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Application of Adomian Decomposition Method for Solving a Class of Diffusion Equation

[1] Manjak N. H , [2] A. K Mishra And [3] Rakiya M. K .A
[1]Mathematical Sciences Programme, Abubakar Tafawa Balewa University, Bauchi, Nigeria
[2]Department of Mathematics, Gombe State University, Gombe, Nigeria
[3]Department of Computer Science, Federal Polytechnic Bauchi, Nigeria

Abstract

In this paper we adopt the Adomian Decomposition Method (ADM) for approximating solutions of linear and nonlinear diffusion (parabolic) equations. The solution is calculated in the form of a series with easily computable components. The series solutions of the differential equations considered by the method in this work converge to the exact solutions which illustrate how effective the ADM is in solving such problems.

Keywords: *Nonlinear parabolic equations; Diffusion equation; Adomian polynomial; Adomian decomposition method.*

1. Introduction

Recently a great deal of interest has been focused on the applications of Adomian's decomposition method to solve a wide variety of stochastic and deterministic problems. Adomian's gold is to find a method to unify linear and nonlinear, ordinary and partial differential equations for solving initial and boundary value problems. Afrouzi and Khademloo [6].

Various physical systems can be described by linear and nonlinear parabolic equations. These linear and nonlinear models, as well as their solutions are of fundamental importance for applied sciences.

Our objective in this paper is the desire to obtain analytic solutions to such linear and nonlinear parabolic equations.

The decomposition method yields rapidly convergent series solutions by using a few iterations for both linear and nonlinear deterministic and stochastic equations. The advantage of this method is that it provides a direct scheme for solving the problem without the need for linearization, perturbation, massive computation and any transformation of the governing differential equations. Mustafa Inc [8].

2. Analysis of the Method

In this section, we demonstrate the main algorithm of ADM for linear and nonlinear parabolic equations with initial conditions, we consider the equation

$$\frac{\partial u}{\partial u} = \frac{\partial^2 u}{\partial x^2} + N(u) + g(x,t).$$
$$(x,t)\varepsilon[a,b]\times(0,T) \qquad (2.1)$$

With the following initial condition

$$u(x,0) = f(x), \qquad (2.2)$$

Where, Nu is the nonlinear term. We are looking for a solution satisfying equation (2.1) under condition (2.2). The decomposition method consists of approximating the solution of (2.1) as an infinite series.

$$u(x,t) = \sum_{n}^{\infty} u_n(x,t) \qquad (2.3)$$

Decomposing N (the nonlinear operator) as

$$N(u) = \sum_{n}^{\infty} A_n(u_0, u_1, u_{2,} ..., u_n) \qquad (2.4)$$

Where the A_n's are the Adomian polynomials, Adomian [1], and are calculated owing to the basic formula, Adomian et al [4], Adomian [2-3] and Manjak[7].

$$A_n = A_n(u_0, u_1, u_2 ... u_n) = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{\kappa=0}^{\infty}\lambda^{\kappa}u_{\kappa}\right)\right]_{\lambda=0} \quad n = 1, 2, 3... \quad ... (2.5)$$

From which

$$A_0 = Nu_0$$

$$A_1 = u_1\left(\frac{d}{du_0}\right)N(u_0)$$

$$A_2 = u_2\left(\frac{d}{du_0}\right)N(u_0) + \left(\frac{u_1^2}{2!}\right)\left(\frac{d^2}{du_0^2}\right)N(u_0)$$

$$A_3 = u_3\left(\frac{d}{du_0}\right)N(u_0) + u_1u_2\left(\frac{d^2}{du_0^2}\right)N(u_0)$$

$$\vdots + \left(\frac{u_1^3}{3!}\right)\left(\frac{d^3}{du_0^3}\right)N(u_0)$$

$\vdots$

The A_n `s can be written in the following convenient way

$$A_n = \sum_{\nu=1}^{n} c(\nu, n) f^{\nu}(u_0), n \geq 1,$$

Applying the decomposition method, Mustafa [8], it is convenient to re-write Eq. (2.1) in the standard operator form as

$$L_t u = L_{xx}u + Nu + g(x,t) \qquad (2.6)$$

Where, $L_t = \frac{\partial}{\partial t}; L_{xx} = \frac{\partial^2}{\partial x^2}; N$ is the nonlinear operator and $g(x,t)$ is the forcing term.

The decision as to which operator to solve in a multidimensional problem is made on the basis of the best known conditions and possibly also on the basis of the operator of the lowest order to minimize integration. Bellman and Adomian [9].

The inverse operator L_t^{-1} exists and it can conveniently be taken as the definite integral with respect to t, i.e $L_t^{-1}(,) = \int_0^t (,)dt$ which is a one- fold definite integral, since L_t is a first- order operator.

Operating with L_t^{-1} on both side of (2.6) yields.

$$L_t^{-1}L_t u = L_t^{-1}g(x,t) + L_t^{-1}Nu + L_t^{-1}L_{xx}u \quad (2.7)$$

So that

$$u(x,t) = u(x,0) + L_t^{-1}g(x,t) + L_t^{-1}Nu + L_t^{-1}L_{xx}u \quad (2.8)$$

Substituting initial condition (2.2) into Eq. (2.8) to have

$$u(x,t) = f(x) + L_t^{-1}g(x,t) + L_t^{-1}Nu + L_t^{-1}L_{xx}u \quad (2.9)$$

Substituting (2.3) and (2.4) into (2.9) we have

$$u(x,t) = f(x) + L_t^{-1}g(x,t) + L_t^{-1}(\sum_{n=0}^{\infty}A_n) + L_t^{-1}L_{xx}(\sum_{n=o}^{\infty}u_n) \quad ... \quad (2.10)$$

From (2.10) the Adomian decomposition scheme is defined by the recurrent relation

$$u_0(x,t) = f(x) + L_t^{-1}g(x,t)$$

And, $u_{n+1}(x,t) = L_t^{-1}A_n + L_t^{-1}L_{xx}u_n$ for n= 0, 1, 2, ...

From which $u_1(x,t) = L_t^{-1}A_0 + L_t^{-1}L_{xx}u_0$

$$u_2(x,t) = L_t^{-1}A_1 + L_t^{-1}L_{xx}u_1$$

$\vdots$

We can determine the components u_n as is necessary to enhance the desired accuracy for the approximation. So, the n-terms approximation can be used to approximate the solution.

That is $\varphi_n = \sum_{i=0}^{n-1} u_i$.

3. Some Analytic Solutions

To give a clear view of our study and to illustrate the above discussed technique, we will consider both linear and nonlinear diffusion equations in this section.

Problem 3.1

Consider the following one dimensional linear heat (heat) equation given by

$$k(x)u_t = u_{xx} \tag{3.1}$$

Subject to the initial boundary conditions

$$u(x,0) = g(x), 0 < x < 1 \tag{3.2}$$

$$u(0,t) = h_1(t), u(1,t) = h_2(t)$$

We let $k(x) = 4$ and $g(x) = \sin \pi x$.

For if $k(x) = k^2$, a constant, then the exact solution of (3.1) is given by

$u(x,t) = e^{\frac{-t\pi^2}{k^2}} \sin \pi x$ Bellman and Adomian [5].

Now, using Adomian decomposition method, re-write equation (3.1) in the general operator form as

$$4L_t u = L_{xx} u \tag{3.3}$$

Where $L_t = \frac{\partial}{\partial t}; L_{xx} = \frac{\partial^2}{\partial x^2}$

Operating with L_t^{-1} on equation (3.3) to have

$$L_t^{-1} L_t u = L_t^{-1}(\frac{1}{4} L_{xx} u) \tag{3.4}$$

Evaluating the L.H.S of (3.4) and substitute back into (3.4) to yield

$$u(x,t) = u(x,0) + L_t^{-1}(\frac{1}{4} L_{xx} \sum_{n=0}^{\infty} u_n) \tag{3.5}$$

From which we obtain the recurrent relation

$$u_o = u(x,0) = \sin \pi x, and, u_{n+1} = L_t^{-1}(\frac{1}{4} L_{xx} u_n), n = 0,1,2,...$$

From which $u_1 = L_t^{-1}(\frac{1}{4} L_{xx} u_0)$

$$u_2 = L_t^{-1}(\frac{1}{4} L_{xx} u_1)$$

$$\vdots$$

$$u_1 = L_t^{-1}[\frac{1}{4} L_{xx}(\sin \pi x)] = -\frac{t\pi^2 \sin \pi x}{4}$$

$$u_2 = L_t^{-1}[\frac{1}{4} L_{xx}(-\frac{t\pi^2 \sin \pi x}{4})] = \frac{t^2\pi^4 \sin \pi x}{32}$$

$$u_3 = L_t^{-1}[\frac{1}{4} L_{xx}(-\frac{t^2\pi^4 \sin \pi x}{32})] = \frac{t^3\pi^6 \sin \pi x}{384}$$

$$\vdots$$

$$\vdots$$

And so in; in this manner the rest of the components of the decomposition series (2.3) can be obtained. The solution for the heat (diffusion) equation (3.1) in a series form is given by

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = u_0 + u_1 + u_2 + u_3 + ...$$

Therefore, the solution obtained from the method will be

$$u(x,t) = \sin \pi x(1 - \frac{t\pi^2}{4} + \frac{t^2\pi^4}{32} + \frac{t^3\pi^6}{384} + ...)$$

$$= \sin \pi x\left(1 - \frac{\left(\frac{t\pi^2}{4}\right)}{1!} + \frac{\left(\frac{t\pi^4}{4}\right)^2}{2!} - \frac{\left(\frac{t\pi^2}{4}\right)^3}{3!} + \ldots\right)$$

$$u(x,t) = \sin \pi x\left(e^{\frac{-t\pi^2}{4^2}}\right)$$

Yielding the same result as the exact solution of the differential equation, where

$$k(x) = k^2 = 4.$$

Problem 3.2

Consider the following nonlinear reaction-diffusion equation

$$u_t - u_{xx} = u^2 - (u_x)^2 \qquad (3.6)$$

subject to the initial condition

$u(x,0) = e^x$ Mustafa [8].

Re-writing (3.6) in the general form, we have

$$L_t u - L_{xx} = Nu \qquad (3.7)$$

Where, $L_t = \frac{\partial}{\partial t}, L_{xx} = \frac{\partial^2}{\partial x^2}$ and $Nu = u^2 - (u_x)^2$

In this case, the operator $L_t = \frac{\partial}{\partial t}$ with inverse $L^{-1}_t = \int_0^t (.)dt$. ... (3.8)

Applying the inverse operator L^{-1}_t in (3.8), we have

$$L_t^{-1} L_t u = L_t^{-1} Nu + L_t^{-1} L_{xx} u \qquad (3.9)$$

Evaluate the L.H.S of (3.9) and substitute back into the equation to obtain

$$u(x,t) = u(x,0) + L_t^{-1} \sum_{n=0}^{\infty} A_n + L_t^{-1} L_{xx} \sum_{n=0}^{\infty} u_n \qquad (3.10)$$

From (3.10) the solution by the decomposition method consists of the following scheme:

$$u_0 = u(x,0) = e^x$$

$u_{n+1} = L_t^{-1} A_n + L_t^{-1} L_{xx} u_n, n \geq 0$, From which

$$u_1 = L_t^{-1} A_0 + L_t^{-1} L_{xx} u_0,$$
$$u_2 = L_t^{-1} A_1 + L_t^{-1} L_{xx} u_1,$$
$$u_3 = L_t^{-1} A_2 + L_t^{-1} L_{xx} u_2, \ldots$$

Whence,

$$Nu = f(u, u_x) = u^2 - u_x^2$$

$$Nu_0 = f(u_0, u_{0_x}) = u_0^2 - (u_{0_x})^2 = e^{2x} - e^{2x}$$

And $u_1 = L_t^{-1}[u_o^2 - (u_{0_x})^2] + L_t^{-1} L_{xx}[u_0]$

$$= L_t^{-1}[e^{2x} - e^{2x}] + L_t^{-1} L_{xx}[e^x]$$

$$= L_t^{-1}[e^x] = \int e^x dt = te^x$$

Ie $u_1 = te^x / 1!$

$$u_2 = L_t^{-1} A_1 + L_t^{-1} L_{xx} u_1$$

And $A_n = \sum_{i=1}^{n} c(i,n) f^i u_0, n = 1,2,3,\ldots$

i.e.

$$A_1 = \sum_{i=1}^{n} c(i,n) f^i u_0 = c(1,1) f^i u_0 = 2u_1[u_0 - u_{0_x}]$$

for $$f(u_0, u_{0_x}) = u_0^2 - u_{0_x}^2.$$

$$\therefore f'(u_0, u_{0_x}) = 2u_0 - 2u_{0_x} = 2(u_0 - u_{0_x})$$

And $$A_1 = 2(u_0 - u_{0_x}) = 2(te^x)[e^x - e^x] = 0$$

Therefore,

$$u_2 = L_t^{-1} L_{xx} u_1 = L_t^{-1} L_{xx}[te^x] = \int te^x dt = \frac{t^2}{2} e^x = \frac{t^2}{2!} e^x$$

$$u_3 = L_t^{-1} A_2 + L_t^{-1} L_{xx} u_2$$

From $$A_n = \sum_{i=1}^{n} c(i,n) f^i u_0, n = 1,2,3,...$$

$$A_2 = \sum_{i=1}^{2} c(i,2) f^i u_0,$$ i,e

$$A_2 = c(1,2) f' u_0 + c(2,2) f'' u_0$$

But $$f(u_0) = f(u_0, u_{0_x}) = u_0^2 - u_{0_x}^2.$$

Therefore

$$f'(u_0) = f'(u_0, u_{0_x}) = 2(u_0 - u_{0_x})$$ and

$$f''(u_0) = f''(u_0, u_{0_x}) = 2(1-1) = 0$$

We obtain $$A_2 = u_2 2[u_0 - u_{0_x}]$$

$$= L_t^{-1}[\frac{t^2}{2} e^x] + L_t^{-1}[\frac{t^2}{2} e^x 2(e^x - e^x)]$$

$$= L_t^{-1}[\frac{t^2}{2} e^x] + 0$$

$$= \int \frac{t^2}{2} e^x dt = \frac{t^3}{3!} e^x$$

i.e $$u_3 = \frac{t^3}{3!} e^x$$

in a similar manner, other terms of the series can be generated.

The solution u(x,t)of the reaction- diffusion equation (3.6) in series form is then given by

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = u_0 + u_1 + u_2 + u_3 + ...$$

$$= e^x + te^x + \frac{t^2}{2} e^x + \frac{t^3}{3!} e^x + ... = e^x[1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + ... = e^x e^t.$$

This gives the exact solution by

$$u(x,t) = e^{x+t}.$$

In view of the forgoing, some important conclusions can be made here.

4. Conclusion

Linear and non parabolic equations play an important role in applied sciences. The basic goal of this paper has been to employ ADM for studying this model. The goal has been achieved by deriving the exact solutions for linear and nonlinear cases by using few terms of the series only. The decomposition introduces a significant development and improvement in the field of solution methods and this makes the scheme powerful and gives a wider applicability. The method avoids the difficulties and massive computational work that usually arise from other classical methods. It gives more realistic series solution that converges very rapidly in physical problems.

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