

Applications of Representation Theory of Symmetric Groups in Voting Theory

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Abstract - We show how representation theory of symmetric groups can be applied in voting theory. This is an algebraic perspective where we view voting profiles as elements of $\mathbb{Q}S_n$. We discuss various election procedures like positional voting procedures, approval voting procedures etc.

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1. INTRODUCTION

Donald Saari has systematically developed a powerful geometric approach to understand, construct, and explain paradoxes that occur in voting ([1]-[4]&[6]-[10]). Saari's geometric approach encodes the collection of votes from an election as a vector, which we call a profile, and as per his approach election procedures are related to linear transformations. By focusing on specific geometric structures in this vector space setting, Saari has resolved many of the difficult combinatorial obstructions associated with voting analysis. In doing so, he focuses on specialized roles played by specific subspaces of the vector space of profiles. His geometric approach has led to an impressive number of unexpected results in voting theory. He refined understanding of many fundamental results in the field. In this paper, we describe an algebraic perspective for viewing profiles as elements of well-studied $\mathbb{Q}S_n$ -modules.

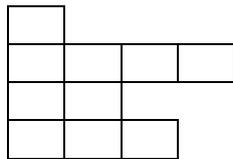
With the help of Theorem 1, we address the relationship between positional voting and approval voting (Theorem 2). By using only simple combinatorial objects like tabloids and some basic ideas from representation theory like Schur's Lemma, we discuss and extend some of Saari's well-known results. For example, we discuss result concerning the important relationship between the Borda count and pairwise voting when voters return full rankings of the candidates (Theorem 6). We discuss extension of this result to a situation in which voters return partial rankings of the candidates (Theorem 9). In the process, we construct an infinite family of "Borda-like" voting procedures.

Approaching the study of voting from an algebraic perspective can be incredibly helpful in understanding many different voting procedures. In fact, we see this paper is a step toward algebraic voting theory. The ideas presented here will be very useful to voting theorists.

2. VOTING ON TABLOIDS

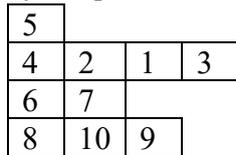
A *composition* of a positive integer n is a sequence $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)$ of positive integers whose sum is n . A *partition* of n is a weakly decreasing sequence $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)$ of positive integers whose sum is n . For example, (4,5,2,3) is a composition of 14 whereas (5,4,3,2) is a composition of 14 which is also a partition of 14.

The *Ferrers diagram* of shape λ is a left justified array of boxes with λ_i boxes in the i^{th} row.



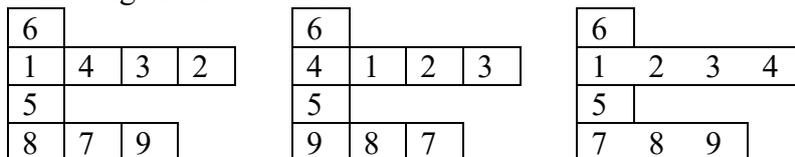
Ferrers diagram of shape (1,4,2,3)

A *Young tableau* of shape λ is obtained by filling the boxes in the Ferrers diagram of shape λ by the positive integers $1, 2, \dots, n$ without repetition.



Young tableau of shape (1,4,2,3)

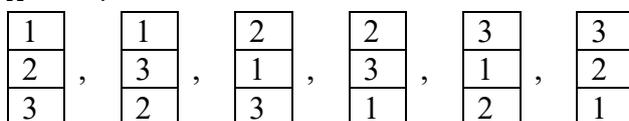
Two Young tableaux are called *equivalent* if their corresponding rows contain the same set of positive integers. A *Young tabloid* of shape λ is an equivalence class of tableau under this relation. We denote tabloid by a representative tableau whose vertical dividers within each row are removed. For convenience, we will select the representative tableau whose entries are in ascending order in each row.



Two equivalent tableaux and their tabloid.

We will denote the set of tabloids of shape λ by X^λ . We can use tabloids to index voting data. For example, suppose there are n candidates in an election say c_1, c_2, \dots, c_n . Asking voters to vote by returning a list of candidates in order of preference, from most to least favoured is equivalent to asking the voter to choose a tabloid from the set $X^{(1,1,\dots,1)}$.

Consider there are 3 candidates. Then each voter must return one of the following tabloids from $X^{(1,1,1)}$:



In this case, a voter who returns the penultimate tabloid prefers c_3 to c_1 to c_2 , whereas the voter who returns first tabloid prefers c_1 to c_2 to c_3 .

The information of voters asked to vote for their favourite candidate can be obtained from the choices they made from $X^{(1,1,\dots,1)}$ by focusing only on top ranked candidates. It may be easier to let them choose from the set $X^{(1,n-1)}$. For example, $n = 5$, then we are asking voters to choose one of the following tabloids from $X^{(1,4)}$.



5			
1	2	3	4

If a voter chooses fourth tabloid above, he/she is saying that his/her favourite candidate is c_4 and he/she is indifferent regarding his/her vote to candidates c_1, c_2, c_3, c_5 .

When voters are asked to provide rankings of candidates by selecting a tabloid from $X^{(1,1,\dots,1)}$, then we say that they are providing *full rankings* of candidates; if they are selecting tabloids from X^λ where $\lambda \neq (1,1, \dots, 1)$ then we say that they are providing *partial rankings* of the candidates.

Suppose we want to determine the winner of an election based on choosing tabloids from X^λ , we must know the number of votes received by each $x \in X^\lambda$.

Let $\mathbf{p}: X^\lambda \rightarrow \mathbb{N}$ be the function such that $\mathbf{p}(x)$ is the number of votes voted for the tabloid x .

This function \mathbf{p} is called a *profile*. Hereafter, we refer to every function $\mathbf{p}: X^\lambda \rightarrow \mathbb{Q}$ a profile. The set M^λ of all functions $\mathbf{p}: X^\lambda \rightarrow \mathbb{Q}$ forms a vector space over \mathbb{Q} where vector addition and scalar multiplication are defined as follows:

$$(\mathbf{p} + \mathbf{q})(x) = \mathbf{p}(x) + \mathbf{q}(x) \ \& \ (\alpha\mathbf{p})(x) = \alpha\mathbf{p}(x) \ \forall \mathbf{p}, \mathbf{q} \in M^\lambda, \forall \alpha \in \mathbb{Q}.$$

The dimension of M^λ is $|X^\lambda|$, & the indicator functions form a basis for M^λ where indicator function on X^λ is defined as follows:

$$\delta_x(y) = \begin{cases} 1 & , y = x \\ 0 & , y \neq x \end{cases}$$

$$\forall x, y \in X^\lambda.$$

We can view a profile $\mathbf{p} \in M^\lambda$ as a formal linear combination of the tabloids in X^λ where the coefficient of the tabloid x is $\mathbf{p}(x)$.

For example: If $n = 3$ and there are 20 voters, then our profile looks like

$$4 \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array} + 0 \begin{array}{|c|} \hline 1 \\ \hline 3 \\ \hline 2 \\ \hline \end{array} + 3 \begin{array}{|c|} \hline 2 \\ \hline 1 \\ \hline 3 \\ \hline \end{array} + 4 \begin{array}{|c|} \hline 2 \\ \hline 3 \\ \hline 1 \\ \hline \end{array} + 3 \begin{array}{|c|} \hline 3 \\ \hline 1 \\ \hline 2 \\ \hline \end{array} + 6 \begin{array}{|c|} \hline 3 \\ \hline 2 \\ \hline 1 \\ \hline \end{array}$$

Where four voters choose the first list, zero voters choose the second list, three voters choose the third list, four voters choose the fourth list, three voters choose the fifth list and six voters choose the sixth list. It is useful to view profiles as formal linear combination of tabloids. We can also view profiles as column vectors in $\mathbb{Q}^{|X^\lambda|}$. For this, we choose the indicator functions of M^λ as a basis and order the tabloids according to lexicographic ordering of the tabloids.

For example: We can encode the above profile as a vector

$$\mathbf{p} = \begin{bmatrix} 4 \\ 0 \\ 3 \\ 4 \\ 3 \\ 6 \end{bmatrix} \begin{array}{l} 1 \ 2 \ 3 \\ 1 \ 3 \ 2 \\ 2 \ 1 \ 3 \\ 2 \ 3 \ 1 \\ 3 \ 1 \ 2 \\ 3 \ 2 \ 1 \end{array}$$

We will refer this example as “*three-candidates and twenty-voters*” in our paper.

3. POSITIONAL VOTING PROCEDURES

After knowing what profiles look like, we turn to voting procedures that assign points to each candidate based on their position in a voter's choice of a tabloid.

Suppose $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)$ is a composition of n . Take a vector $\mathbf{w} = [w_1, w_2, \dots, w_m]^t \in \mathbb{Q}^m$ and suppose a profile $\mathbf{p} \in M^\lambda$ is given to us. The vector \mathbf{w} is called a *weighting vector* and we use it to assign points to each candidate and we declare a candidate *winner* if he/she receives the most points. We do this as follows.

For each tabloid x , if candidate c_i is in row j of x , then the voter will be given $w_j p(x)$ points. The total number of points assigned to candidate c_i will be determined by summing over all tabloids $x \in X^\lambda$. This will be referred as a *positional voting procedure of type λ based on \mathbf{w}* .

For example, let us consider "three-candidates and twenty-voters" example mentioned above. In a positional voting procedure with weighting vector $\mathbf{w} = [1, s, 0]^t$, where $0 \leq s \leq 1$, the candidate c_1 will receive $(4 \times 1) + (6 \times s) + (10 \times 0) = 4 + 6s$ points, the candidate c_2 will receive $(7 \times 1) + (10 \times s) + (3 \times 0) = 7 + 10s$ points, the candidate c_3 will receive $(9 \times 1) + (4 \times s) + (7 \times 0) = 9 + 4s$ points. We encode these points in the result vector $\mathbf{r} = [4 + 6s, 7 + 10s, 9 + 4s]^t$.

We use the parameter s in this example to highlight one of the things that makes voting theory very interesting: For a fixed profile \mathbf{p} , the outcomes of an election can vary rapidly with the choice of an election procedure. See Saari [9].

If $s = 0$ in the above example, we have the well-known plurality voting procedure "vote for your favourite". In this case, c_3 wins the election with the result vector $\mathbf{r} = [4, 7, 9]^t$.

If $s = \frac{1}{2}$, c_2 wins the election with the result vector $\mathbf{r} = [7, 12, 11]^t$.

One beautiful feature of positional voting is that we can realize the result vector as the product of a matrix say $T_{\mathbf{w}}$ and the given profile. The $(i, j)^{th}$ entry of $T_{\mathbf{w}}$ is the number of points awarded to candidate c_i based on his/her position in the j^{th} tabloid.

e.g. If \mathbf{p} and \mathbf{w} are as above, then

$$T_{\mathbf{w}}(\mathbf{p}) = \begin{pmatrix} 1 & 1 & s & 0 & s & 0 \\ s & 0 & 1 & 1 & 0 & s \\ 0 & s & 0 & s & 1 & 1 \end{pmatrix} \begin{bmatrix} 4 \\ 0 \\ 3 \\ 4 \\ 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 4 + 6s \\ 7 + 10s \\ 9 + 4s \end{bmatrix} = \mathbf{r}$$

We can view every positional voting method of type λ based on a weighting vector \mathbf{w} as a linear transformation $T_{\mathbf{w}}: M^\lambda \rightarrow M^{(1, n-1)}$ since we may use the tabloids in $X^{(1, n-1)}$ to index the set of candidates.

e.g. we have $T_{[1 \ s \ 0]^t}: M^{(1,1,1)} \rightarrow M^{(1,2)}$ in above example.

Suppose $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)$ is a composition of n . There are $n!$ Tableaux of shape λ . We can think of a tableau as a full ranking of the candidates by reading the entries of a tableau from left to right, top to bottom.

e.g. The tableau

1	4
3	5
2	

 Corresponds to the full ranking

1
4
3
5
2

This is useful, because we can view a vote for the tabloid

1	4
3	5
2	

as coming from any tableau whose full ranking of the candidates corresponds to one of the tableaux in the equivalence class of the above tabloid.

e.g. Above tabloid corresponds to any of the following full rankings:

1	1	4	4
4	4	1	1
3	5	3	5
5	3	5	3
2	2	2	2

We can view profiles $\mathbf{p} \in M^\lambda$ as elements of $M^{(1,1,\dots,1)}$ in the following manner.

Let f denote the number of full rankings corresponding to a tabloid of shape λ .

Define $\iota: M^\lambda \rightarrow M^{(1,1,\dots,1)}$ (think like inclusion map) by mapping each tabloid (i.e. each indicator function) to the $\frac{1}{f}$ times sum of its corresponding full rankings.

e.g. the above tabloid will be mapped to

$$\frac{1}{4} \left(\begin{array}{|c|} \hline 1 \\ \hline 4 \\ \hline 3 \\ \hline 5 \\ \hline 2 \\ \hline \end{array} + \begin{array}{|c|} \hline 1 \\ \hline 4 \\ \hline 5 \\ \hline 3 \\ \hline 2 \\ \hline \end{array} + \begin{array}{|c|} \hline 4 \\ \hline 1 \\ \hline 3 \\ \hline 5 \\ \hline 2 \\ \hline \end{array} + \begin{array}{|c|} \hline 4 \\ \hline 1 \\ \hline 5 \\ \hline 3 \\ \hline 2 \\ \hline \end{array} \right)$$

Let $\mathbf{p} \in M^\lambda$, and define $\bar{\mathbf{p}} = \iota(\mathbf{p})$. Equivalently, we can view each profile $\mathbf{p} \in M^\lambda$ as a profile $\bar{\mathbf{p}} \in M^{(1,1,\dots,1)}$ that is constant on the equivalence classes that form the tabloids of shape λ .

Corresponding to weighting vector $\mathbf{w} = [w_1, w_2, \dots, w_m]^t$ associated to λ , we define $\bar{\mathbf{w}}$ to be the weighting vector in \mathbb{Q}^n whose first λ_1 entries are equal to w_1 , whose next λ_2 entries are equal to w_2 , and so on. Then we have $T_{\mathbf{w}}(\mathbf{p}) = T_{\bar{\mathbf{w}}}(\bar{\mathbf{p}})$.

e.g. Suppose $\lambda = (2,1)$, $\mathbf{w} = [2,0]^t$, and

$$\mathbf{p} = 6 \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \hline \end{array} + 7 \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline \end{array} + 8 \begin{array}{|c|c|} \hline 2 & 3 \\ \hline 1 & \\ \hline \end{array}$$

Then $\bar{\mathbf{w}} = [2,2,0]^t$,

$$\bar{\mathbf{p}} = 3 \left(\begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array} + \begin{array}{|c|} \hline 2 \\ \hline 1 \\ \hline 3 \\ \hline \end{array} \right) + \frac{7}{2} \left(\begin{array}{|c|} \hline 1 \\ \hline 3 \\ \hline 2 \\ \hline \end{array} + \begin{array}{|c|} \hline 3 \\ \hline 1 \\ \hline 2 \\ \hline \end{array} \right) + 4 \left(\begin{array}{|c|} \hline 2 \\ \hline 3 \\ \hline 1 \\ \hline \end{array} + \begin{array}{|c|} \hline 3 \\ \hline 2 \\ \hline 1 \\ \hline \end{array} \right)$$

$$\text{And } T_{\mathbf{w}}(\mathbf{p}) = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & 2 \end{pmatrix} \begin{bmatrix} 6 \\ 7 \\ 8 \end{bmatrix} = \begin{pmatrix} 2 & 2 & 2 & 0 & 2 & 0 \\ 2 & 0 & 2 & 2 & 0 & 2 \\ 0 & 2 & 0 & 2 & 2 & 2 \end{pmatrix} \begin{bmatrix} 3 \\ 7 \\ 2 \\ 3 \\ 4 \\ 7 \\ 2 \\ 4 \end{bmatrix} = T_{\bar{\mathbf{w}}}(\bar{\mathbf{p}}).$$

4. REPRESENTATION THEORY AND THE SYMMETRIC GROUP

Now we can view profiles as vectors and each positional voting procedure as a linear transformation. Now we will show that each map $T: M^\lambda \rightarrow M^{(1,n-1)}$ is a $\mathbb{Q}S_n$ -module homomorphism. Let us explain this. Let λ be a composition of n . S_n acts naturally on the set X^λ of tabloids of shape λ by permuting the entries of the tabloids.

e.g. if $n = 6$ and $\sigma = (15)(2346)$, then

$$\sigma \begin{array}{|c|c|c|c|} \hline 1 & 4 & 5 & 6 \\ \hline 2 & 3 & & \\ \hline \end{array} = \begin{array}{|c|c|c|c|} \hline \sigma(1) & \sigma(4) & \sigma(5) & \sigma(6) \\ \hline \sigma(2) & \sigma(3) & & \\ \hline \end{array} = \begin{array}{|c|c|c|c|} \hline 5 & 6 & 1 & 2 \\ \hline 3 & 4 & & \\ \hline \end{array}$$

We can extend the action of S_n on X^λ to action of S_n on profile space M^λ by defining $(\sigma \mathbf{p})(x) = \mathbf{p}(\sigma^{-1}x)$. Again, we can extend this action to an action of the group ring $\mathbb{Q}S_n$ on M^λ . Here, if $a = \sum_{\sigma \in S_n} a_\sigma \sigma$, then $(a\mathbf{p})(x) = \sum_{\sigma \in S_n} a_\sigma \mathbf{p}(\sigma^{-1}x)$.

The action of $\mathbb{Q}S_n$ on M^λ turns M^λ into a $\mathbb{Q}S_n$ -module. We will describe implications of this realization as far as voting is concerned.

Note that, for each element $a \in \mathbb{Q}S_n$, \exists a linear transformation $L_a: M^\lambda \rightarrow M^\lambda$ defined by $L_a(\mathbf{p}) = a\mathbf{p}$.

Let $End(M^\lambda)$ be the ring of linear transformations from M^λ to M^λ . Then this defines a ring homomorphism $\rho: \mathbb{Q}S_n \rightarrow End(M^\lambda)$. The homomorphism ρ is a representation of $\mathbb{Q}S_n$ since each element $a \in \mathbb{Q}S_n$ may be represented by a linear transformation $\rho(a) = L_a \in End(M^\lambda)$. If we restrict ρ to elements of S_n , then the images are all invertible linear transformations. i.e. We have a representation $\rho \downarrow_{S_n}: S_n \rightarrow Aut(M^\lambda) = GL(M^\lambda)$.

M^λ is a $\mathbb{Q}S_n$ -module tells us that we can write M^λ as a direct sum of $\mathbb{Q}S_n$ -modules. The decomposition $M^\lambda = M_1 \oplus M_2 \oplus \dots \oplus M_k$ holds such that for all $a \in \mathbb{Q}S_n$, and $\mathbf{p}_i \in M_i$, $a\mathbf{p}_i \in M_i$.

A submodule U of module M is called simple if $U \neq \{\mathbf{0}\}$ and the only submodules of U are $\{\mathbf{0}\}$ and U itself. Upto isomorphism, there are only a finite number of distinct $\mathbb{Q}S_n$ -modules. These simple modules can be indexed by partitions of n . Here if μ is a partition of n , then we denote corresponding simple module by S^μ .

We have $M^{(1,2)} \cong S^{(3)} \oplus S^{(2,1)}$ and $M^{(1,1,1)} \cong S^{(3)} \oplus S^{(2,1)} \oplus S^{(2,1)} \oplus S^{(1,1,1)}$.

Consider the following decomposition of $M^{(1,2)}$:

$$M^{(1,2)} = \left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right) \oplus \left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right)$$

These subspaces are invariant under the action of $\mathbb{Q}S_3$. The first subspace, the span of all ones vector is an invariant under the action of $\mathbb{Q}S_3$ because the entries in this vector are all equal and permuting the entries of such vector keeps the vector unchanged. The second subspace is the orthogonal complement of the first subspace with respect to the usual dot product. This is the subspace of vectors whose entries sum to zero. Since sum of permuted entries will also be zero, it is also a submodule of $M^{(1,2)}$. Both subspaces are simple $\mathbb{Q}S_3$ -modules. The first space is isomorphic to $S^{(3)}$. In general, $S^{(n)}$ denote this trivial one dimensional $\mathbb{Q}S_n$ -module and every element of S_n acts as identity linear transformation on this space. The second space is isomorphic to $S^{(2,1)}$, in general, $S^{(n-1,1)}$ is the $(n-1)$ -dimensional simple module which is isomorphic to the orthogonal complement of the all-ones vector.

We have $M^{(1,n-1)} \cong M^{(n-1,1)}$ as $\mathbb{Q}S_n$ -modules. (see [11] for detailed description).

Now we relate this to positional voting. Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)$ be a composition of n , let $\mathbf{w} = [w_1, w_2, \dots, w_m]^t$ be a weighting vector and consider the positional voting procedure

$T_w: M^\lambda \rightarrow M^{(1,n-1)}$ of type λ based on w . Note that the linear transformation T_w is a $\mathbb{Q}S_n$ -module homomorphism. i.e. if $a \in \mathbb{Q}S_n$, then $T_w(ap) = aT_w(p)$.

If $\sigma \in S_n$, we are simply asking that $T_w(\sigma p) = \sigma T_w(p)$ which is same as saying that “if we permute the labels of the candidates then we have to only permute the original scores accordingly.” This notion is called *neutrality* (see e.g. [18]). We extend this notion to the action of the entire group ring.

Viewing T_w as a $\mathbb{Q}S_n$ -module homomorphism is helpful.

If $T: M \rightarrow N$ is a module homomorphism, then $\text{Ker}(T)$ is a submodule of M , and the image of T is a submodule of N . If we define the effective space $E(T)$ of T to be the orthogonal complement with respect to the usual dot product of the kernel of T , then $E(T) \cong T(M)$ as modules.

Knowing about $E(T_w)$, might be useful to say something about T_w .

Now let us see important result about module homomorphisms.

Schur’s Lemma: Every nonzero module homomorphism between simple modules is an isomorphism.

Let us make use of Schur’s Lemma in study of voting. Consider positional voting procedure $T_w: M^\lambda \rightarrow M^{(1,n-1)}$. As discussed earlier, the module $M^{(1,n-1)}$ is isomorphic to the direct sum of simple modules $S^{(n)}$ and $S^{(n-1,1)}$. This means that if simple submodule U of M^λ is not isomorphic to $S^{(n)}$ or $S^{(n-1,1)}$ then it must be in the kernel of T_w , i.e. such a submodule contains the information which will have no effect on the results of the election.

For example, if $n = 3$, and we have $T_w: M^{(1,1,1)} \rightarrow M^{(1,2)}$, then since $M^{(1,2)} \cong S^{(3)} \oplus S^{(2,1)}$ and $M^{(1,1,1)} \cong S^{(3)} \oplus S^{(2,1)} \oplus S^{(2,1)} \oplus S^{(1,1,1)}$ we know that the kernel of T_w must contain exactly one copy of $S^{(1,1,1)}$ and $S^{(2,1)}$.

Note that there is a bijection between full rankings in $X^{(1,1,\dots,1)}$ and permutations in S_n , where we map the tabloid

i_1
i_2
\vdots
i_{n-1}
i_n

to the permutation σ with property $\sigma(j) = i_j$.

We may view each profile $p \in M^{(1,1,\dots,1)}$ as an element of $\mathbb{Q}S_n$. Here, we replace each full ranking with its associated permutation. This means that if $p \in M^\lambda$ then we may view $\bar{p} \in M^{(1,1,\dots,1)}$ as an element of $\mathbb{Q}S_n$ which is constant on left cosets of subgroup of S_n that fixes the tabloid containing the tableau corresponding to the identity of S_n .

Our positional voting procedures are $\mathbb{Q}S_n$ -module homomorphisms as well as the results of profiles acting on weighting vectors. If $p \in \mathbb{Q}S_n$ is a profile and $w \in \mathbb{Q}^n \cong M^{(1,n-1)}$ is a weighting vector, then $T_w(p) = pw$.

Note that in our “three-candidates and twenty-voters” example above, if $e \in S_3$ is the identity and if we use usual cycle notation for other element of S_3 , then

$$\begin{aligned}
 T_w(\mathbf{p}) &= \begin{pmatrix} 1 & 1 & s & 0 & s & 0 \\ s & 0 & 1 & 1 & 0 & s \\ 0 & s & 0 & s & 1 & 1 \end{pmatrix} \begin{bmatrix} 4 \\ 0 \\ 3 \\ 4 \\ 3 \\ 6 \end{bmatrix} \\
 &= 4 \begin{bmatrix} 1 \\ s \\ 0 \end{bmatrix} + 3 \begin{bmatrix} s \\ 1 \\ 0 \end{bmatrix} + 4 \begin{bmatrix} 0 \\ 1 \\ s \end{bmatrix} + 3 \begin{bmatrix} s \\ 0 \\ 1 \end{bmatrix} + 6 \begin{bmatrix} 0 \\ s \\ 1 \end{bmatrix} \\
 &= (4e + 3(12) + 4(123) + 3(132) + 6(13)) \begin{bmatrix} 1 \\ s \\ 0 \end{bmatrix} \\
 &= \mathbf{pw}.
 \end{aligned}$$

Let us make a few observations about weighting vectors whose entries sum to zero. We know that $M^{(1,n-1)} \cong S^{(n)} \oplus S^{(n-1,1)}$. Therefore, we can write $\mathbf{w} = \mathbf{1}_w + \hat{\mathbf{w}}$, where $\mathbf{1}_w \in S^{(n)}$ is the projection of \mathbf{w} onto the all-ones vector $\mathbf{1}$ and $\hat{\mathbf{w}} \in S^{(n-1,1)}$ is the projection of \mathbf{w} onto the orthogonal $(n-1)$ -dimensional subspace of vectors whose entries sum to zero. e.g. if $n = 3$ and $\mathbf{w} = [1, s, 0]^t$, then

$$\mathbf{1}_w = \left[\frac{1+s}{3}, \frac{1+s}{3}, \frac{1+s}{3} \right]^t \text{ and } \hat{\mathbf{w}} = \left[\frac{2-s}{3}, \frac{2s-1}{3}, \frac{-1-s}{3} \right]^t.$$

Since $T_w(\mathbf{p}) = \mathbf{pw} = \mathbf{p}(\mathbf{1}_w + \hat{\mathbf{w}}) = \mathbf{p}\mathbf{1}_w + \mathbf{p}\hat{\mathbf{w}}$ and $\mathbf{p}\mathbf{1}_w \in S^{(n)}$, the summand $\mathbf{p}\hat{\mathbf{w}}$ contains all of the information that will determine the outcome of the election. $\mathbf{p}\mathbf{1}_w$ is simply some multiple of all-ones vector and will therefore not differentiate between any of the candidates. Therefore, we will focus on weighting vectors $\mathbf{w} \in M^{(1,n-1)} = \mathbb{Q}^n$ such that $\mathbf{w} = \hat{\mathbf{w}}$, i.e. weighting vectors whose entries sum to zero. A vector in \mathbb{Q}^n whose entries sum to zero is called a sum-zero vector. We use a sum-zero weighting vector \mathbf{w} because the results vector $\mathbf{r} = T_w(\mathbf{p})$ will also be the sum-zero vector. Here, winner is still the candidate who receives the most points.

Theorem 1: "Let $n \geq 2$, and let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)$ be a partition of n . Suppose that $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_k$ form a linearly independent set of weighting vectors in \mathbb{Q}^n such that $\overline{\mathbf{w}}_1, \overline{\mathbf{w}}_2, \dots, \overline{\mathbf{w}}_k$ are sum-zero vectors. If $\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_k$ are any sum-zero results vectors in \mathbb{Q}^n , then there exist infinitely many profiles $\mathbf{p} \in M^\lambda$ such that $T_{\mathbf{w}_i}(\mathbf{p}) = \mathbf{r}_i$ for all i such that $1 \leq i \leq k$."(see Theorem 1 in [5])

5. APPROVAL VOTING

In the approval voting procedure, we ask a voter to return an unordered list of the candidates to whom voter approves. A candidate receives a point for each time he/she appears on such a list, and the candidate who receives the most points is declared winner.

We assume that if a voter has to return a fully ranked list of the candidates, then the candidates whom our voter approves of would form the top portion of voter's list. We therefore, imagine a situation in which we ask each voter to return a fully ranked list of the candidates together with a cutoff point. Candidates above the cutoff are approved by our voter and those below the cutoff are not. We denote the cutoff point in a tableau with a blank space separating the approved candidates from other candidates.

In approval voting, the voter may approve all of the candidates or none of the candidates but such a preference will have no impact on the outcome of the election. Therefore, we assume that voters approve at least one but not all of the candidates.

1	1	2	2	3	3
2	3	1	3	1	2
3	2	3	1	2	1
1	1	2	2	3	3
2	3	1	3	1	2
3	2	3	1	2	1

Tableaux with cutoffs for approval voting
 In approval voting, our ranked profile looks like

$$p = \begin{matrix} p_1 \\ p_2 \\ \vdots \\ p_{n-1} \end{matrix}$$

where p_i corresponds to those voters who approved i candidates. Approval voting can be considered like it is made up of several positional voting systems simultaneously.

For example, consider the weighting vector $a_i = \{1,1, \dots, 1,0,0, \dots, 0\}$ with i ones and $n - i$ zeros. The sum-zero portion of results of the election using approval voting is given by

$$p_1 \hat{a}_1 + p_2 \hat{a}_2 + \dots + p_{n-1} \hat{a}_{n-1} = r_{app}$$

Whereas for positional voting with respect to weighting vector w , the sum-zero portion of the results is given by $(p_1 + p_2 + \dots + p_{n-1}) \hat{w} = r_{pos}$.

Theorem 2 (see theorem 2 in [5]): “Let $n \geq 3$, let r_{app} and r_{pos} be any two sum-zero results vectors in \mathbb{Q}^n , let w be any nontrivial sum-zero weighting vector in \mathbb{Q}^n . Then there exist infinitely many ranked approval profiles

$$p = \begin{matrix} p_1 \\ p_2 \\ \vdots \\ p_{n-1} \end{matrix}$$

Such that the approval voting outcome of p is r_{app} and the positional voting outcome with respect to w is r_{pos} ”.

6. Equivalent Weighting Vectors and Effective Spaces

Now, let us put an equivalence relation on weighting vectors. Two weighting vectors are said to be equivalent iff they provide the same ordinal rankings for all profiles.

Now let us focus on fully ranked situation. We denote the all-ones vector in \mathbb{Q}^n by $\mathbf{1}$. Two weighting vectors \mathbf{w} and \mathbf{x} are said to be equivalent iff $\exists \alpha, \beta \in \mathbb{Q}$ such that $\alpha > 0$ and $\mathbf{x} = \alpha\mathbf{w} + \beta\mathbf{1}$.

This definition of equivalence relation is inspired by the fact that for any positive rational number α , \mathbf{w} is equivalent to $\alpha\mathbf{w}$ since, $\forall \mathbf{p} \in M^{(1,1,\dots,1)}$, the ordinal rankings provided by $T_{\mathbf{w}}(\mathbf{p})$ and $T_{\alpha\mathbf{w}}(\mathbf{p})$ are same because all the entries in $T_{\alpha\mathbf{w}}(\mathbf{p})$ are α times the entries of $T_{\mathbf{w}}(\mathbf{p})$.

If $\mathbf{x} = \alpha\mathbf{w} + \beta\mathbf{1}$, where $\alpha, \beta \in \mathbb{Q}$ and $\alpha > 0$, the ordinal rankings provided by $T_{\mathbf{x}}(\mathbf{p})$ and $T_{\mathbf{w}}(\mathbf{p})$ are same because the addition of $\beta\mathbf{1}$ to $\alpha\mathbf{w}$ changes each candidate's score by exactly the same amount.

$\mathbf{w} \sim \mathbf{x}$ iff there is a positive rational number $\gamma \in \mathbb{Q}$ such that $\widehat{\mathbf{w}} = \gamma\widehat{\mathbf{x}}$ i.e. the sum-zero component of \mathbf{w} is a positive multiple of the sum-zero component of \mathbf{x} . Two weighting vectors \mathbf{w} and \mathbf{x} will give the same outcome iff $\mathbf{w} \sim \mathbf{x}$.

Theorem3:(Theorem 2.3.1 in [6]) "Let $n \geq 2$, and \mathbf{w} and \mathbf{x} be weighting vectors in \mathbb{Q}^n . The ordinal rankings of $T_{\mathbf{w}}(\mathbf{p})$ and $T_{\mathbf{x}}(\mathbf{p})$ will be the same $\forall \mathbf{p} \in M^{(1,1,\dots,1)}$ iff $\mathbf{w} \sim \mathbf{x}$."

It is beneficial to view theorem 3 in terms of effective spaces as is done by Saari (see,[7]- [10]).

The effective space $E(T)$ of T is the orthogonal complement $Ker(T)^\perp$ of kernel of T . If \mathbf{w} is a weighting vector we denote the effective space of $T_{\mathbf{w}}$ by $E(\mathbf{w})$ instead of $E(T_{\mathbf{w}})$. Being a $\mathbb{Q}S_n$ -submodule of the profile space $\mathbb{Q}S_n$, the effective space $E(\mathbf{w})$ of any nontrivial sum-zero vector \mathbf{w} is isomorphic to $S^{(n-1,1)}$. If $\mathbf{w} \in \mathbb{Q}^n$ has a nontrivial projection onto the all-ones vector and $\widehat{\mathbf{w}} \neq 0$, then $E(\mathbf{w}) \cong S^{(n)} \oplus S^{(n-1,1)}$. If \mathbf{w} is a non-zero multiple of all-ones vector then $E(\mathbf{w}) \cong S^{(n)}$.

Theorem4. (Theorem 4 in [5]) Two nontrivial sum-zero weighting vectors \mathbf{w} and \mathbf{x} share the same effective space iff $\mathbf{w} \sim \mathbf{x}$ or $\mathbf{w} \sim -\mathbf{x}$; otherwise, their effective spaces intersect only at $\{\mathbf{0}\}$.

\mathbf{w} is orthogonal to \mathbf{x} if the dot product of \mathbf{w} and \mathbf{x} is zero. We denote this by $\mathbf{w} \perp \mathbf{x}$.

Let U and W be subspaces of a vector space V such that $u \perp w \forall u \in U, w \in W$ then we write $U \perp W$.

We can view permutations $\sigma \in S_n$ as tableau in $M^{(1,1,\dots,1)}$. e.g. the permutation (135)(24) corresponds to the tableau

3
4
5
2
1

in $M^{(1,1,\dots,1)}$. With respect to the permutation σ , the candidate j occupies the position given by $\sigma^{-1}(j)$. If $\mathbf{w} = [w_1, w_2, \dots, w_n]^t$ is a weighting vector then the candidate j will receive $w_{\sigma^{-1}(j)}$ points.

Theorem 5 (see Theorem 5 in [5]):

Two sum-zero weighting vectors have orthogonal effective spaces if and only if the vectors themselves are orthogonal under the dot product.

7. THE BORDA COUNT

The Borda Count is the positional voting procedure for n candidates that uses the weighting vector $\mathbf{w} = [n - 1, n - 2, \dots, 2, 1, 0]^t$.

Consider the Copeland method for running an election. This is based on information regarding head-to-head contests between the candidates. For each candidate c_i , let $l(i)$ and $w(i)$ denote the number of head-to-head contests lost and won respectively, by c_i . In Copeland's method, winner is the candidate whose difference $w(i) - l(i)$ is largest. *Condorcet winner* is a candidate who defeats all other candidates in head-to-head contests.

In Copeland method, all of the results can be obtained from the image of a map $P: M^{(1,1,\dots,1)} \rightarrow M^{(1,1,n-2)}$ which is called the pairs map. Pairs map obtains all of the necessary information regarding pairs of candidates in head-to-head contests. P has the defining characteristic that it maps a tabloid \mathbf{u} in $X^{(1,1,\dots,1)}$ to the sum of all tabloids in $X^{(1,1,n-2)}$ in which the candidates ranked first and second are ranked in the same order as in \mathbf{u} .

e.g. suppose $n = 4$, then the image of the tabloid

$$\begin{array}{|c|} \hline 4 \\ \hline 3 \\ \hline 2 \\ \hline 1 \\ \hline \end{array} \text{ is } \begin{array}{|c|} \hline 4 \\ \hline 3 \\ \hline 1 \ 2 \\ \hline \end{array} + \begin{array}{|c|} \hline 4 \\ \hline 2 \\ \hline 1 \ 3 \\ \hline \end{array} + \begin{array}{|c|} \hline 4 \\ \hline 1 \\ \hline 2 \ 3 \\ \hline \end{array} + \begin{array}{|c|} \hline 3 \\ \hline 2 \\ \hline 1 \ 4 \\ \hline \end{array} + \begin{array}{|c|} \hline 3 \\ \hline 1 \\ \hline 2 \ 4 \\ \hline \end{array} + \begin{array}{|c|} \hline 2 \\ \hline 1 \\ \hline 3 \ 4 \\ \hline \end{array}$$

For given profile $\mathbf{p} \in M^{(1,1,\dots,1)}$, the scores for Copeland method can be determined from the images of \mathbf{p} under the pairs map P . We need coefficients of $P(\mathbf{p})$ to determine the winner of each head-to-head contests.

We want to find whether there is any relationship between a map $T_{\mathbf{w}}$, and the pairs map P . Let $T: V \rightarrow W$ and $T': V \rightarrow U$ be linear transformations defined on the vector space V . T' is said to be recoverable from T if \exists a linear transformation $R: W \rightarrow U$ such that $T' = R \cdot T$.

T' is recoverable from T iff $\text{Ker}(T) \subseteq \text{Ker}(T')$ which occurs iff $E(T') \subseteq E(T)$.

Let us find those weighting vectors $\mathbf{w} \in \mathbb{Q}^n$ for which $T_{\mathbf{w}}$ is recoverable from P . For this we will focus on effective spaces of positional voting procedures and pairs map P .

We know that the pairs map $P: M^{(1,1,\dots,1)} \rightarrow M^{(1,1,n-2)}$ is a $\mathbb{Q}S_n$ -module homomorphism. Now we will make use of Schur's Lemma. Now let us turn our attention to effective spaces of P . The codomain $M^{(1,1,n-2)}$ of pairs map P has following decomposition into simple submodules:

$$M^{(1,1,n-2)} \cong S^{(n)} \oplus S^{(n-1,1)} \oplus S^{(n-1,1)} \oplus S^{(n-2,2)} \oplus S^{(n-2,1,1)}.$$

$E(P) \cong S^{(n)} \oplus S^{(n-1,1)} \oplus S^{(n-2,1,1)}$ is decomposition of effective space of P . Since there is only one copy of $S^{(n-1,1)}$ in decomposition of $E(P)$ it follows that there are at most two equivalence classes of weighting vectors whose effective spaces are contained in $E(P)$. Such equivalence classes contain Borda Count and its negative.

Theorem 6 (Theorem 6 in [5]): “Let $n \geq 2$, and let $\mathbf{w} \in \mathbb{Q}^n$ be a nontrivial weighting vector. The map $T_{\mathbf{w}}$ is recoverable from the pairs map P iff \mathbf{w} or $-\mathbf{w}$ is equivalent to the Borda Count”.

8. ANALOGOUS TO THE BORDA COUNT

If voters are asked to return fully ranked ballots, then the Borda Count and its negative are unique nontrivial positional voting procedures that are recoverable from the pairs map P . In this section, we will study the situations in which voters do not return fully ranked ballots or it is infeasible to ask voters to rank all of the candidates.

Consider “rank-only-your- $tp-k$ ” situation where $\lambda = (1, 1, \dots, 1, n - k) = (1^k, n - k)$. Now we want to generalize the pairs map $P: M^{(1,1,\dots,1)} \rightarrow M^{(1,1,n-2)}$ and theorem 6 to the “rank-only-your- $tp-k$ ” situation. Here, we get more than one “Borda-like” equivalence class of weighting vectors.

We generalize the pairs map P as follows:

Let $0 \leq \tau \leq 1$, and define $P_{\tau}^k: M^{(1^k, n-k)} \rightarrow M^{(1,1,n-2)}$ as we defined pairs map P in the full ranking case, except now, if two candidates c_i and c_j are tied for last place, then we assign both of the ordered pairs (c_i, c_j) and (c_j, c_i) the value τ . By letting τ be a parameter, we are able to consider simultaneously an infinite number of the pairs map P .

For example, suppose $n = 4$ and $\lambda = (1, 1, 2)$. Then the image of the tabloid

1
2
3 4

is

$$\begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 3 & 4 \\ \hline \end{array} + \begin{array}{|c|} \hline 1 \\ \hline 3 & 4 \\ \hline 2 & 4 \\ \hline \end{array} + \begin{array}{|c|} \hline 1 \\ \hline 4 & 3 \\ \hline 2 & 3 \\ \hline \end{array} + \begin{array}{|c|} \hline 2 \\ \hline 3 & 4 \\ \hline 1 & 4 \\ \hline \end{array} + \begin{array}{|c|} \hline 2 \\ \hline 4 & 3 \\ \hline 1 & 3 \\ \hline \end{array} + \tau \begin{array}{|c|} \hline 3 \\ \hline 4 & 2 \\ \hline 1 & 2 \\ \hline \end{array} + \tau \begin{array}{|c|} \hline 4 \\ \hline 3 & 2 \\ \hline 1 & 2 \\ \hline \end{array}$$

Now we will see for which partial weighting vectors $\mathbf{w} = [w_1, w_2, \dots, w_{k+1}]^t$, $T_{\mathbf{w}}$ is recoverable from P_{τ}^k . Define $\mathbf{b} = [b_1, b_2, \dots, b_{k+1}]^t$ as the partial weighting vector corresponding to $\lambda = (1^k, n - k)$ where $b_i = n - i$ and $b_{k+1} = \frac{(n-k-1)}{2}$.

We define $\mathbf{b}^{\tau} = [b_1^{\tau}, b_2^{\tau}, \dots, b_{k+1}^{\tau}]$ where $b_i^{\tau} = b_i$ for $1 \leq i \leq k$ and $b_{k+1}^{\tau} = 2\tau b_{k+1}$.

If $\tau = \frac{1}{2}$, then we get $\mathbf{b} = \mathbf{b}^{\tau}$.

Here $T_{\mathbf{b}}$ and $T_{\mathbf{b}^{\tau}}$ are recoverable from P_{τ}^k .

Proposition 7: “The weighting vectors $\bar{\mathbf{b}}$ and $\bar{\mathbf{b}}^{\tau}$ are equivalent iff $\tau = \frac{1}{2}$.” (see proposition 7 in [5]).

Proposition 8: “If $\tau = \frac{1}{2}$, then the image of $P_{\tau}^k: M^{(1^k, n-k)} \rightarrow M^{(1,1,n-2)}$ contains exactly one copy of the simple module $S^{(n-1,1)}$.” (see proposition 8 in [5]).

Theorem 9: “Let w be a partial weighting vector with respect to $\lambda = (1^k, n - k)$ where $1 \leq k \leq n - 2$. The positional map T_w is recoverable from the map P_τ^k iff \bar{w} is a linear combination of \widehat{b} and $\widehat{b^\tau}$ ”. (see theorem 9 in [5]).

9. CONCLUSION

Representation theory of symmetric groups has applications in different voting procedures like positional voting procedures, approval voting procedures and it can be used to decide the winner of an election.

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