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Application of Sowunmi's Lemma on the stability Analysis of a Two Dimensional Infection Age – Structured Mathematical Model of HIV/AIDS

BAWA, Musa
Department of Mathematics/ Computer Science,
Ibrahim Badamasi Babangida University, Lapai, Nigeria.

Abstract--**The Sowunmi's lemma is used to analyse the stability or otherwise of the zero equilibrium state of a two dimensional infection age structured mathematical model of the dynamics of HIV/AIDS. Two compartments of the susceptible and infected members were considered. The susceptables are virus free but prone to infection through specific transmission pattern that is, coming in contact with infected body fluids such as blood, sexual fluids and breast milk. The system gave rise to a set of model equations with one ordinary differential equation and one partial differential equation. The zero equilibrium state and the corresponding characteristics equation were obtained. The latter which arises from the perturbation of the equilibrium state of the system is of transcendental form, from which the lemma applies. The result revealed that to sustain the population, the birth rate must be greater than the death rate among others. This is consistent with the work of Bawa[7].**

1. INTRODUCTION

Mathematical modeling can be defined as the process of creating a mathematical representation of some phenomena in order to gain a better understanding of them. It is therefore an abstraction of reality into the world of mathematics. Any phenomenon which has the ability to grow or decay over time can be represented by a mathematical model and then solved analytically where feasible or in several cases tools of advanced calculus and functional analysis are employed to study and interpret the dynamics. According to Benyah [9], mathematical modeling is an evolving process, as new insight is gained the process begins again as additional factors are considered.

The research work considers the work of Bawa[7],and applied the Sawunmi's lemma to analyse the stability of the zero equilibrium state of the aged structured model which resulted in a system of ordinary and partial differential equations

In this work, the population is partitioned into two compartments of the susceptible S(t), which is the class of members that are virus – free but are prone to infection as they interact with the infected class.The infected class I(t) consists of members that contracted the virus and are at various stages of infection. This class is structured by the infection age, with the density function ρ(t, a) where 't' is the time and 'a' is the infection age.

There is a maximum infection age T at which a member of the infected class must leave the compartment via death i.e. when a = T for $0 \leq a \leq T$. However, a member of the class could still die by natural causes at a rate μ, which is also applicable to the susceptible class S(t).

Members of S(t) move into I(t) at a rate α due to negative change in behavior. The gross death rate via infection is given by $\sigma(a) = \mu + \delta \tan \frac{\pi a}{2TK}$, δ is additional burden from infection while K is a control parameter associated with the measure of slowing down the death of the infected member, which give the victims longer life – span. A high rate of 'K' will imply high effectiveness of such measure and vice – versa.

It is assumed that while the new births in S(t) are born there in, the off-springs of I(t) are divided between S(t) and I(t) in the proportion θ and $1 - \theta$ respectively that is, $1 - \theta$ of the off-springs of I(t) are born with the virus.

2. THE MODEL EQUATIONS

$$S^1 = (\beta - \mu)s(t) + \theta\beta I(t) - \propto s(t)I(t) \qquad (1)$$

$$\text{and } I(t) = \int_0^T \rho(t, a)da, \; 0 \leq a \leq T \qquad (2)$$

$$\frac{\partial \rho}{\partial t}(t, a) + \frac{\partial \rho}{\partial a}(t, a) + (t + \sigma(a))\rho(t, a) = 0 \qquad (3)$$

$$\text{Where } \sigma(a) = \mu + \delta \tan \frac{\pi a}{2TK} \qquad (4)$$

$$\rho(t, 0) = B(t) = \propto s(t)I(t) + (1-\theta)\beta I(t) \qquad (5)$$

$$\text{and } \rho(0, a) = \varphi(a) \qquad (6)$$

$$S(0) = S_0, \; I(0) = I_0 \qquad (7)$$

With the parameters given by

β = natural birth rate for the population;

μ = natural death rate for the population.

$\propto$ = rate of contracting the HIV virus.

σ(a) = gross death rate of the infected class.

δ = additional burden from infection.

K = measure of the effectiveness of efforts at slowing down the death of infected members.

θ = the proportion of the off-springs of the infected which are virus free at birth $0 \le \theta \le 1$.

T = maximum infection age i.e. when a = T the infected member dies of the disease.

3. EQUILIBRIUM STATES

At the equilibrium states, let

$S(0) = x,\ I(0) = y$ (8)

if $\rho(t, a) = \varphi(a)$ (9)

from (1.2), $y = \int_0^T \varphi(a)$ (10)

from (5), $\varphi(0) = \beta(0) = \propto xy + (1-\theta)\beta y$ (11)

Substituting (9) to (11) into (1) and (3)

$(\beta-\mu)\, x + \theta\beta y - \propto xy = 0$ (12)

$\frac{d\varphi\varnothing(a)}{da} + \sigma(a)\varphi(a) = 0$ (13)

$\frac{d\varnothing\varphi(a)}{\varphi(a)} = -\sigma(a)da$ (14)

Integrating both sides

$\varphi(a) = \varphi(0) \exp\{-\int_0^a \sigma(s)\, ds\}$ (15)

Let $\pi(a) = \exp\{-\int_0^a \sigma(s)ds\}$ (16)

That is,

$\varphi(a) = \varphi(0)\pi(a)$ (17)

and

$y = \varphi(0)\int_0^T \pi(a)da = \varphi(0)\bar{\pi}$ (18)

Using (11) and (18)

$$y = (\propto xy + (1-\theta)\,\beta y)\,\bar{\pi} \quad (19)$$

From (12) and (19).

$$x = \frac{1}{\propto}(1 - (1-\theta)\,\beta\bar{\pi}) \quad (20)$$

Substituting (20) in (12)

$$y = \frac{(\beta-\mu)\frac{1}{\propto}(1-(1-\theta)\beta\bar{\pi})}{[\,(1-(1-\theta)\beta\bar{\pi})-\theta\beta]} \quad (21)$$

Hence, the zero equilibrium state, is $(x,y) = (0,0)$ and the non-zero equilibrium state is given by (20) and (21).

4. THE CHARACTERISTICS EQUATION

As in Akinwande [3], let the equilibrium state be perturbed as follows:

$S(t) = x + p(t),\ p(t) = \bar{p}e^{\lambda t}$ (22)

$I(t) = y + q(t);\ q(t) = \bar{q}e^{\lambda t}$ (23)

Let $\rho(t. a) = \varphi(a) + \eta(a)\, e^{\lambda t}$ (24)

With $\bar{q} = \int_0^T (a)da$ (25)

Substituting (22) to (25) into the model equations (1) and (3)

$\lambda)d\frac{d}{dt}\,(x + \bar{p}\,e^{\lambda t}) = (\beta-\mu)\ (x+\bar{p}\,e^{\lambda t}) + \theta\beta(y+\bar{q}e^{\lambda t}) - \propto (x+\bar{p}\,e^{\lambda t})(y+\bar{q}\,e^{\lambda t})$

$\lambda\bar{p}\,e^{\lambda t} = (\beta-\mu)\ x + (\beta-\mu)\,\bar{p}\,e^{\lambda t} + \theta\beta y + \theta\beta\bar{q}e^{\lambda t} - \propto xy - \propto x\bar{q}\,e^{\lambda t} - \propto y\bar{p}\,e^{\lambda t} - \propto\bar{p}\bar{q}\,e^{2\lambda t}$

From equation (12) and neglecting terms of order 2, we have;

$\lambda\,\bar{p}\,e^{\lambda t} = (\beta-\mu)\ \bar{p}e^{\lambda t} + \theta\beta\bar{q}\,e^{\lambda t} - \propto x\bar{q}\,e^{\lambda t} - \propto y\bar{q}\,e^{\lambda t}$ or $(\beta-\mu-\propto y-\lambda)\bar{p} + (\theta\beta-\propto x)\bar{q} = 0$ (26)

Substituting (24) into (3), gives

$\frac{d\rho}{dt}(\varphi(a)\ \eta(a)\ e^{\lambda t}] + \frac{d}{da}[\varphi(a)+\eta(a)\ e^{\lambda t}] + \sigma(a)[\varphi(a)+\eta(a)e^{\lambda t}] = 0$

That is,

$\lambda\eta(a)e^{\lambda t} + \frac{d\varphi(a)}{da} + e^{\lambda t}\frac{d}{da}\eta(a) + \sigma(a)\varphi(a) + \sigma(a)\eta(a)e^{\lambda t} = 0$

Since

$\frac{d\varphi(a)}{da} + \sigma(a)\varphi(a) = 0$

Then

$\lambda(a)e^{\lambda t} + e^{\lambda t}\frac{d}{da}\eta(a) + \sigma(a)\eta(a)e^{\lambda t} = 0$

$\frac{d}{da}\eta^{(a)} + (\sigma(a) + \lambda)\ \eta(a) = 0$ (27)

Solving the Ordinary Differented Equation (27), gives

$$\frac{d\eta(a)}{\eta(a)} = -(\sigma(a)+\lambda)da \qquad (28)$$

$$\eta(a) = \eta(0)\exp\{-\int_0^a(\sigma(s)+\lambda)ds\} \qquad (29)$$

Integrating (29) over [0, T] gives

$$\bar{q} = \eta(0)\int_0^T[\exp\{-\int_0^a(\sigma(s)+\lambda)ds\}]\,da$$

or $\bar{q} = \eta(0)\,b(\lambda)$ (30)

Since $\bar{q} = \eta(0)\,b(\lambda)$, where $b(\lambda) = \int_0^T[\exp\{-\int_0^a(\sigma(s)+\lambda)ds\}\,da$ (31)

$\eta(0)$ is calculated as follows:

From (11), $\varphi(0) = \propto xy + (1-\theta)\beta y$.

and (24) $\rho(t, a) = \varphi(a) + \eta(a)e^{\lambda t}$

But $\rho(t, 0) = \beta(t) = \varphi(0) + \eta(0)e^{\lambda t}$ (32)

From (5), $\beta(t) = \propto s(t) + (1-\theta)\beta I(t)$

Substituting (22) to (25) into (5) and using (11) and (32)

$$B(t) = \propto(x + \bar{p}e^{\lambda t})(y + \bar{q}e^{\lambda t}) + (1-\theta)\beta(y + \bar{q}e^{\lambda t})$$

$$= \propto xy + \propto\bar{p}ye^{\lambda t} + \propto x\bar{q}e^{\lambda t} + \propto\bar{p}\bar{q}e^{2\lambda t} + (1-\theta)\beta y + (1-\theta)\beta\bar{q}e^{\lambda t} \qquad (33)$$

Compare this with (32) using (11) for $\varphi(0)$ gives

$$\propto xy + (1-\theta)\beta y + \eta(0)e^{\lambda t} = \propto xy + \propto\bar{p}ye^{\lambda t} + \propto x\bar{q}e^{\lambda t} + \propto\bar{p}\bar{q}e^{2\lambda t} + (1-\theta)\beta y + (1-\theta)\beta\bar{q}e^{\lambda t}$$

neglecting terms of order 2

$$\eta(0) = \propto\bar{p}y + \propto\bar{q}\chi + (1-\theta)\beta\bar{q} \qquad (34)$$

Substituting $\eta(0)$ in (30)

$$\bar{q} = (\propto y\bar{p} + \propto x\bar{q} + (1-\theta)\beta\bar{q})\,b(\lambda) \qquad (35)$$

$$\propto y\bar{p} + [(\propto x + (1-\theta)\beta)\,b(\lambda) - 1)]\,\bar{q} = 0 \qquad (36)$$

Using (26) and (36), we obtain the Jacobian determinant for the system with the eigen value λ

$$\begin{vmatrix} \beta - \mu - \propto y - \lambda & \theta\beta - \propto x \\ \alpha y & (\propto x + (1-\theta)\beta)b(\lambda) - 1 \end{vmatrix} = 0 \qquad (37)$$

and the characteristics equation is given by:

$$(\beta - \mu - \propto y - \lambda)\,[(\propto x + (1-\theta)\beta)b(\lambda) - 1] - \propto y(\theta\beta - \propto x) = 0 \qquad (38)$$

5. SOWUNMI'S LEMMA

Suppose the characteristics equation arising from the perturbation of the equilibrium state of a dynamical system is of the transcendental form

$$\int_0^\infty e^{-\lambda t}w(t)dt = 1 \qquad (39)$$

Where λ is the eigen value and $w(t)$ is some continuous function of t. Let

$$g(u) = \int_0^\infty e^{-ut}|w(t)|dt \qquad (40)$$

Be the real part of the expression obtained from the left hand side of (39) by setting $\lambda = u \pm iv$, then a necessary condition for the local asymptotic stability of the equilibrium state of the system is given by

$$g(0) < 1 \qquad (41)$$

The equilibrium or steady state is unstable if

$$g(0) > 1 \qquad (42)$$

Akinwande[5].

Proof:

Let $\lambda = u \pm iv$, then (39) gives the pair of equations

$$\int_0^\infty e^{-ut}w(t)\cos vt\,dt = 1 \qquad (43)$$

And

$$\int_0^\infty e^{-ut}w(t)\sin vt\,dt = 0 \qquad (44)$$

From stability theories [1], [2], [4], [6], [8], [9] and [10], a necessary and sufficient condition for local asymptotic stability is for the real part of the eigen value to be in the negative half plane, that is

$$\mathrm{Re}\,\lambda = u < 0$$

From the real part of (43), we derive the function $g(u)$ given by

$$g(u) = \int_0^\infty e^{-ut}|w(t)|dt \quad (45)$$

Which is an exponentially monotone decreasing function of u. Further more, every solution of (45) is less or equal to the solution of (43). The solution U_c will be negative if

$$\int_0^\infty |w(t)|dt < 1 \quad (46)$$

And positive

$$\int_0^\infty |w(t)|dt > 1 \quad (47)$$

6. STABILITY ANALYSIS OF THE ZERO EQUILIBRIUM STATE

The characteristic equation takes the form

$$(\beta - \mu - \lambda)[(1-\theta)\beta b(\lambda)-1] = 0 \quad (48)$$

Which gives

$$(\beta - \mu - \lambda) = 0 \text{ or } (1-\theta)\beta b(\lambda)-1 = 0 \quad (49)$$

$$\lambda_1 = \beta - \mu \quad (50)$$

That is $\lambda_1 < 0$, if $\beta < \mu$

We now apply the sawunmi's lemma [ ], on the transcendental equation in (49). The lemma requires that if the transcendental equation can be expressed in the form

$g(\lambda) = 1,$ where λ is the eigen value.

Then the steady state of the system is unstable if

$$g(0) > 1$$

And a necessary condition for the local asymptotic stability of the equilibrium state is given by

$$g(0) < 1$$

From (49)

$$(1-\theta)\beta b(\lambda) = 1 \quad (51)$$

This gives

$$g(\lambda) = (1-\theta)\beta b(\lambda)$$

For stability of the zero equilibrium state, the inequalities

$$\lambda_1 = \beta - \mu < 0 \quad (52)$$

And

$$g(0) = (1-\theta)\beta b(0) < 1 \quad (53)$$

Should hold simultaneously

But

$$b(\lambda) = \int_0^T \exp\left[-\int_0^a (\sigma(s)+\lambda)ds\right]da$$

So that

$$b(0) = \int_0^T \exp\left[-\int_0^a \sigma(s)ds\right]da$$

Inequality (53) becomes

$$(1-\theta)\beta\int_0^T \exp\left[-\int_0^a \sigma(s)ds\right]da < 1$$

Let

$$D_1 = (1-\theta)\beta\int_0^T \exp\left[-\int_0^a \sigma(s)ds\right]da$$

Then

$$D_1 < 1 \quad (54)$$

Hence, the constraints on the parameters which can guarantee stability are the inequality (54) together with $\beta < \mu$.

CONCLUSION

The zero equilibrium state will be stable when the birth rate is less than the death rate in addition to meeting the requirement of inequality (54). This is consistent with the work of Bawa[7].

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