a)}\left(u-\frac{\left(a^2-\lambda(1+a)+1-2a\right)}{4}\right),$$

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$$v^2 = -\frac{(1-a)^2(1+a)}{(\lambda+1+a)}\left(u - \frac{\left(\left(1-2a+a^2\right)-\lambda(1+a)\right)}{4}\right),$$

$$v^2 = -\frac{(1-a)^2(1+a)}{(\lambda+1+a)}\left(u - \frac{\left((1-a)^2-\lambda(1+a)\right)}{4}\right),$$

which has its vertex at the point $\left(\frac{\left((1-a)^2-\lambda(1+a)\right)}{4}, 0\right)$.

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