Applications Of Meromorphic Univalent Functions Associated With Differential Subordination

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Abstract

In this paper authors introduced subclasses $D_p^*(\alpha,\beta)$ of meromorphic univalent functions in the punctured unit disk $D^* = \{z: 0 < |z| < 1\}=D\backslash\{0\}$. By using the method of differential subordinations, we derive some certain properties of meromorphically univalent functions.

Key Words *Analytic function, Subordination, Meromorphically univalent function, convolution.*

1. Introduction

Let S denote the class of functions of the form:

(1) $f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n$,

Which are analytic and univalent in the punctured unit disc D.

$D^* = \{z: 0 < |z| < 1\}=D\backslash\{0\}$.

Let $f(z)$ and $g(z)$ be analytic in D, then, we say that $f(z)$ is subordinate to $g(z)$ in D.

Where $f(z) \prec g(z)$, if there exists an analytic function $h(z)$ in D, such that $|h(z)| \le |z|$ and $f(z) = g[h(z)]$, $(z \in D)$. If $g(z)$ is univalent in D then the subordination $f(z) \prec g(z)(D) \Leftrightarrow f(0) = g(0)$ and $f(D) \subset g(D)$.

Let $q(z) = 1 + q_1 z + q_2 z + \cdots\cdots$ be analytic in D, $\left(-\frac{1}{2} \le \beta < \alpha \le \frac{1}{2}\right)$ such that

(2) $q(z) \prec \frac{1+2\alpha z}{1+2\beta z}$ $(z \in D)$ If and only if

(3) $\left|q(z) - \frac{1-4\alpha\beta}{1-4\beta^2}\right| < \frac{2(\alpha-\beta)}{1-4\beta^2}$, $\left(-\frac{1}{2} \le \beta < \alpha \le \frac{1}{2}\right)$.

(4) $Re\ q(z) > \frac{1-2\alpha}{2}$, $(2\beta = -1,\ z \in D)$.

Several authors recently proved some interesting properties of meromorphically univalent functions. In the present topic, we are going to prove some subordination properties for the class S.

When $g(z) = f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} b_n z^n$, We define the Hadamard product (convolution) of f(z) and g(z) by

$(f * g)(z) = f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n b_n z^n$.

Where $(m \in N_0 = N \cup \{0\}, z \in D)$.

We define a linear operator by

$\Omega_1^m(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon)f(z)$

$= \frac{1}{z} + \sum_{n=0}^{\infty}\left[\frac{\sigma(\xi+\eta)(1+n)}{\varepsilon(\delta-\upsilon)} + 1\right]^m a_n z^n$

$= \left(\psi_{\sigma,\xi,\eta,l}^{1,m} * f\right)(z)$.

Where

$\psi_{\sigma,\xi,\eta,\varepsilon,\delta,\upsilon}^{1,m}(z) = \frac{1}{z} + \sum_{n=0}^{\infty}[\ell(1+n)+1]^m z^n$

& $\begin{pmatrix} a > 0, m \in N_0 = N \cup \{0\}, z \in D, \xi \ge 0, \eta \ge 0, \\ 0 < \sigma \le \frac{1}{2}, 0 < \varepsilon \le \frac{1}{2}, \delta > 0, \upsilon \ge 0, \delta > \upsilon \end{pmatrix}$.

For simplicity throughout the paper we are using $\Omega_1^m(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon)f(z) = \Omega f(z)$

and $\ell = \frac{\sigma(\xi+\eta)}{\varepsilon(\delta-\upsilon)} \ne 0$.

It is easy to verify that

(5) $\sigma(\xi+\eta)z[\Omega_1^m(\sigma,\eta,\xi,l)f(z)]' = \Omega f(z)$
$-[\sigma(\xi+\eta)+l]\Omega f(z)$.

We note that

$\Omega_1^0(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon)f(z) = f(z)$ and

$\Omega_1^1\left(\frac{1}{2},1,1,1,\frac{1}{2},1\right)f(z) = \frac{[z^2 f(z)]'}{z}$
$= 2f(z) + zf'(z)$.

The above operators are analytic in D and satisfy the following condition

$Re\{q(z)\} > 0, \quad (z \in D).$

for $k \in N, \epsilon_k = exp\left(\frac{2\pi i}{k}\right),$

(6) $f_{1,k}^{m}[\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;z]$

$= \frac{1}{k}\sum_{j=0}^{k-1}\epsilon_k^{j,1}[\Omega f(z)](\epsilon_k^{j}z)$

$= \frac{1}{z} + \cdots \qquad (f \in S).$

$g_{1,k}^{m}[\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;z]$

$= \frac{1}{2}[\Omega f(z)] + \overline{\Omega f(\bar{z})}$

$= \frac{1}{z} + \cdots \qquad (g \in S).$

$h_{1,k}^{m}[\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;z]$

$= \frac{1}{2}[\Omega f(z)] + \overline{\Omega f(-\bar{z})} = \frac{1}{z} + \cdots \qquad (h \in S).$

for $k = 1$ we have,

$f_{1,1}^{m}[\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;z] = \Omega f(z).$

We now introduce and investigate the following subclasses of the class S of meromorphically univalent functions.

1.1. Definitions

Definition 1.1.1 A function $f \in S$ is said to be in the class $F_{1,k}^{m}(\propto;\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;z)$ if it satisfies the following subordination condition:

$$-\frac{z\left[(1+\propto)(\Omega f(z))'(z)+\propto\left(\Omega_1^{m+1}(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon)f\right)'(z)\right]}{p\left[(1+\propto)f_{1,k}^{m}(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;z)+\propto f_{1,1}^{m+1}(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;z)\right]} \prec \varphi(z).$$

$\left[\propto \geq 0, z \in D, f \in F_{1,k}^{m}(\propto;\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;z), f_{1,1}^{m+1}[\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;z] \neq 0\right].$

For simplicity we can write

$F_{1,k}^{m}(0;\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;\varphi) = F_{1,k}^{m}(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;\varphi).$

Where

$$\begin{pmatrix} m \in N_0 = N \cup \{0\}, z \in D, \xi \geq 0, \eta \geq 0, \\ 0 < \sigma \leq \frac{1}{2}, 0 < \varepsilon \leq \frac{1}{2}, \delta > 0, \upsilon \geq 0, \delta > \upsilon \end{pmatrix}.$$

Definition 1.1.2 A function $f \in S$ is said to be in the class $G_1^{m}(\propto;\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;z)$ if it satisfies the following subordination condition:

$$\frac{-z\left[(1+\propto)(\Omega f(z))'(z)+\propto\left(\Omega_1^{m+1}(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon)f\right)'(z)\right]}{\left[(1+\propto)g_1^{m}(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;z)+\propto g_1^{m+1}(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;z)\right]} \prec \varphi(z)$$

Where $(z \in D, \propto \geq 0), g \in G_1^{m}(\propto;\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;z),$

and $g_1^{m+1}[\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;z] \neq 0.$

$\therefore G_1^{m}(0;\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;\varphi) = G_1^{m}(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;\varphi).$

Definition 1.1.3 A function $f \in S$ is said to be in the class $H_1^{m}(\propto;\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;z)$ if it satisfies the following subordination condition:

$$-\frac{z\left[(1+\propto)(\Omega f(z))'(z)+\propto\left(\Omega_1^{m+1}(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon)f\right)'(z)\right]}{\left[(1+\propto)h_1^{m}(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;z)+\propto h_1^{m+1}(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;z)\right]} \prec \varphi(z).$$

$z \in D, \propto \geq 0, h \in H_1^{m}(\propto;\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;z)$

and $h_1^{m+1}[\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;z] \neq 0.$

$\therefore H_1^{m}(0;\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;\varphi) = H_1^{m}(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;\varphi).$

Where

$$\begin{pmatrix} a > 0, m \in N_0 = N \cup \{0\}, z \in D, \xi \geq 0, \eta \geq 0, \\ 0 < \sigma \leq \frac{1}{2}, 0 < \varepsilon \leq \frac{1}{2}, \delta > 0, \upsilon \geq 0, \delta > \upsilon \end{pmatrix}.$$

Definition 1.1.4 A function $f \in S$ is said to be in the class $\mathfrak{I}_{1,k}^{m}(\propto;\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;z)$ if it satisfies the following subordination condition

$$-\frac{z\left[(1+\propto)(\Omega f(z))'(z)+\propto\left(\Omega_1^{m+1}(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon)f\right)'(z)\right]}{\left[(1+\propto)l_{1,k}^{m}(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;z)+\propto l_{1,1}^{m+1}(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;z)\right]} \prec \varphi(z).$$

$z \in D, \propto \geq 0, l \in \mathfrak{I}_{1,k}^{m}(\propto;\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;z)$

and $\ell_{1,1}^{m+1}[\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;z] \neq 0.$

For simplicity we can write

$\mathfrak{I}_{1,k}^{m}(0;\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;\varphi) = \mathfrak{I}_{1,k}^{m}(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;\varphi).$

Where

$$\begin{pmatrix} a > 0, m \in N_0 = N \cup \{0\}, z \in D, \xi \geq 0, \eta \geq 0, \\ 0 < \sigma \leq \frac{1}{2}, 0 < \varepsilon \leq \frac{1}{2}, \delta > 0, \upsilon \geq 0, \delta > \upsilon \end{pmatrix}.$$

Definition 1.1.5 A function $f \in S$ is said to be in the class $\mathfrak{U}_{1,k}^{m}(\propto;\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;z)$ if it satisfies the following subordination condition:

$$-\frac{z\left[(1+\propto)(\Omega f(z))'(z)+\propto\left(\Omega_1^{m+1}(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon)f\right)'(z)\right]}{\left[(1+\propto)u_{1,k}^{m}(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;z)+\propto u_{1,1}^{m+1}(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;z)\right]} \prec \varphi(z).$$

$z \in D, \propto \geq 0, u \in \mathfrak{U}_{1,k}^{m}(\propto;\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;z)$

and $u_{1,k}^{m+1}[\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;z] \neq 0.$

$\therefore \mathfrak{U}_{1,k}^{m}(0;\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;\varphi) = \mathfrak{U}_{1,k}^{m}(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;\varphi).$

Where

$$\begin{pmatrix} a > 0, m \in N_0 = N \cup \{0\}, z \in D, \xi \geq 0, \eta \geq 0, \\ 0 < \sigma \leq \frac{1}{2}, 0 < \varepsilon \leq \frac{1}{2}, \delta > 0, \upsilon \geq 0, \delta > \upsilon \end{pmatrix}.$$

Definition 1.1.6 A function $f \in S$ is said to be in the class $\mathfrak{X}_{1,k}^{m}(\propto;\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;z)$ if it satisfies following

$$-\frac{z\left[(1+\propto)(\Omega f(z))'(z)+\propto\left(\Omega_1^{m+1}(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon)f\right)'(z)\right]}{\left[(1+\propto)x_{1,k}^{m}(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;z)+\propto x_{1,1}^{m+1}(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;z)\right]} \prec \varphi(z).$$

$z \in D, \propto \geq 0, x \in \mathfrak{X}_{1,k}^{m}(\propto;\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;z)$

and $x_{1,k}^{m+1}[\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;z] \neq 0.$

$\therefore \mathfrak{X}_{1,k}^{m}(0;\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;\varphi) = \mathfrak{X}_{1,k}^{m}(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;\varphi).$

Where

$$\begin{pmatrix} a > 0, m \in N_0 = N \cup \{0\}, z \in D, \xi \geq 0, \eta \geq 0, \\ 0 < \sigma \leq \frac{1}{2}, 0 < \varepsilon \leq \frac{1}{2}, \delta > 0, \upsilon \geq 0, \delta > \upsilon \end{pmatrix}.$$

1.2. Remarks

Remark 1.2.1 Putting $\xi = \eta = l = 1, \sigma = \frac{1}{2}, m = 0,$ $k = 2,$ and $\varphi(z) = \frac{1+z}{1-z}.$

In definition 1.1.1, we have the class

$$Re\left\{-\frac{z\left[(1+3\propto)f'(z)+\propto\left(zf'(z)\right)\right]'}{(1+3\propto)T_s f(z)+\propto z[T_s f(z)]'}\right\} > 0,$$

where $T_s f(z) = \frac{1}{2}[f(z) - f(-z)].$

Remark 1.2.2 For $\propto = 0,$ $\xi = \eta = 1, \sigma = \frac{1}{2}$ we have the class

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$F_{1,k}^{m}\left(0;\frac{1}{2},1,1,\varepsilon,\delta,\mathcal{v};\varphi\right)=F_{p,k}^{m}(\varepsilon,\delta,\mathcal{v};\varphi)$.

Where the class $F_{1,k}^{m}(\varepsilon,\delta,\mathcal{v};\varphi)$ consisting of functions $f(z)\in S$, this satisfies the following subordination condition $-\frac{z[\Omega_1^m(\varepsilon,\delta,\mathcal{v})f]'(z)}{pf_{1,k}^{m}(\varepsilon,\delta,\mathcal{v};z)}\prec\varphi(z)$.

Where $\varphi\in P$,

$f_{1,k}^{m}[\varepsilon,\delta,\mathcal{v};z]$
$=\frac{1}{k}\sum_{j=0}^{k-1}\epsilon_k^j[\Omega_1^m(\sigma,\eta,\xi,\varepsilon,\delta,\mathcal{v})f](\epsilon_k^j z)\neq 0$,

and

$$\begin{pmatrix}a>0,m\in N_0=N\cup\{0\},z\in D,\xi\geq 0,\eta\geq 0,\\ 0<\sigma\leq\frac{1}{2},0<\varepsilon\leq\frac{1}{2},\delta>0,\mathcal{v}\geq 0,\delta>\mathcal{v}\end{pmatrix}.$$

Remark 1.2.3 For $\propto=0$, $\xi=\eta=l=1,\sigma=\frac{1}{2}$ we have the class $F_{1,k}^{m}\left(0;\frac{1}{2},1,1,1/2,1,1;\varphi\right)=F_{1,k}^{m}(\varphi)$. Where the class $F_{1,k}^{m}(\varphi)$ consisting of functions $f(z)\in S$ which satisfies the following subordination condition

$-\frac{z[I_1^m f]'(z)}{f_{1,k}^{m}(z)}\prec\varphi(z)$, where $\varphi\in P$

$f_{1,k}^{m}[l;z]=\frac{1}{k}\sum_{j=0}^{k-1}\epsilon_k^j[I_1^m f](\epsilon_k^j z)\neq 0$,

and

$$\begin{pmatrix}a>0,m\in N_0=N\cup\{0\},z\in D,\xi\geq 0,\eta\geq 0,\\ 0<\sigma\leq\frac{1}{2},0<\varepsilon\leq\frac{1}{2},\delta>0,\mathcal{v}\geq 0,\delta>\mathcal{v}\end{pmatrix}.$$

Remark 1.2.4 Putting $m=0,k=2,\xi=\eta=l=1$, $\sigma=\frac{1}{2}$ and $\varphi(z)=\frac{1+z}{1-z}$.

In definition 1.1.3, we have the class

$$Re\left\{-\frac{z\left[(1+3\propto)f'(z)+\propto\left(zf'(z)\right)\right]'}{(1+3\propto)T_{sc}f(z)+\propto z[T_s f(z)]'}\right\}>0$$

where $T_{sc}f(z)=\frac{1}{2}[f(z)-\overline{f(-z)}]$ and

$$\begin{pmatrix}a>0,m\in N_0=N\cup\{0\},z\in D,\xi\geq 0,\eta\geq 0,\\ 0<\sigma\leq\frac{1}{2},0<\varepsilon\leq\frac{1}{2},\delta>0,\mathcal{v}\geq 0,\delta>\mathcal{v}\end{pmatrix}.$$

1.3. Preliminary Lemmas

Lemma 1.3.1 Let $\ell(z)$ be analytic and starlike univalent in D with $h(0)=0$. If g(z) is analytic in D and $zg'(z)\prec\ell(z)$,

then $g(z)\prec g(0)+\int_0^z\frac{h(t)}{t}dt$.

Lemma 1.3.2 Let q(z) be analytic and other than constant in D with $q(0)=1$. $0<|z_0|<1$and $Re\,q(z_0)=min_{|z|\leq|z_0|}Re\,q(z)$,

then $z_0q'(z_0)\leq-\frac{|1-q(z_0)|^2}{2[1-Re\,q(z_0)]}$.

Lemma 1.3.3 Let $d,r\in C$; and $\emptyset(z)$ is convex and univalent in D with

$\emptyset(0)=1$ and $Re[d\emptyset(z)+r]>0$.

If $q(z)$ is analytic in D with $q(0)=1$, then the following subordination:

$$q(z)+\frac{zq'(z)}{dq(z)+r}\prec\emptyset(z)\Rightarrow q(z)\prec\emptyset(z)\qquad(z\in D).$$

Lemma 1.3.4 Let $d,r\in C$; and $\emptyset(z)$ is convex and univalent in D with $\emptyset(0)=1$, $Re[d\emptyset(z)+r]>0$.

Also let $(z)\prec\emptyset(z)$ $(z\in D)$.

If $q(z)$ is analytic in D with $q(0)=1$, then the following subordination

$$q(z)+\frac{zq'(z)}{dq(z)+r}\prec\emptyset(z)\Rightarrow q(z)\prec\emptyset(z)\qquad(z\in D).$$

Lemma 1.3.5 Let $f\in F_{1,k}^{m}(\alpha;\sigma,\xi,\eta,\varepsilon,\delta,\mathcal{v};\varphi)$ then

$$\frac{z\left[(1+\propto)\left(f_1^m(\sigma,\eta,\xi,\varepsilon,\delta,\mathcal{v})f\right)'(z)+\propto\left(f_1^{m+1}(\sigma,\eta,\xi,\varepsilon,\delta,\mathcal{v})f\right)'(z)\right]}{-\left[(1+\propto)f_{1,k}^{m}(\sigma,\xi,\eta,\varepsilon,\delta,\mathcal{v};z)+\propto f_{1,1}^{m+1}(\sigma,\xi,\eta,\varepsilon,\delta,\mathcal{v};z)\right]}$$

$\prec\varphi(z)$ (z ∈ D).

$$\begin{pmatrix}a>0,m\in N_0=N\cup\{0\},z\in D,\xi\geq 0,\eta\geq 0,\\ 0<\sigma\leq\frac{1}{2},0<\varepsilon\leq\frac{1}{2},\delta>0,\mathcal{v}\geq 0,\delta>\mathcal{v}\end{pmatrix}.$$

Proof For $(j\in\{0,1,2,\cdots,k-1\})$ we have obtained

$f_{1,k}^{m}[\sigma,\xi,\eta,l;z]=$
$\frac{1}{k}\sum_{j=0}^{k-1}\epsilon_k^j[\Omega_1^m(\sigma,\eta,\xi,\varepsilon,\delta,\mathcal{v})f](\epsilon_k^j z)$.

Hence

(7) $f_{1,k}^{m}[\sigma,\xi,\eta,\varepsilon,\delta,\mathcal{v};\epsilon_k^j z]$
$=\frac{1}{k}\sum_{n=0}^{k-1}\epsilon_k^n[\Omega_1^m(\sigma,\eta,\xi,\varepsilon,\delta,\mathcal{v})f](\epsilon_k^{n+j}z)$
$=\epsilon_k^{-j}\frac{1}{k}\sum_{n=0}^{k-1}\epsilon_k^{(n+j)}[\Omega_1^m(\sigma,\eta,\xi,\varepsilon,\delta,\mathcal{v})f](\epsilon_k^{n+j}z)$
$=\epsilon_k^{-j}f_{1,k}^{m}[\sigma,\xi,\eta,\varepsilon,\delta,\mathcal{v};z]$

And

(8) $\left[f_{1,k}^{m}[\sigma,\xi,\eta,\varepsilon,\delta,\mathcal{v};z]\right]'$
$=\frac{1}{k}\sum_{n=0}^{k-1}\epsilon_k^{2j}[f_1^m(\sigma,\eta,\xi,\varepsilon,\delta,\mathcal{v})f]'(\epsilon_k^j z)$.

Replacing m by $m+1$ in (A) we get

(9) $f_{1,k}^{m+1}[\sigma,\xi,\eta,\varepsilon,\delta,\mathcal{v};\epsilon_k^j z]$
$=\epsilon_k^{-j}f_{1,k}^{m+1}[\sigma,\xi,\eta,\varepsilon,\delta,\mathcal{v};z]$.

Replacing m by $m+1$ in (B) we get

(10) $\left[f_{1,k}^{m+1}[\sigma,\xi,\eta,\varepsilon,\delta,\mathcal{v};z]\right]'$
$=\frac{1}{k}\sum_{n=0}^{k-1}\epsilon_k^{2j}[f_1^{m+1}(\sigma,\eta,\xi,\varepsilon,\delta,\mathcal{v})f]'(\epsilon_k^j z)$.

From (7) and (10) we obtained

$$-\frac{z\left[(1+\propto)\left(f_{1,k}^{m}(\sigma,\eta,\xi,\varepsilon,\delta,\mathcal{v})f\right)'(z)+\propto\left(f_{1,k}^{m+1}(\sigma,\eta,\xi,\varepsilon,\delta,\mathcal{v})f\right)'(z)\right]}{\left[(1+\propto)f_{1,k}^{m}(\sigma,\xi,\eta,\varepsilon,\delta,\mathcal{v};z)+\propto f_{1,k}^{m+1}(\sigma,\xi,\eta,\varepsilon,\delta,\mathcal{v};z)\right]}$$

$$=\frac{-1}{k}\sum_{j=0}^{k-1}\left\{\frac{\epsilon_k^{2j}z\left[(1+\propto)\left(I_1^m(\sigma,\eta,\xi,\varepsilon,\delta,\mathcal{v})f\right)'\left(\epsilon_k^j z\right)\right]}{(1+\propto)f_{1,k}^{m}(\sigma,\xi,\eta,\varepsilon,\delta,\mathcal{v};z)+\propto f_{1,k}^{m+1}(\sigma,\xi,\eta,\varepsilon,\delta,\mathcal{v};z)}+\frac{\epsilon_k^{2j}z\left[\propto\left(I_1^{m+1}(\sigma,\eta,\xi,\varepsilon,\delta,\mathcal{v})f\right)'\left(\epsilon_k^j z\right)\right]}{(1+\propto)f_{1,k}^{m}(\sigma,\xi,\eta,\varepsilon,\delta,\mathcal{v};z)+\propto f_{1,k}^{m+1}(\sigma,\xi,\eta,\varepsilon,\delta,\mathcal{v};z)}\right\}=$$

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$$-\frac{1}{k}\sum_{j=0}^{k-1}\left\{\frac{\epsilon_k^j z\left[(1+\propto)\left(I_1^m(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon)f\right)'\left(\epsilon_k^j z\right)\right]}{(1+\propto)f_{1,k}^m\left(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;\epsilon_k^j z\right)+\propto f_{1,k}^{m+1}\left(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;\epsilon_k^j z\right)}\right.$$
$$\left.+\frac{\epsilon_k^j z\left[\propto\left(I_1^{m+1}(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon)f\right)'\left(\epsilon_k^j z\right)\right]}{(1+\propto)f_{1,k}^m\left(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;\epsilon_k^j z\right)+\propto f_{1,k}^{m+1}\left(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;\epsilon_k^j z\right)}\right\}$$

It is clear that

$$\frac{\epsilon_k^j z\left[(1+\propto)\left(I_1^m(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon)f\right)'\left(\epsilon_k^j z\right)+\propto\left(I_1^{m+1}(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon)f\right)'\left(\epsilon_k^j z\right)\right]}{-\left[(1+\propto)f_{1,k}^m\left(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;\epsilon_k^j z\right)+\propto f_{1,k}^{m+1}\left(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;\epsilon_k^j z\right)\right]}$$
$\prec \emptyset(z)$.

Noting that $\emptyset(z)$ is convex and univalent in D we conclude that Lemma 1.3.5 holds true. Using equations (5) and (6) we get

$$z\left(f_{1,k}^m(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon)f\right)'$$
$$+\left(1+\frac{l}{\sigma(\xi+\eta)}\right)f_{1,k}^m(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;z)$$
$$=\frac{l}{\sigma(\xi+\eta)k}\sum_{j=0}^{k-1}\epsilon_k^j\left[\Omega_1^{m+1}(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon)f\right]\left(\epsilon_k^j z\right)$$
$$=\frac{l}{\sigma(\xi+\eta)}f_{1,k}^{m+1}(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon;z)\qquad (f\in S).$$

Let $f\in F_{1,k}^m(\alpha;\sigma,\eta,\xi,l;\emptyset)$ and suppose

$$\psi(z)=-\frac{z\left[f_{1,k}^m(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon;z)\right]'}{p f_{1,k}^m(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon;z)}.$$

Then $\psi(z)$ is analytic in D and $\psi(0)=1$.

Hence $1+\frac{1}{b}-\psi(z)=\frac{1}{b}\frac{f_{1,k}^{m+1}(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon;z)}{f_{1,k}^m(\sigma,\eta,\xi,l;z)}$

$\therefore z\left[f_{1,k}^{m+1}(\sigma,\eta,\xi,l;z)\right]'=$
$-b\left\{z\psi'(z)+1+\frac{1}{b}-\psi(z)\right\}f_{1,k}^m(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon;z)$
$(z\in D^*)$.

From above relations we obtained

$$\frac{z\left[(1+\propto)\left(f_{1,k}^m(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon)f\right)'(z)+\propto\left(f_{1,k}^{m+1}(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon)f\right)'(z)\right]}{-\left[(1+\propto)f_{1,k}^m(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;z)+\propto f_{1,k}^{m+1}(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;z)\right]}=$$
$$\frac{(1+\propto)\psi(z)f_{1,k}^m(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;z)+\alpha\cdot b\left[1+\frac{1}{b}-\psi(z)\right]\psi(z)f_{1,k}^m(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon)}{(1+\propto)f_{1,k}^m(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;z)+\alpha\, b\left[1+\frac{1}{b}-p\psi(z)\right]f_{1,k}^m(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon)}$$
$$=\frac{(1+\propto)\psi(z)+\alpha b\left\{z\psi'(z)+\left[1+\frac{1}{b}-\psi(z)\right]\psi(z)\right\}}{(1+\propto)+\alpha b\left[1+\frac{1}{b}-\psi(z)\right]}$$
$$=\frac{\alpha b z\psi'(z)+\left\{(1+\propto)+\alpha b\left[1+\frac{1}{b}-\psi(z)\right]\right\}\cdot\psi(z)}{(1+\propto)+\alpha b\left[1+\frac{1}{b}-\psi(z)\right]}.$$

$\psi(z)+\frac{z\psi'(z)}{\frac{1}{b\alpha}+2\frac{1}{b}+1-\psi(z)}\prec\varphi(z).\,(z\in D).$ Since

$$Re\left(\frac{1}{b\alpha}+2b+1-\psi(z)\right)>0$$

and by Lemma 3 we get

$$\psi(z)=-\frac{z\left(f_{1,k}^m(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon;z)\right)'}{p f_{1,k}^m(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon;z)}\prec\varphi(z)\qquad (z\in D).$$

Lemma 1.3.6: Let $f\in G_{1,k}^m(\propto;\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;\varphi)$ then

$$-\frac{z\left[(1+\propto)\left(g_1^m(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon)f\right)'(z)+\propto\left(g_1^{m+1}(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon)f\right)'(z)\right]}{\left[(1+\propto)g_1^m[\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;z]+\propto g_1^{m+1}[\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;z]\right]}$$
$\prec\varphi(z)\qquad(z\in D)$.

Therefore,if $\emptyset\in$ P with

$$Re\left[\frac{1}{b}\left(2+\frac{1}{a}\right)+(1-\varphi(z))\right]>0.$$

Where

$$\begin{pmatrix}a>0,m\in N_0=N\cup\{0\},z\in D,\xi\geq 0,\eta\geq 0,\\ 0<\sigma\leq\frac{1}{2},0<\varepsilon\leq\frac{1}{2},\delta>0,\upsilon\geq 0,\delta>\upsilon,\end{pmatrix}.$$
$$-\frac{z\left(g_1^m(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon;z)\right)'}{g_1^m(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon;z)}\prec\varphi(z)\qquad(z\in D).$$

Lemma 1.3.7: Let $f\in F_{1,k}^m(\propto;\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;\varphi)$ then

$$-\frac{z\left[(1+\propto)\left(h_1^m(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon)f\right)'(z)+\propto\left(h_1^{m+1}(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon)f\right)'(z)\right]}{\left[(1+\propto)h_1^m[\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;z]+\propto h_1^{m+1}[\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;z]\right]}$$
$\prec\varphi(z)\qquad(z\in D)$.

Therefore, if $\emptyset\in P$ with

$$Re\left[\frac{1}{b}\left(2+\frac{1}{a}\right)+(1-\varphi(z))\right]>0.$$

Where

$$\begin{pmatrix}a>0,m\in N_0=N\cup\{0\},z\in D,\xi\geq 0,\eta\geq 0,\\ 0<\sigma\leq\frac{1}{2},0<\varepsilon\leq\frac{1}{2},\delta>0,\upsilon\geq 0,\delta>\upsilon\end{pmatrix}.$$
$$-\frac{z\left(h_1^m(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon;z)\right)'}{p h_1^m(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon;z)}\prec\varphi(z)\qquad(z\in D).$$

2. Main Results

Section 1

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Some properties of meromorphically univalent functions

Theorem 2.1:

Let $0<a\leq 1$ and $0<b<1$. If $f(z)\in S$ satisfies $f(z)\neq 0$ $z\in D^*$ and

(11) $\left|\frac{1}{zf(z)}\left(\frac{zf'(z)}{f(z)}\right)+1\right|<\delta\qquad$ (z ∈ D).

Where δ is the minimum positive root of the equation

(12) $\frac{a}{2}sin\left(\pi\frac{b}{2}\right)x^2-x+\left(1-\frac{a}{2}\right)sin\left(\pi\frac{b}{2}\right)=0.$

Then

(13) $\left|arg\left(zf(z)-\frac{a}{2}\right)\right|<\pi\frac{b}{2}\qquad(z\in D).$

The bound b is the best possible for each $0<a\leq 1$

Proof: Let

(14) $g(x)=\frac{a}{2}sin\left(\pi\frac{b}{2}\right)x^2-x$
$+\left(1-\frac{a}{2}\right)sin\left(\pi\frac{b}{2}\right).$

It is clear that the Equation

$\frac{a}{2}sin\left(\pi\frac{b}{2}\right)x^2-x+\left(1-\frac{a}{2}\right)sin\left(\pi\frac{b}{2}\right)=0$

have two positive roots.

Since

$g(0)=\frac{a}{2}sin\left(\pi\frac{b}{2}\right)\cdot(0)^2-0+\left(1-\frac{a}{2}\right)sin\left(\pi\frac{b}{2}\right)$
$=\left(1-\frac{a}{2}\right)sin\left(\pi\frac{b}{2}\right)>0$
$g(1)=\frac{a}{2}sin\left(\pi\frac{b}{2}\right)\cdot(1)^2-1+\left(1-\frac{a}{2}\right)sin\left(\pi\frac{b}{2}\right)$
$=\frac{a}{2}sin\left(\pi\frac{b}{2}\right)+\frac{a}{2}sin\left(\pi\frac{b}{2}\right)$
$=2\frac{a}{2}sin\left(\pi\frac{b}{2}\right)<0.$

Hence we get

(15) $0 < \frac{a}{2-a}\delta \le \delta < 1$

Put

(16) $zf(z) = \frac{a}{2} + \left(1 - \frac{a}{2}\right) q(z)$.

Then from the assumption of the theorem, we see that $q(z)$ is analytic in D with $q(0) = 1$ and $\frac{a}{2} + \left(1 - \frac{a}{2}\right) q(z) \neq 0$ for all $(z \in D)$. Taking the logarithmic differentiations on both sides of

$zf(z) = \frac{a}{2} + \left(1 - \frac{a}{2}\right) q(z)$,

we obtained

(17) $\frac{zf'(z)}{f(z)} + 1 = \frac{(2-a)zq'(z)}{a+(2-a)q(z)}$

(18) $zf(z)\left[\frac{zf'(z)}{f(z)} + 1\right] = \frac{(2-a)zq'(z)}{[a+(2-a)q(z)]^2}$ $(z \in D)$.

Thus the inequality

$\left|\frac{1}{zf(z)}\left(\frac{zf'(z)}{f(z)}\right) + 1\right| < \delta$ $(z \in D)$

is equivalent to next equation as given in (19).

(19) $\frac{(2-a)zq'(z)}{[a+(2-a)q(z)]^2} \prec \delta z$.

By using Lemma 1, above inequality leads to

(20) $\int_0^z \frac{(2-a)q'(t)}{[a+(2-a)q(t)]^2} dt \prec \delta z$.

Or to

$1 - \frac{2}{a+(2-a)q(z)} \prec \delta z$.

In view of above results it can be written as

(21) $q(z) \prec \frac{1+\frac{a}{2-a}\delta z}{1-\delta z}$. Now by taking $\propto = \frac{a}{2-a} \cdot \frac{\delta}{2}$ and $\beta = -\frac{\delta}{2}$ in (1.2), we have

$\left|arg\left(zf(z) - \frac{a}{2}\right)\right|$

$= |arg\ q(z)| < arc\ sin\left(\frac{2\delta}{2-a+a\delta^2}\right) = \pi\frac{b}{2}$. $(z \in D)$.

Because of $g(\delta) = 0$. This proves statement.

Next, we consider the function f(z) defined by

$f(z) = \frac{z^{-1}}{1-\delta z}$ $(z \in D^*)$.

It is easy to see that

$\left|zf(z)\left[\frac{zf'(z)}{f(z)} + p\right]\right| = |\delta z| < \delta$ $(z \in D)$.

Since $zf(z) - \frac{a}{2} = \frac{1+\frac{a}{2-a}\delta z}{1-\delta z}$.

It follows from (3) that

$sup_{z\in D}\left|arg\left(zf(z) - \frac{a}{2}\right)\right|$

$= arc\ sin\left(\frac{2\delta}{2-a+a\delta^2}\right) = \pi\frac{b}{2}$.

Hence, we conclude that the bound b is the best possible for each $a \in (0,1]$.

Next, we derive the following.

Theorem 2.2: If $f(z) \in S$ satisfies $f(z) \neq o, (z \in D^*)$ and

(22) $Re\left[zf(z)\left[\frac{zf'(z)}{f(z)} + 1\right]\right] < \varepsilon$ $(z \in D)$.

(23) $0 < \varepsilon < \frac{1}{2\log 2}$

then

(24) $Re\frac{1}{zf(z)} > 1 - 2\varepsilon \cdot log2$ $(z \in D)$

The above inequality holds good.

Proof: Let

(25) $q(z) = zf(z)$

Then $q(z)$ is analytic in D with $q(0) = 1$ and $q(z) \neq 0$ for $z \in D$. In accordance with (17) and (20), we obtained

$1 - \frac{zq'(z)}{\varepsilon q^2(z)} \prec \frac{1+z}{1-z}$. Let

(26) $z\left[\frac{1}{q(z)}\right]' \prec \frac{2\varepsilon z}{1-z}$. Now by Lemma 1, we obtain

$\frac{1}{q(z)} \prec 1 - 2\varepsilon \cdot log(1-z)$.

Since the function $1 - 2\varepsilon \cdot log(1-z)$ is convex univalent in D and

$Re[1 - 2\varepsilon \cdot log(1-z)] > 1 - 2\varepsilon \cdot log\,2$ $(z \in D)$.

From $z\left[\frac{1}{q(z)}\right]' \prec \frac{2\varepsilon z}{1-z}$ we obtained the inequality

$Re\frac{1}{zf(z)} > 1 - 2\varepsilon \cdot log2$.

To show that the bound

$Re\frac{1}{zf(z)} > 1 - 2\varepsilon \cdot log2$ $(z \in D)$.

Cannot be increased, we consider

$f(z) = \frac{1}{z[1-2\varepsilon \cdot log(1-z)]}$ $(z \in D^*)$.

We can verify that the function $f(z)$ satisfies the inequality

$Re\left[zf(z)\left[\frac{zf'(z)}{f(z)} + 1\right]\right] < \varepsilon$ $(z \in D)$.

On the other hand we have

$Re\ zf(z) \to 1 - 2\varepsilon \cdot log2\ \ as\ \ z \to -1$.

Hence the Theorem holds good.

Theorem 2.3: Let $f(z) \in S$ satisfies $f(z) \neq o$,

(27) $(z \in D^*)$.If $\left|Im\left\{\frac{zf'(z)}{f(z)}[zf(z) - \tau]\right\}\right| < \sqrt{2}\ \tau$ $(z \in D, \tau > 0)$. Then

(28) $Re\ zf(z) > 0$ $(z \in D)$.

Proof: Let q(z) in D be defined as

$Re\ zf(z) = q(z)$ then $q(0) = 1, q(z) \neq 0$,

And

(29) $\frac{zf'(z)}{f(z)}(zf(z) - \tau) = [q(z) - \tau] \cdot \left[\frac{zq'(z)}{q(z)} - 1\right]$

(30) $Re\ q(z) > 0$, $|z| < |z_0|$ and $q(z_0) = ib$

where b is real and $b \neq 0$. Then by Lemma 1.2.2 we have

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(31) $\quad z_0 \cdot q'(z_0) \le \frac{-(1-b^2)}{2}$.

Thus it follows from above obtained results that

(32) $\quad J_o = Im\left[\frac{z_0 f'(z_0)}{f(z_0)}(z.f(z)-\tau)\right]$

$= -b + \frac{\tau}{b} z_0 \cdot q'(z_0)$

In accordance with $\tau > 0$ and from the statements (2.2.1) and (2.2.2) we obtained

(33) $\quad J_0 \ge \frac{-[\tau+(\tau+2)b^2]}{2b} \ge \sqrt{\tau(\tau+2)} \quad (b>0)$

(34) $\quad J_0 \le \frac{[\tau+(\tau+2)b^2]}{2b} \le -\sqrt{\tau(\tau+2)} \quad (b>0)$

But both The inequalities obtained above contradict the assumption given below.

$\left|Im\left\{\frac{zf'(z)}{f(z)}[zf(z)-\tau]\right\}\right| < \sqrt{2}\,\tau \quad (z \in D)$ and $\tau > 0$. Therefore we have, $\operatorname{Re} q(z) > 0$ for all $(z \in D)$. This shows that $Re\, zf(z) > 0 \quad (z \in D)$. Theorem Holds true.

Section 2

Subclasses of meromorphically univalent Functions associated with Generalized multiplier transformations

2.1. Inclusion relationships

Theorem 2.1.1: Let $\varphi \in P$ with

$Re\left[\frac{1}{b}\left(2+\frac{1}{\alpha}\right)+(1-\varphi(z))\right] > 0.$

Where

$\begin{pmatrix} a>0, m \in N_0 = N \cup \{0\}, z \in D, \xi \ge 0, \eta \ge 0, \\ 0<\sigma \le \frac{1}{2}, 0<\varepsilon \le \frac{1}{2}, \delta > 0, v \ge 0, \delta > v \end{pmatrix}.$

Then

$F_{1,k}^m(\propto;\sigma,\xi,\eta,\varepsilon,\delta,v;\varphi) \subset F_{1,k}^m(\sigma,\xi,\eta,\varepsilon,\delta,v;\varphi).$

Proof: Let $f \in F_{1,k}^m(\propto;\sigma,\xi,\eta,\varepsilon,\delta,v;\varphi)$ and

$q(z) = -\frac{z\left(I_1^m(\sigma,\eta,\xi,\varepsilon,\delta,v)f\right)'}{pI_1^m(\sigma,\eta,\xi,\varepsilon,\delta,v;z)} \qquad (z \in D).$

Then $q(z)$ is analytic in D and $q(0)=1$ hence

$q(z) f_{1,k}^m(\sigma,\xi,\eta,\varepsilon,\delta,v;z)$

$= -\frac{1}{b} I_1^{m+1}(\sigma,\xi,\eta,\varepsilon,\delta,v)f(z)$

$+\left(\frac{1}{b}+1\right) I_1^m(\sigma,\xi,\eta,\varepsilon,\delta,v)f(z).$

Differentiating both sides we get

$zq'(z) + \left(\frac{1}{b}+1+\frac{z\left(f_{1,k}^m(\sigma,\eta,\xi,\varepsilon,\delta,v;z)\right)'}{f_{1,k}^m(\sigma,\eta,\xi,\varepsilon,\delta,v;z)}\right) q(z)$

$= \frac{-1}{b} \cdot \frac{z\left(I_1^{m+1}(\sigma,\eta,\xi,\varepsilon,\delta,v)f(z)\right)'}{f_{1,k}^m(\sigma,\eta,\xi,\varepsilon,\delta,v;z)}$

$\therefore -\frac{z\left[(1+\propto)\left(I_1^m(\sigma,\eta,\xi,\varepsilon,\delta,v)f\right)'(z)+\propto\left(I_1^{m+1}(\sigma,\eta,\xi,\varepsilon,\delta,v)f\right)'(z)\right]}{\left[(1+\propto)f_{1,k}^m(\sigma,\xi,\eta,\varepsilon,\delta,v;z)+\propto f_{1,k}^{m+1}(\sigma,\xi,\eta,\varepsilon,\delta,v;z)\right]}$

$= \frac{(1+\propto)q(z)f_{1,k}^m(\sigma,\xi,\eta,\varepsilon,\delta,v;z)}{(1+\propto)f_{1,k}^m(\sigma,\xi,\eta,\varepsilon,\delta,v;z)+\alpha b\left[1+\frac{1}{b}-\psi(z)\right]f_{1,k}^m(\sigma,\eta,\xi,\varepsilon,\delta,v)}$

$+\frac{\alpha b\left[zq'(z)+1+\frac{1}{b}-p\psi(z)\right]q(z)f_{1,k}^m(\sigma,\eta,\xi,\varepsilon,\delta,v)}{(1+\propto)f_{1,k}^m(\sigma,\xi,\eta,\varepsilon,\delta,v;z)+\alpha b\left[1+\frac{1}{b}-\psi(z)\right]f_{1,k}^m(\sigma,\eta,\xi,\varepsilon,\delta,v)}$

$= \frac{(1+\propto)q(z)+\alpha b\left\{zq'(z)+\left[1+\frac{1}{b}-\psi(z)\right]q(z)\right\}}{(1+\propto)+\alpha b\left[1+\frac{1}{b}-\psi(z)\right]}$

$= \frac{\frac{\alpha\cdot\sigma(\xi+\eta)}{\varepsilon(\delta-v)}zq'(z)+\left\{(1+\propto)+b\alpha\left[1+\frac{1}{b}-\psi(z)\right]\right\}\cdot q(z)}{(1+\propto)+\alpha b\left[1+\frac{1}{b}-\psi(z)\right]}$

$= q(z) + \frac{zq'(z)}{\frac{1}{b\alpha}+2\frac{1}{b}+1-\psi(z)} \prec \varphi(z) \qquad (z \in D).$

Since $Re\left(\frac{1}{b\alpha}+2\frac{1}{b}+1-p\psi(z)\right) > 0$

And by Lemma 3 we get

$\psi(z) = -\frac{z\left(f_{1,k}^m(\sigma,\eta,\xi,\varepsilon,\delta,v;z)\right)'}{pf_{1,k}^m(\sigma,\eta,\xi,\varepsilon,\delta,v;z)} \prec \varphi(z) \qquad (z \in D).$

By Lemma 1.2.2 we find that $q(z) \prec \varphi(z)$ $(z \in D)$.

$\therefore\ f \in F_{1,k}^m(\propto;\sigma,\xi,\eta,\varepsilon,\delta,v;\varphi)$

$\Rightarrow F_{1,k}^m(\propto;\sigma,\xi,\eta,l;\varphi) \subset F_{1,k}^m(\sigma,\xi,\eta,\varepsilon,\delta,v;\varphi).$

Corollary 2.1.1: Let φ ∈ P with

$Re\left[\frac{1}{b}\left(2+\frac{1}{\alpha}\right)+(1-\varphi(z))\right] > 0.$

Where

$\begin{pmatrix} a>0, m \in N_0 = N \cup \{0\}, z \in D, \xi \ge 0, \eta \ge 0, \\ 0<\sigma \le \frac{1}{2}, 0<\varepsilon \le \frac{1}{2}, \delta > 0, v \ge 0, \delta > v \end{pmatrix}.$

Then

$G_1^m(\propto;\sigma,\xi,\eta,\varepsilon,\delta,v;\varphi) \subset G_1^m(\sigma,\xi,\eta,\varepsilon,\delta,v;\varphi).$

Corollary 2.1.2: Let $\varphi \in P$ with

$Re\left[\frac{1}{b}\left(2+\frac{1}{\alpha}\right)+(1-\varphi(z))\right] > 0.$

Where

$\begin{pmatrix} a>0, m \in N_0 = N \cup \{0\}, z \in D, \xi \ge 0, \eta \ge 0, \\ 0<\sigma \le \frac{1}{2}, 0<\varepsilon \le \frac{1}{2}, \delta > 0, v \ge 0, \delta > v \end{pmatrix}.$

Then

$H_1^m(\propto;\sigma,\xi,\eta,\varepsilon,\delta,v;\varphi) \subset H_1^m(\sigma,\xi,\eta,\varepsilon,\delta,v;\varphi).$

Theorem 2.1.2: Let $\varphi \in P$ with

$Re\left[\frac{\varepsilon(\delta-v)}{\sigma(\xi+\eta)}\left(2+\frac{1}{\alpha}\right)+(1-\varphi(z))\right] > 0.$

Where

$\begin{pmatrix} a>0, m \in N_0 = N \cup \{0\}, z \in D, \xi \ge 0, \eta \ge 0, \\ 0<\sigma \le \frac{1}{2}, 0<\varepsilon \le \frac{1}{2}, \delta > 0, v \ge 0, \delta > v \end{pmatrix}.$

Then

$\mathfrak{I}_{1,k}^m(\propto;\sigma,\xi,\eta,\varepsilon,\delta,v;\varphi) \subset \mathfrak{I}_{1,k}^m(\sigma,\xi,\eta,\varepsilon,\delta,v;\varphi).$

Proof Let $f \in \mathfrak{I}_{1,k}^m(\propto;\sigma,\xi,\eta,\varepsilon,\delta,v;\varphi)$and suppose that $q(z) = -\frac{z\left(I_1^m(\sigma,\eta,\xi,\varepsilon,\delta,v;z)\right)'}{p\ell_{1,k}^m(\sigma,\eta,\xi,\varepsilon,\delta,v;z)} \qquad (z \in D).$

Thus $q(z)$ is analytic in D and $q(0)=1$.

$\therefore q(z)\ell_{1,k}^m(\sigma,\eta,\xi,\varepsilon,\delta,v;z)$

$= \frac{-1}{b} I_1^{m+1}(\sigma,\xi,\eta,\varepsilon,\delta,v)f(z)$

$+\left(\frac{1}{b}+1\right) I_1^m(\sigma,\xi,\eta,\varepsilon,\delta,v)f(z).$

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Differentiating both sides we get

$$zq'(z)+\left(\frac{1}{b}+1+\frac{z\left(\ell_{1,k}^{m}(\sigma,\eta,\xi,\varepsilon,\delta,v;z)\right)'}{\ell_{1,k}^{m}(\sigma,\eta,\xi,\varepsilon,\delta,v;z)}\right)q(z)$$

$$=-\frac{1}{b}\cdot\frac{z\left(I_1^{m+1}(\sigma,\eta,\xi,\varepsilon,\delta,v)f(z)\right)'}{\ell_{1,k}^{m}(\sigma,\eta,\xi,\varepsilon,\delta,v;z)}$$

$$\varphi(z)=-\frac{z\left(\ell_{1,k}^{m}(\sigma,\eta,\xi,\varepsilon,\delta,v;z)\right)'}{\ell_{1,k}^{m}(\sigma,\eta,\xi,\varepsilon,\delta,v;z)}\qquad(z\in D).$$

$\therefore q(z)\prec\varphi(z)\qquad(z\in D).$

This implies that $f\in\mathfrak{J}_{1,k}^{m}(\propto;\sigma,\xi,\eta,\varepsilon,\delta,v;\varphi)$.

Corollary 2.1.3: Let φ ∈ P with

$$Re\left[\frac{1}{b}\left(2+\frac{1}{\alpha}\right)+(1-\varphi(z))\right]>0.$$

Where

$$\begin{pmatrix}a>0,m\in N_0=N\cup\{0\},z\in D,\xi\geq0,\eta\geq0,\\0<\sigma\leq\frac{1}{2},0<\varepsilon\leq\frac{1}{2},\delta>0,v\geq0,\delta>v\end{pmatrix}.$$

Then

$\mathfrak{U}_1^{m}(\propto;\sigma,\xi,\eta,\varepsilon,\delta,v;\varphi)\subset\mathfrak{U}_1^{m}(\sigma,\xi,\eta,\varepsilon,\delta,v;\varphi)$.

Corollary 2.1.4: Let $\varphi\in P$ with

$$Re\left[\frac{1}{b}\left(2+\frac{1}{\alpha}\right)+p(1-\varphi(z))\right]>0.$$

Where

$$\begin{pmatrix}a>0,m\in N_0=N\cup\{0\},z\in D,\xi\geq0,\eta\geq0,\\0<\sigma\leq\frac{1}{2},0<\varepsilon\leq\frac{1}{2},\delta>0,v\geq0,\delta>v\end{pmatrix}.$$

Then

$\mathfrak{X}_1^{m}(\propto;\sigma,\xi,\eta,\varepsilon,\delta,v;\varphi)\subset\mathfrak{X}_1^{m}(\sigma,\xi,\eta,\varepsilon,\delta,v;\varphi)$.

Corollary 2.1.5: Let φ ∈ P with

$$Re\left[\frac{1}{b}\left(2+\frac{1}{\alpha}\right)+(1-\varphi(z))\right]>0.$$

Where

$$\begin{pmatrix}a>0,m\in N_0=N\cup\{0\},z\in D,\xi\geq0,\eta\geq0,\\0<\sigma\leq\frac{1}{2},0<\varepsilon\leq\frac{1}{2},\delta>0,v\geq0,\delta>v\end{pmatrix}.$$

Then

$F_{1,k}^{m+1}(\sigma,\xi,\eta,\varepsilon,\delta,v;\varphi)\subset F_{1,k}^{m}(\sigma,\xi,\eta,\varepsilon,\delta,v;\varphi)$.

Corollary 2.1.6: Let φ ∈ P with

$$Re\left[\frac{1}{b}\left(2+\frac{1}{\alpha}\right)+(1-\varphi(z))\right]>0.$$

Where

$$\begin{pmatrix}a>0,m\in N_0=N\cup\{0\},z\in D,\xi\geq0,\eta\geq0,\\0<\sigma\leq\frac{1}{2},0<\varepsilon\leq\frac{1}{2},\delta>0,v\geq0,\delta>v\end{pmatrix}.$$

Then

$G_1^{m+1}(\sigma,\xi,\eta,\varepsilon,\delta,v;\varphi)\subset G_1^{m}(\sigma,\xi,\eta,\varepsilon,\delta,v;\varphi)$.

Corollary 2.1.7: Let φ ∈ P with

$$Re\left[\frac{1}{b}\left(2+\frac{1}{\alpha}\right)+(1-\varphi(z))\right]>0.$$

Where

$$\begin{pmatrix}a>0,m\in N_0=N\cup\{0\},z\in D,\xi\geq0,\eta\geq0,\\0<\sigma\leq\frac{1}{2},0<\varepsilon\leq\frac{1}{2},\delta>0,v\geq0,\delta>v\end{pmatrix}.$$

Then

$H_1^{m+1}(\sigma,\xi,\eta,\varepsilon,\delta,v;\varphi)\subset H_1^{m}(\sigma,\xi,\eta,\varepsilon,\delta,v;\varphi)$.

Corollary 2.1.8: Let φ ∈ P with

$Re\left[\frac{1}{b}\left(2+\frac{1}{\alpha}\right)+(1-\varphi(z))\right]>0.$ Where

$$\begin{pmatrix}a>0,m\in N_0=N\cup\{0\},z\in D,\xi\geq0,\eta\geq0,\\0<\sigma\leq\frac{1}{2},0<\varepsilon\leq\frac{1}{2},\delta>0,v\geq0,\delta>v\end{pmatrix}.$$

Then

$\mathfrak{J}_1^{m+1}(\sigma,\xi,\eta,\varepsilon,\delta,v;\varphi)\subset\mathfrak{J}_1^{m}(\sigma,\xi,\eta,\varepsilon,\delta,v;\varphi)$.

Corollary 2.1.9: Let $\varphi\in P$ with

$$Re\left[\frac{1}{b}\left(2+\frac{1}{\alpha}\right)+(1-\varphi(z))\right]>0.$$

Where

$$\begin{pmatrix}a>0,m\in N_0=N\cup\{0\},z\in D,\xi\geq0,\eta\geq0,\\0<\sigma\leq\frac{1}{2},0<\varepsilon\leq\frac{1}{2},\delta>0,v\geq0,\delta>v\end{pmatrix}.$$

Then

$\mathfrak{U}_1^{m+1}(\sigma,\xi,\eta,\varepsilon,\delta,v;\varphi)\subset\mathfrak{U}_1^{m}(\sigma,\xi,\eta,\varepsilon,\delta,v;\varphi)$.

Corollary 2.1.10: Let φ ∈ P with

$$Re\left[\frac{\varepsilon(\delta-v)}{\sigma(\xi+\eta)}\left(2+\frac{1}{\alpha}\right)+(1-\varphi(z))\right]>0.$$

Where

$$\begin{pmatrix}a>0,m\in N_0=N\cup\{0\},z\in D,\xi\geq0,\eta\geq0,\\0<\sigma\leq\frac{1}{2},0<\varepsilon\leq\frac{1}{2},\delta>0,v\geq0,\delta>v\end{pmatrix}.$$

Then $\mathfrak{X}_1^{m+1}(\sigma,\xi,\eta,\varepsilon,\delta,v;\varphi)\subset\mathfrak{X}_1^{m}(\sigma,\xi,\eta,\varepsilon,\delta,v;\varphi)$.

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2.2. Integral Representation

In this section we are going to prove integral representations associated with the function classes $F_{1,k}^{m}(\sigma,\xi,\eta,\varepsilon,\delta,v;\varphi)$, $G_{1,k}^{m}(\sigma,\xi,\eta,\varepsilon,\delta,v;\varphi)$ and $H_{1,k}^{m}(\sigma,\xi,\eta,\varepsilon,\delta,v;\varphi)$.

Theorem 2.2.1: Let $f\in F_{1,k}^{m}(\sigma,\xi,\eta,\varepsilon,\delta,v;\varphi)$ then

$$F_{1,k}^{m}(\sigma,\xi,\eta,\varepsilon,\delta,v;z)=$$

$$z^{-1}exp\left(-\frac{1}{k}\sum_{j=0}^{k-1}\int_0^z\frac{\varphi\left[w\left(\epsilon_k^j z\right)\right]-1}{\zeta}d\zeta\right).$$

Where $f_{1,k}^{m}[\sigma,\xi,\eta,\varepsilon,\delta,v;z]$

$$=\frac{1}{k}\sum_{j=0}^{k-1}\epsilon_k^j\left[\Omega_1^m(\sigma,\eta,\xi,\varepsilon,\delta,v)f\right]\left(\epsilon_k^j z\right)$$

$$=\frac{1}{z}+\cdots\qquad(f\in S),$$

$w(z)$ is analytic in D with $w(0)=0$ and $|w(z)|<1.\quad(z\in D)$.

Proof: Suppose that $f\in F_{1,k}^{m}(\sigma,\xi,\eta,\varepsilon,\delta,v;\varphi)$.

$$-\frac{z\left(I_1^m(\sigma,\eta,\xi,\varepsilon,\delta,v)f\right)'\left(\epsilon_k^j z\right)}{f_{1,k}^{m}(\sigma,\eta,\xi,\varepsilon,\delta,v;z)}=\varphi(w(z))\qquad(z\in D).$$

Where $w(z)$ is analytic in D with $w(0)=0$ and $|w(z)|<1\qquad(z\in D)$.

Replacing z by $\epsilon_k^j z\quad(j=0,1,2,\cdots)$ above equation holds true.

That is

$$\frac{\epsilon_k^j z\left(I_1^m(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon)f\right)'\left(\epsilon_k^j z\right)}{-p f_{1,k}^m(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon;z)} = \varphi\left(w\left(\epsilon_k^j z\right)\right) \qquad (z \in D).$$

We note that

$$f_{1,k}^m\left(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon;\epsilon_k^j z\right) = \epsilon_k^{-j} f_{1,k}^m(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon;z)$$

Letting $(j = 0,1,2,\cdots\cdots)$, successively and summing the resultant equations we obtained

$$\frac{z\left(f_1^m(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon;z)\right)'}{-p f_{1,k}^m(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon;z)} = \frac{1}{k}\sum_{j=0}^{k-1}\varphi\left(w\left(\epsilon_k^j z\right)\right) \qquad (z \in D).$$

And

$$\frac{\left(f_1^m(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon;z)\right)'}{f_{1,k}^m(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon;z)} + \frac{1}{z} = \frac{-1}{k}\sum_{j=0}^{k-1}\frac{\varphi\left(w\left(\epsilon_k^j z\right)\right)-1}{z}.$$

Upon integration which yields,

$$\log\left(z f_{1,k}^m(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon;z)\right) = -\frac{1}{k}\sum_{j=0}^{k-1}\int_0^z \frac{\varphi\left[w\left(\epsilon_k^j z\right)\right]-1}{\zeta}\,d\zeta.$$

Taking exponential theorem holds true.

Theorem 2.2.2: Let $f \in F_{1,k}^m(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;\varphi)$ then

$$I_1^m(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon)f(z) = -\int_0^z\left[\begin{array}{c}\varsigma^{-2}\varphi\left(w(\varsigma)\right)\\ \times\exp\left(-\frac{1}{k}\sum_{j=0}^{k-1}\int_0^\varsigma \frac{\varphi\left[w\left(\epsilon_k^j z\right)\right]-1}{\zeta}\,d\zeta\right)\end{array}\right]d\varsigma,$$

Where $w(z)$ is analytic in D with $w(0) = 0$ and $|w(z)| < 1$ $(z \in D)$.

Proof: Suppose that $f \in F_{1,k}^m(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;\varphi)$ then

$$\left(I_1^m(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon)f\right)'(z) = -\frac{F_{1,k}^m(\sigma,\xi,\eta,l;z)}{z}\cdot\varphi(w(z))$$

$$= -z^{-2}\varphi(w(z))\cdot\exp\left(-\frac{1}{k}\sum_{j=0}^{k-1}\int_0^z \frac{\varphi\left[w\left(\epsilon_k^j z\right)\right]-1}{\zeta}\,d\zeta\right).$$

Integrating above equation, theorem holds true.

Theorem 2.2.3: Let $f \in F_{1,k}^m(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;\varphi)$ then

$$I_1^m(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon)f(z) = -\int_0^z\left[\varsigma^{-2}\varphi(w_2(\varsigma))\cdot \exp\left(-\frac{1}{k}\sum_{j=0}^{k-1}\int_0^z \frac{\varphi[w_1(\zeta)]-1}{\zeta}\,d\zeta\right)\right]d\varsigma,$$

Where $w_j(z)$ is analytic in D with $w_j(0) = 0$ and $\left|w_j(z)\right| < 1$ $(z \in D, j = 1,2)$.

Proof: Suppose that $f \in F_{1,k}^m(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;\varphi)$ then

$$-\frac{z\left(f_{1,k}^m(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon;z)\right)'}{p f_{1,k}^m(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon;z)} = \varphi(w_1(z)). \qquad (z \in D)$$

Where $w_1(z)$ is analytic in D with $w_1(0) = 0$ and $|w_1(z)| < 1.$ $(z \in D)$

Thus by applying method of the proof of theorem 2.2.3 we find that

$$f_{1,k}^m(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon;z)$$

$$= z^{-1}.\exp\left(-\int_0^z \frac{\varphi[w_1(\zeta)]-1}{\zeta}\,d\zeta\right).$$

From above equations we get

$$\left(I_1^m(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon)f\right)'(z) = -\frac{f_{1,k}^m(\sigma,\eta,\xi,l;z)}{z}\cdot\varphi(w_2(z))$$

$$= -z^{-2}\cdot\varphi(w_2(z))\cdot\exp\left(-\int_0^z \frac{\varphi[w_1(\zeta)]-1}{\zeta}\,d\zeta\right).$$

Where $w_j(z)$ is analytic in D with $w_j(0) = 0$ and $\left|w_j(z)\right| < 1$ $(z \in D, j = 1,2)$. Integrating both sides of above integral, we readily approach to the assertion of above theorem.

Corollary 2.2.1: Let $f \in G_1^m(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;\varphi)$ then

$$g_1^m(\sigma,\eta,\xi\varepsilon,\delta,\upsilon;z) = z^{-1}.\exp\left(\frac{-1}{z}\int_0^z \frac{\varphi[w(\zeta)]+\overline{\varphi[w(\bar\zeta)]}-2}{\zeta}\,d\zeta\right)$$

Where $g_1^m[\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;z]$

$$= \frac{1}{2}\left[\Omega_1^m(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon)f(z)\right] + \overline{\Omega_1^m(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon)f(\bar z)} = \frac{1}{z}+\cdots \qquad (z \in D).$$

Where $w(z)$ is analytic in D with $w(0) = 0$ and $|w(z)| < 1$ $(z \in D)$

Corollary 2.2.2: Let $f \in G_1^m(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;\varphi)$ then

$$I_1^m(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon)f(z) = -\int_0^z\left[\varsigma^{-2}\varphi(w(\varsigma)) \times\exp\left(\frac{-1}{z}\int_0^z \frac{\varphi[w(\zeta)]+\overline{\varphi[w(\bar\zeta)]}-2}{\zeta}\,d\zeta\right)\right]dz.$$

Where $w(z)$ is analytic in D with $w(0) = 0$ and $|w(z)| < 1$ $(z \in D)$

Corollary 2.2.3: Let $f \in H_1^m(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;\varphi)$ then

$$h_1^m(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon;z) = z^{-1}.\exp\left(\frac{-1}{z}\int_0^z \frac{\varphi[w(\zeta)]-\overline{\varphi[w(\bar\zeta)]}-2}{\zeta}\,d\zeta\right)$$

Where

$$h_1^m[\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;z] = \frac{1}{2}\left[\Omega_1^m(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon)f(z)\right] + \overline{\Omega_1^m(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon)f(-\bar z)} = \frac{1}{z}+\cdots \qquad (h \in S)$$

Where $w(z)$ is analytic in D with $w(0) = 0$ and $|w(z)| < 1.$ $(z \in D)$

Corollary 2.2.4: Let $f \in H_1^m(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;\varphi)$ then

$$I_1^m(\sigma,\eta,\xi,\varepsilon,\delta,\upsilon)f(z) = -\int_0^z \varsigma^{-2}\varphi(w(\varsigma)).\exp\left(\frac{-1}{z}\int_0^z \frac{\varphi[w(\zeta)]-\overline{\varphi[w(\bar\zeta)]}-2}{\zeta}\,d\zeta\right)d\varsigma$$

Where $w(z)$ is analytic in D with $w(0) = 0$ and $|w(z)| < 1.$ $(z \in D)$

2.3. Convolution Properties

In this section we are going to derive several convolution properties for the function classes $F_{1,k}^m(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;\varphi), G_{1,k}^m(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;\varphi)$ and $H_{1,k}^m(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;\varphi)$.

Theorem 2.3.1: Let $f \in F_1^m(\sigma,\xi,\eta,\varepsilon,\delta,\upsilon;\varphi)$ then

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$$f(z)=\Big[-\int_0^z \varsigma^{-2}\,\varphi\big(w(\varsigma)\big)$$
$$\times\, exp\left(\frac{-1}{k}\sum_{j=0}^{k-1}\int_0^{\zeta}\frac{\varphi\big[w\big(\epsilon_k^j\zeta\big)\big]-1}{\zeta}d\zeta\right)d\varsigma\Big]$$
$$*\sum_{n=0}^{\infty}\left[\frac{\varepsilon(\delta-\mathscr{v})}{\sigma(\xi+\eta)+\varepsilon(\delta+\mathscr{v})}\right]^m z^{n-1}.$$

Where $w(z)$ is analytic in D with $w(0)=0$ and $|w(z)|<1 \quad (z\in D)$.

Proof: Since we know that linear operator given by

$$\Omega_1^m(\sigma,\eta,\xi,\varepsilon,\delta,\mathscr{v})f(z)$$
$$=\frac{1}{z}+\sum_{n=0}^{\infty}[\mathscr{b}(1+n)+1]^m a_n z^n$$
$$=\big(\psi_{\sigma,\xi,\eta,\varepsilon,\delta,\mathscr{v}}^{1,m}*f\big)(z)$$

Where $\psi_{\sigma,\xi,\eta,\varepsilon,\delta,\mathscr{v}}^{1,m}(z)=\frac{1}{z}+\sum_{n=0}^{\infty}[\mathscr{b}(1+n)+1]^m z^n$ and $I_1^m(\sigma,\xi,\eta,\varepsilon,\delta,\mathscr{v})f(z)=$

$$-\int_0^z\Big[\varsigma^{-2}\varphi\big(w(\varsigma)\big)$$
$$\times\, exp\left(-\frac{1}{k}\sum_{j=0}^{k-1}\int_0^{\varsigma}\frac{\varphi\big[w\big(\epsilon_k^j z\big)\big]-1}{\zeta}d\zeta\right)\Big]d\varsigma$$
$$-\int_0^z\varsigma^{-2}\,\varphi(w(\varsigma))exp\left(\frac{-1}{k}\sum_{j=0}^{k-1}\int_0^{\zeta}\frac{\varphi\big[w\big(\epsilon_k^j\zeta\big)\big]-1}{\zeta}d\zeta\right)d\varsigma$$
$$\left\{\sum_{n=0}^{\infty}\left[\frac{\varepsilon(\delta-\mathscr{v})}{\sigma(\xi+\eta)+\varepsilon(\delta+\mathscr{v})}\right]^m z^{n-1}\right\}*f(z).$$

Thus from above equation, we can easily get

$$f(z)=\Big\{-\int_0^z\varsigma^{-2}\,\varphi\big(w(\varsigma)\big)$$
$$\times\, exp\left(\frac{-1}{k}\sum_{j=0}^{k-1}\int_0^{\zeta}\frac{\varphi\big[w\big(\epsilon_k^j\zeta\big)\big]-1}{\zeta}d\zeta\right)d\varsigma\Big\}$$
$$*\sum_{n=0}^{\infty}\left[\frac{\varepsilon(\delta-\mathscr{v})}{\sigma(\xi+\eta)+\varepsilon(\delta+\mathscr{v})}\right]^m z^{n-1}.$$

Theorem 2.3.2: Let $f\in F_1^m(\sigma,\xi,\eta,\varepsilon,\delta,\mathscr{v};\varphi)$ then

$$f(z)=\Big\{-\int_0^z\varsigma^{-2}\,\varphi\big(w_2(\varsigma)\big)$$
$$\times\, exp\left(\frac{-1}{k}\sum_{j=0}^{k-1}\int_0^{\zeta}\frac{\varphi[w_1(\zeta)]-1}{\zeta}d\zeta\right)d\varsigma\Big\}$$
$$*\sum_{n=0}^{\infty}\left[\frac{\varepsilon(\delta-\mathscr{v})}{\sigma(\xi+\eta)+\varepsilon(\delta+\mathscr{v})}\right]^m z^{n-1}.$$

Where $w_j(z)$ is analytic in D with $w_j(0)=0$ and $|w_j(z)|<1\ (z\in D, j=1,2)$.

Proof: Since we know that linear operator given by,

$$\Omega_1^m(\sigma,\eta,\xi,\varepsilon,\delta,\mathscr{v})f(z)$$
$$=\frac{1}{z}+\sum_{n=0}^{\infty}[\mathscr{b}(1+n)+1]^m a_n z^n$$
$$=\big(\psi_{\sigma,\xi,\eta,\varepsilon,\delta,\mathscr{v}}^{1,m}*f\big)(z).$$

Where $\psi_{\sigma,\xi,\eta,\varepsilon,\delta,\mathscr{v}}^{1,m}(z)=\frac{1}{z}+\sum_{n=0}^{\infty}[\mathscr{b}(1+n)+1]^m z^n$ and $I_1^m(\sigma,\xi,\eta,\varepsilon,\delta,\mathscr{v})f(z)$

$$=-\int_0^z\begin{bmatrix}\varsigma^{-2}\varphi\big(w_2(\varsigma)\big)\\ \times\, exp\left(-\frac{1}{k}\sum_{j=0}^{k-1}\int_0^{z}\frac{\varphi[w_1(\zeta)]-1}{\zeta}d\zeta\right)\end{bmatrix}d\varsigma.$$

We obtained

$$-\int_0^z\varsigma^{-2}\,\varphi\big(w_2(\varsigma)\big)$$
$$\times\, exp\left(-\sum_{j=0}^{k-1}\int_0^{\zeta}\frac{\varphi\big[w_1\big(\epsilon_k^j\zeta\big)\big]-1}{\zeta}d\zeta\right)d\varsigma$$
$$=\left[\sum_{n=0}^{\infty}\left[\frac{\sigma(\xi+\eta)}{\varepsilon(\delta-\mathscr{v})}+1\right]^m z^{n-1}\right]*f(z).$$

Thus from above result, we can easily prove the Theorem.

Theorem 2.3.3 Let $f\in S$ and $\varphi\in P$. Then $f\in F_{1,k}^m(\sigma,\xi,\eta,\varepsilon,\delta,\mathscr{v};\varphi)$ if and only if

$$f(z)*\begin{Bmatrix}\big(-z^{-1}+\sum_{n=1}^{\infty}[\mathscr{b}+1]^m(n-1)z^{n-1}\big)+\varphi\big(e^{i\theta}\big)\times\\ \big(z^{-1}+\sum_{n=1}^{\infty}[\mathscr{b}+1]^m z^{n-1}\big)*\left(\frac{1}{k}\sum_{n=0}^{\infty}\frac{1}{z^1(1-e^{\nu}z)}\right)\end{Bmatrix}\neq 0$$

$(z\in D^*; 0\leq\theta<2\pi)$.

Proof: Suppose that $f\in F_{1,k}^m(\sigma,\xi,\eta,\varepsilon,\delta,\mathscr{v};\varphi)$. since the following subordination condition:

$$-\frac{z\big(I_1^m(\sigma,\eta,\xi,\varepsilon,\delta,\mathscr{v})f\big)'(z)}{pf_{p1k}^m(\sigma,\eta,\xi,\varepsilon,\delta,\mathscr{v};z)}\prec\varphi(z).\qquad(z\in D).$$

It is equivalent to $-\frac{z\big(I_1^m(\sigma,\eta,\xi,\varepsilon,\delta,\mathscr{v})f\big)'(z)}{pf_{1,k}^m(\sigma,\eta,\xi,\varepsilon,\delta,\mathscr{v};z)}\neq\varphi(e^{i\theta})$ $(z\in D; 0\leq\theta<2\pi)$.

It is easy to verify that the above condition can be written as follows:

(35) $\quad z\big(I_1^m(\sigma,\eta,\xi,\varepsilon,\delta,\mathscr{v})f\big)'(z)$
$+f_{1,k}^m(\sigma,\eta,\xi,\varepsilon,\delta,\mathscr{v};z)\varphi(e^{i\theta})\neq 0.$

On the other hand we find from (10) that

(36) $\quad z\big(I_1^m(\sigma,\eta,\xi,\varepsilon,\delta,\mathscr{v})f\big)'(z)=$
$\big(-z^{-1}+\sum_{n=1}^{\infty}[\mathscr{b}+1]^m(n-1)z^{n-1}\big)*f(z)$

Moreover, from the definition of $f_{1,k}^m(\sigma,\eta,\xi,\varepsilon,\delta,\mathscr{v};z)$, we obtained

(37) $\quad f_{1,k}^m(\sigma,\eta,\xi,\varepsilon,\delta,\mathscr{v};z)$
$=I_1^m(\sigma,\eta,\xi,\varepsilon,\delta,\mathscr{v})f(z)*\left(\frac{1}{k}\sum_{n=0}^{\infty}\frac{1}{z(1-e^{\nu}z)}\right)$
$=\big(z^{-1}+\sum_{n=1}^{\infty}[\mathscr{b}+1]^m z^{n-1}\big)*\left(\frac{1}{k}\sum_{n=0}^{\infty}\frac{1}{z(1-e^{\nu}z)}\right)$
$*f(z)$. Substituting (36) and (37) in (35) we can easily arrive at the convolution property asserted by given theorem. In view of Corollaries 2.2.2 and 2.2.4 and by applying the method similar to method of Theorem 2.2.1 we can easily obtain the following result for the function classes $G_1^m(\sigma,\eta,\xi,l)$ and $H_1^m(\sigma,\eta,\xi,\varepsilon,\delta,\mathscr{v})$.

Corollary 2.3.1: Let $f\in G_{1,k}^m(\sigma,\eta,\xi,\varepsilon,\delta,\mathscr{v},\varphi)$, then

$$f(z)=\begin{bmatrix}-\int_0^z\varsigma^{-2}\,\varphi\big(w_2(\varsigma)\big)\\ \times\, exp\left(\frac{-1}{2}\int_0^{\varsigma}\frac{\varphi[w(\zeta)]+\overline{\varphi\big[w(\bar{\zeta})\big]}-2}{\zeta}d\zeta\right)d\varsigma\end{bmatrix}$$
$$*\left(\sum_{n=0}^{\infty}\left[\frac{\varepsilon(\delta+\mathscr{v})}{\sigma(\eta+\xi)+\varepsilon(\delta+\mathscr{v})}\right]^m z^{n-1}\right)$$

Where w(z) is analytic in D with w(0) = 0 and $|w(z)|<1. \quad (z\in D)$.

Corollary 2.3.2: Let $f\in H_{1,k}^m(\sigma,\eta,\xi,\varepsilon,\delta,\mathscr{v},\varphi)$, Then

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$$f(z)=\begin{bmatrix} -\int_0^z \varsigma^{-2}\,\varphi\big(w_2(\varsigma)\big) \\ exp\left(\frac{-1}{2}\int_0^{\varsigma}\frac{\varphi[w(\zeta)]+\overline{\varphi[w(\bar{\zeta})]}-2}{\zeta}d\zeta\right)d\varsigma \end{bmatrix}$$
$$*\left(\sum_{n=0}^{\infty}\left[\frac{\varepsilon(\delta-\vartheta)}{\sigma(\eta+\xi)+\varepsilon(\delta+\vartheta)}\right]^m z^{n-1}\right).$$

Where $w(z)$ is analytic in D with $w(0)=0$ and $|w(z)|<1.\quad(z\in D)$.

Corollary 2.3.3: Let $f\in S$ and $\varphi\in P$. Then $f\in G_{1,k}^m(\sigma,\xi,\eta,\varepsilon,\delta,\vartheta;\varphi)$ if and only if

$$f*\left[(-z^{-1}+\sum_{n=1}^{\infty}[b+1]^m(n-1)z^{n-1})+\frac{\varphi(e^{i\theta})}{2}I(z)\right]$$
$$+\frac{\varphi(e^{i\theta})}{2}\overline{(I*f)(\bar{z})}\neq 0\qquad(z\in D^*;0\le\theta<2\pi).$$

Where I(z) is given by
$I(z)=z^{-1}+\sum_{n=1}^{\infty}[b+1]^m z^{n-1}$.

Corollary 2.3.4: Let $f\in S$ and $\varphi\in P$. Then $f\in H_1^m(\sigma,\xi,\eta,\varepsilon,\delta,\vartheta;\varphi)$ if and only if

$$f*\begin{bmatrix}(-z^{-1}+\sum_{n=1}^{\infty}[b+1]^m(n-1)z^{n-1}) \\ +\frac{\varphi(e^{i\theta})}{2}I(z)\end{bmatrix}$$
$$-\frac{\varphi(e^{i\theta})}{2}\overline{(I*f)\overline{(-z)}}\neq 0\qquad(z\in D^*;0\le\theta<2\pi).$$

where $f(z)$ is given by above equation.

Remark Specializing the parameters $\sigma,\xi,\eta,\varepsilon,\delta,\vartheta,m,$ A and B in our results, we obtain corresponding results due to various researchers.

References

[1] R. M. Ali, V. Ravichandran, N. Seenivasagan, "Subordin-ation and ordination of the Liu-Srivastava linear operator on meromorphically functions", Bull Malays Math Sci Soc. 31, 193–207 (2008).
[2] M. K. Aouf, "Certain subclasses of meromprphically multivalent functions associated with generalized hyper-geometric function", Comput. Math. Appl. 55, 494–509 (2008).
[3] N. E. Cho, O. S. Kwon, H. M. Srivastava, "A class of integral operators preserving subordination and super-ordination for meromorphic functions", Appl Math Comput. 193, 463–474 (2007).
[4] J. L. Liu, H.M. Srivastava, "A linear operator and associated families of meromorphically multivalent functions", *J. Math Anal Appl*. 259, 566–581 (2001).
[5] J. L. Liu, H. M. Srivastava, "Classes of meromorphically multivalent functions associated with the generalized hyper-geometric function", math. Comput. Model. 39, 21–34 (2004).
[6] Z. G. Wang, Y. P. Jiang, H. M. Srivastava, "Some sub-classes of meromorphic-ally multivalent functions associated with the generalized hypergeometric function", Comput. Math Appl. 57, 571–586 (2009).
[7] Z. G. Wang, Y. Sun, Z. H. Zhang, "Certain classes of meromorphically multivalent functions", Comput Math Appl. 58, 1408–1417 (2009).
[8] T. J. Suffridge, "Some remarks on convex maps of the unit disk*", Duke Math J.* 37, 775–777 (1970).
[9] S. S. Miller, P. T. Mocanu, "Second order differential inequalities in the complex plane", *J Math Anal Appl*. 65, 289–305 (1978).
[10] et al Xu, "Some properties of meromorphically multi-valent functions", *Journal of Inequalities and Applications* 2012 2012:86.
[11] M. K. Aouf and H. M. Hossen, "New criteria for meromorphic p-valent starlike functions", *Tsukuba J. Math*. 17 (1993) 481–486.
[12] T. Bulboaca, "Differential Subordinations and Super-ordinations", Recent Results (House of Scientific Book Publ., Cluj-Napoca, 2005).
[13] N. E. Cho, O. S. Kwon and H. M Srivastava, "Inclusion and argument properties for certain subclasses of meromorphic functions associated with a family of multiplier transformations", *J. Math. Anal. Appl*. 300 (2004) 505–520.
[14] N. E. Cho, O. S. Known and H. M. Srivastava, "Inclusion relationships for certain subclasses of meromorphic functions associated with a family of multiplier transformations", Integral Transforms Spec. Funct. 16 (2005) 647–659.
[15] P. Eenigenburg, S. S. Miller, P. T. Mocanu and M. O. Reade, "on a Briot- Bouquet differential subordination, in General Mathematics 3", International Series of Numerical Mathematics, Vol. 64 (*Birkh¨auser-Verlag, Basel, 1983*), pp. 339–348; see also Rev. Roumaine Math. Pures Appl. 29 (1984) 567–573.
[16] R. M. El-Ashwah, "A note on certain meromorphic p-valent functions", to appear in Appl. Math. Lett.
[17] S. S. Miller and P. T. Mocanu, "Differential subordinations and univalent functions", *Michigan Math. J.* 28 (1981) 157–171.
[18] S. S. Miller and P. T. Mocanu, "on some classes of first order differential subordinations", *Michigan Math. J*. 32 (1985) 185–195.
[19] S. S. Miller and P. T. Mocanu, "Differential Subordinations", Theory and Applications, Series on Monographs and Textbooks in *Pure and Applied Math, Vol. 225 (Marcel Dekker, New York, 2000).*
[20] K. S. Padmanabhan and R. Parvatham, "Some applications of differential subordination", Bull. Austral. Math. Soc. 32 (1985) 321–330.
[21] H. M. Srivastava, D. G. Yang and N. E. Xu, "Some subclasses of meromorphically multivalent functions associated with a linear operator", Appl. Math. Comput.195 (2008) 11–23.
[22] B. A. Uralegaddi and C. Somanatha, "New criteria for meromorphic starlike functions", *Bull. Austral. Math. Soc.* 43 (1991) 137–140.
[23] B. A. Uralegaddi and C. Somanatha, "Certain differential operators for meromorphic functions", *Houston J. Math*. 17 (1991) 279–284.
[24] Z. Z. Zou and Z. R. Wu, "on meromorphically starlike functions and functions meromorphically starlike with

IJERT

respect to symmetric conjugate points", *J. Math. Anal. Appl.*261 (2001) 17–27.
[25] Z.-Z. Zou and Z. R. Wu, "on functions meromorphically starlike with respect to symmetric points", *J. Math. Res. Exposition* 23 (2003) 71–76.

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