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Analysis of the Outer Region, Wall Region and Fluid-Solid Interface in Turbulent Pipe Flow

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***Abstract*—Theoretical analysis along with empirical relations are used to split turbulent flow into two main regions; the outer region and the wall region. The location of the boundary between the outer region and wall region was found. Results fall within observed range and show the trend in the boundary location with respect to variation in Reynolds number and relative roughness. The study presented also shades much needed light on turbulent pipe flows.**

Keywords*— *Pipe flow; turbulence; friction; viscous sublayer; fluid flow

I. INTRODUCTION

The study of developed turbulent flows in pipes is of great interest, since all flows are turbulent in nature and laminar flows are just idealization based on suitable conditions for simplification. In pipe flows, laminar flows can be assumed when Re ≤ 2000, and assumed to be turbulent when Re > 2000, though there is a transition regime of 2000 < Re < 4000 and under controlled laboratory condition, laminar flows have been observed in flows with Re up to 40000 [1]. Turbulent flows are characterized by fluctuations in velocity in all coordinate directions; however, the time averaged values are usually used in turbulent flow analysis [1]. The turbulent flow may be divided into 3 main regions; viscous sublayer at $0 \leq y^+ \leq 5$, buffer zone at $5 < y^+ < 30$ and outer region at $y^+ \geq 30$ [1] [2]. Where y^+ is the normalized distance from the wall which is defined as

$$y^+ = \frac{u_\tau y}{\upsilon} \quad (1)$$

Where u_τ is the shear velocity defined by $u_\tau = \sqrt{\tau_0/\rho}$, τ_0 is the shear stress at the wall, ρ is the fluid density, y is the linear distance from the wall and υ is the kinematic viscosity [1] [2]. The viscous sublayer has been identified as the region critical for the understanding of wall bounded turbulent flow [2] [3] [4] [5] [6], because experimental studies have shown that the nature of flow close to the wall has an important influence on the wall shear stress and energy as indicated by eddies develop in the viscous sublayer and flow out to the other regions [3] [4] [5]. Understanding the viscous sublayer may also be the key to understanding how Drag Reduction Agents work [7].

Specifically, the velocity profile of the viscous sublayer determines the wall shear stress τ_0 which is an important parameter for describing turbulent flows [2]. In experiments, it is not easy to obtain accurate velocity measurements within the viscous sublayer [2]. Though there is a wealth of experimental information on turbulence close to the wall. Experimental study of viscous sublayer has included visual observation using ultra microscope [3] [8], hot-wire measurements [3], oil-film interferometry [2] [9] [10], and micro-pillar shear-stress sensors [11] [12]. All these experimental methods have limitations and challenges [2] which results in errors in their results. Empirical relations have been developed for the velocity profile in wall bounded turbulent flows for smooth walls and rough walls [1] [6]. The velocity profile for a smooth pipe is given as

$$u^+ = y^+ \quad \text{for} \quad 0 \leq y^+ \leq 5 \quad (2)$$

$$u^+ = 2.44 ln y^+ + 4.9 \quad \text{for} \quad 30 < y^+ \text{and} \ \frac{y}{R} < 0.15 \quad (3)$$

$$u^+_{max} - u^+ = 2.44 ln\left(\frac{R}{y}\right) + 0.8 \ \text{for the outer region} \quad (4)$$

where R is the internal radius of the pipe and u_{max} is the maximum velocity. In the buffer zone ($5 < y^+ < 30$), the two curves (equations (2) and (3)) are merged. For rough pipes, the viscous sublayer has less influence because the protruding wall elements dorminate the generation of turbulence, so the velocity profile becomes a function of pipe roughness, e, and is given as [1]

$$u^+ = 2.44 ln\left(\frac{y}{e}\right) + 8.5 \quad \text{for} \quad 30 < y^+ \text{and} \ \frac{y}{R} < 0.15 \quad (5)$$

The maximum velocity is given as [1]

$$u^+_{max} = 2.44 ln\left(\frac{u_\tau R}{\upsilon}\right) + 5.7 \quad \text{for smooth pipes} \quad (6)$$

$$\text{and} \quad u^+_{max} = 2.44 ln\left(\frac{R}{e}\right) + 9.3 \quad \text{for rough pipes} \quad (7)$$

The wall shear stress, if not measured directly for example by the new micro-pillar shear-stress sensor, can be found by using pressure sensors and the equation

$$\tau_0 = -\frac{R}{2}\frac{dP}{dx} \quad (8)$$

where dP/dx is the pressure gradient. τ_0 can also be found using the coefficient of friction, f, with the equation [1]

$$\tau_0 = \frac{1}{8}\rho f V^2 \quad (9)$$

where V is the average velocity. Additionally, a simpler velocity profile which adequately describes the turbulent flow velocity in a pipe is the power-law profile given as [1]

$$\frac{u}{u_{max}} = \left(\frac{y}{R}\right)^{1/n} \quad (10)$$

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where the maximum velocity is given as

$$u_{max} = \frac{(n+1)(2n+1)}{2n^2}V \tag{11}$$

and n is an integer between 5 and 10 related to the coefficient of friction by the empirical relation

$$n = \frac{1}{\sqrt{f}} \tag{12}$$

The power law fails to adequately describe velocity in the wall region and would yield infinite shear stress at the wall [1].

There is insufficient theoretical analysis to interpret the experimental observations of turbulence close to the wall [3]. Some authors have contributed theoretically [3] [13] [14] [15] [16] and the theory has been trying to catch up with experimental observation starting with assumption of laminar flow in the region to the assumption of turbulent flow in the region with the region being pointed as the key source of turbulence. Some of the theories have used assumption that there is slip at the wall [5] [13].

In this paper, the outer region described by the power law is combined with a selected empirical friction model to develop a model for describing the viscous sublayer.

II. METHODS

Turbulent flows in pipes are dependent on the wall condition which is either smooth or rough, so two cases will be treated in the analysis of turbulent flow in pipes.

A. Case 1: Turbulent flow in smooth pipes

The simplest form of the velocity profile for a turbulent flow is the power-law profile [1] which will be adopted for this work, but due to its limitation at the centerline and at the wall, appropriate equations from common solutions of the Navier Stokes equation will be used as presented below

$$u(r) = \begin{cases} u_0 = u_{max}\left(1-\left(\frac{r}{R}\right)^2\right) & 0 \le r \le r_0 \\ u_1 = u_{max}\left(1-\frac{r}{R}\right)^{1/n} & r_0 \le r \le r_1 \\ u_2 = Aln(r) + Br^2 + C & r_1 \le r \le R \end{cases} \tag{13}$$

Where u_{max} is the maximum velocity, A and B are constants, while r_0 and r_1 are the points where the centerline velocity and wall velocity matches the power-law profile respectively. The boundary conditions (BCs) are

$$u_0 = u_1 \qquad @\ r = r_0 \tag{14}$$

$$u_1 = u_2 \qquad @\ r = r_1 \tag{15}$$

$$\frac{du_1}{dr} = \frac{du_2}{dr} \qquad @\ r = r_1 \tag{16}$$

$$u_2 = 0 \qquad @\ r = R \tag{17}$$

Applying the first BC (equation (14)) gives

$$r_0 = 1.2036Rn^{-1.058} \tag{18}$$

Applying BC 2 (equation (15)) gives

$$u_{max}\left(1-\frac{r_1}{R}\right)^{1/n} = Aln(r_1) + Br_1^2 + C \tag{19}$$

Applying BC 3 (equation (15)) gives

$$-\frac{u_{max}}{nR}\left(1-\frac{r_1}{R}\right)^{\frac{1}{n}-1} = \frac{A}{r_1} + 2Br_1 \tag{20}$$

Applying BC 4 (equation (16)) gives

$$Aln(R) + BR^2 + C = 0 \tag{21}$$

Solving equations (19), (20) and (21) gives

$$A = \frac{\left(\frac{u_{max}}{n}\right)\left(\frac{r_1}{R}\right)\left(2n\frac{r_1}{R}\left(1-\frac{r_1}{R}\right)^{1/n} - \left(1-\left(\frac{r_1}{R}\right)^2\right)\left(1-\frac{r_1}{R}\right)^{\frac{1}{n}-1}\right)}{\left(1-\left(\frac{r_1}{R}\right)^2 + 2\left(\frac{r_1}{R}\right)^2 \ln\left(\frac{r_1}{R}\right)\right)} \tag{22}$$

$$B = \frac{\left(\frac{u_{max}}{nR^2}\right)\left(-n\left(1-\frac{r_1}{R}\right)^{1/n} - \left(\frac{r_1}{R}\right)\ln\left(\frac{r_1}{R}\right)\left(1-\frac{r_1}{R}\right)^{\frac{1}{n}-1}\right)}{\left(1-\left(\frac{r_1}{R}\right)^2 + 2\left(\frac{r_1}{R}\right)^2 \ln\left(\frac{r_1}{R}\right)\right)} \tag{23}$$

And

$$C = \frac{\left(\frac{u_{max}}{n}\right)\left(n\left(1-2\left(\frac{r_1}{R}\right)^2 \ln(R)\right)\left(1-\frac{r_1}{R}\right)^{1/n} + \left(\left(\frac{r_1}{R}\right)^2 \ln\left(\frac{r_1}{R}\right) - \left(\frac{r_1}{R}\right)^3 \ln(R)\right)\left(1-\frac{r_1}{R}\right)^{\frac{1}{n}-1}\right)}{\left(1-\left(\frac{r_1}{R}\right)^2 + 2\left(\frac{r_1}{R}\right)^2 \ln\left(\frac{r_1}{R}\right)\right)} \tag{24}$$

Where r_1 and u_{max} are still unknown. u_{max} can be found from the average velocity value as shown below

$$V = \frac{\int_0^{r_0} u_0(r)2\pi r dr + \int_{r_0}^{R} u_1(r)2\pi r dr}{\pi R^2} = \frac{\int_0^{r_0} u_{max}\left(1-\left(\frac{r}{R}\right)^2\right)2\pi r dr + \int_{r_0}^{R} u_{max}\left(1-\frac{r}{R}\right)^{1/n} 2\pi r dr}{\pi R^2}$$

Here it is assumed that the error in using u_1 to cover for u_2 is negligible since integration is involved. Therefore, integrating and solving for u_{max} gives

$$u_{max} = 1.4611Vn^{0.8885} \tag{25}$$

r_1 is found by applying equation (19) at r = R

$$\tau_0 = \frac{1}{8}\rho f V^2 = -\mu \left.\frac{du_2}{dr}\right|_{r=R} = -\mu\left(\frac{A}{R} + 2BR\right) \tag{26}$$

Equations (23) and (23) into (26) gives

$$\frac{1}{8}\rho f V^2 = -\mu\left(\frac{1}{R}\left(\frac{\left(\frac{1.4611Vn^{0.8885}}{n}\right)\left(\frac{r_1}{R}\right)\left(2n\frac{r_1}{R}\left(1-\frac{r_1}{R}\right)^{1/n} - \left(1-\left(\frac{r_1}{R}\right)^2\right)\left(1-\frac{r_1}{R}\right)^{\frac{1}{n}-1}\right)}{\left(1-\left(\frac{r_1}{R}\right)^2 + 2\left(\frac{r_1}{R}\right)^2 \ln\left(\frac{r_1}{R}\right)\right)}\right) + 2\left(\frac{\left(\frac{1.4611Vn^{0.8885}}{nR^2}\right)\left(-n\left(1-\frac{r_1}{R}\right)^{1/n} - \left(\frac{r_1}{R}\right)\ln\left(\frac{r_1}{R}\right)\left(1-\frac{r_1}{R}\right)^{\frac{1}{n}-1}\right)}{\left(1-\left(\frac{r_1}{R}\right)^2 + 2\left(\frac{r_1}{R}\right)^2 \ln\left(\frac{r_1}{R}\right)\right)}\right)R\right) \tag{27}$$

Equation (27) simplifies to

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$$\frac{Re}{23.3776} = -n^2\left(\left(\frac{\left(\frac{r_1}{R}\right)\left(2n\frac{r_1}{R}\left(1-\frac{r_1}{R}\right)^{1/n}-\left(1-\left(\frac{r_1}{R}\right)^2\right)\left(1-\frac{r_1}{R}\right)^{\frac{1}{n}-1}\right)}{n^{0.1115}\left(1-\left(\frac{r_1}{R}\right)^2+2\left(\frac{r_1}{R}\right)^2\ln\left(\frac{r_1}{R}\right)\right)}\right)+ 2\left(\frac{\left(-n\left(1-\frac{r_1}{R}\right)^{1/n}-\left(\frac{r_1}{R}\right)\ln\left(\frac{r_1}{R}\right)\left(1-\frac{r_1}{R}\right)^{\frac{1}{n}-1}\right)}{n^{0.1115}\left(1-\left(\frac{r_1}{R}\right)^2+2\left(\frac{r_1}{R}\right)^2\ln\left(\frac{r_1}{R}\right)\right)}\right)\right) \tag{28}$$

Assuming that r_1 falls in the wall region such that r_1/R ranges from 0.805 to 1, equation (28) can be simplified to

$$\frac{Re}{23.3776} = \frac{17.029n^2 - 168.745n + 443.692}{1-\frac{r_1}{R}} \tag{29}$$

Therefore,

$$\frac{r_1}{R} = 1 - \frac{23.3776}{Re}\left(17.029n^2 - 168.745n + 443.692\right) \tag{30}$$

At this point all unknowns can be estimated for the velocity profile of turbulent flow in a smooth pipe as given by equation (13), so that appropriate analysis can be performed with it. Equation (30) can also be used to estimate r_1/R by using an equation for the friction coefficient for a smooth pipe. An equation similar to that of Petukov [17] was developed using more recent experimental data published in McKeon et al work [18]

$$n = \frac{1}{\sqrt{f}} = 0.762\ln(Re) - 1.3328 \tag{31}$$

(valid for $4000 \leq Re \leq 36000000$)

The value of y^+ corresponding to r_1 can be found from a further simplified form of equation (1)

$$y^+ = \frac{y}{\upsilon}\sqrt{\frac{\tau_0}{\rho}} = \frac{y}{\upsilon}\sqrt{\frac{\rho V^2 f}{8\rho}} = \frac{y}{D}\frac{Re}{2}\sqrt{\frac{f}{2}} = \frac{(R-r)}{D}\frac{Re}{2}\sqrt{\frac{f}{2}} = \frac{\left(1-\frac{r}{R}\right)}{2}\frac{Re}{2}\sqrt{\frac{f}{2}} = \left(\frac{1}{2}-\frac{r}{D}\right)\frac{Re}{2}\sqrt{\frac{f}{2}} \tag{32}$$

B. *Case 2: Turbulent flow in rough pipes*

The simplest form of the velocity profile for a turbulent flow is the power-law profile [1] will also be adopted for this case, and the common solutions of the Navier Stokes equation used in case 1 will also be used as presented below

$$u(r) = \begin{cases} u_0 = u_{max}\left(1-\left(\frac{r}{R}\right)^2\right) & 0 \leq r \leq r_0 \\ u_1 = u_{max}\left(1-\frac{r}{R}\right)^{1/n} & r_0 \leq r \leq r_1 \\ u_2 = A\ln(r) + Br^2 + C & r_1 \leq r \leq \varepsilon \\ u_3 = Er^2 + Fr + G & \varepsilon \leq r \leq R \end{cases} \tag{33}$$

Additionally, a parabolic velocity profile was adopted for the rough region since it very thin and Popovich and Hummel [19] found in their experimental work that the thin layer of fluid at the wall was characterized with a linear velocity gradient, so a parabolic profile over an infinitesimal will look linear and the constant E will be small after the analysis. Compared to **case 1**, E, F and G are new constants to be found and ε is pipe roughness and is known for the pipe material. The boundary conditions (BCs) are

$$u_0 = u_1 \qquad @\ r = r_0 \tag{34}$$

$$u_1 = u_2 \qquad @\ r = r_1 \tag{35}$$

$$\frac{du_1}{dr} = \frac{du_2}{dr} \qquad @\ r = r_1 \tag{36}$$

$$u_2 = u_3 \qquad @\ r = R - \varepsilon \tag{37}$$

$$u_3 = 0 \qquad @\ r = R \tag{38}$$

Applying the first BC 1 (equation (34)) gives

$$r_0 = 1.2036Rn^{-1.058} \tag{39}$$

Applying BC 2 (equation (35)) gives

$$u_{max}\left(1-\frac{r_1}{R}\right)^{1/n} = A\ln(r_1) + Br_1^2 + C \tag{40}$$

Applying BC 3 (equation (36)) gives

$$-\frac{u_{max}}{nR}\left(1-\frac{r_1}{R}\right)^{\frac{1}{n}-1} = \frac{A}{r_1} + 2Br_1 \tag{41}$$

Applying BC 4 (equation (37)) gives

$$A\ln(R-\varepsilon) + B(R-\varepsilon)^2 + C = E(R-\varepsilon)^2 + F(R-\varepsilon) + G \tag{42}$$

Applying BC 4 (equation (38)) gives

$$ER^2 + FR + G = 0 \tag{43}$$

Assuming that R >> ε, equation (42) can be expanded so that the method of matched asymptotic expansion can be applied:

$$A\left(\ln(R) - \frac{\varepsilon}{R} - \frac{\varepsilon^2}{2R^2} - \frac{\varepsilon^3}{3R^3} - \frac{\varepsilon^4}{4R^4} - \cdots - O(\varepsilon^5)\right) + B(R^2 - 2R\varepsilon - \varepsilon^2) + C = E(R^2 - 2R\varepsilon - \varepsilon^2) + F(R-\varepsilon) + G \tag{44}$$

Equating coefficients

$$\varepsilon 0: \quad A\ln(R) + BR^2 + C = 0 \tag{45}$$

$$\varepsilon 1: \quad -\frac{A}{R} - 2BR = -2ER - F \tag{46}$$

$$\varepsilon 2: \quad -\frac{A}{2R^2} + B = E \tag{47}$$

Equations (40), (41) and (45) retains the terms from case one, so the solution for A, B and C are the same as that of **case 1**, but E and F needs to be found by solving equations (46) and (47) simultaneously, we have

$$A = \frac{\left(\frac{u_{max}}{n}\right)\left(\frac{r_1}{R}\right)\left(2n\frac{r_1}{R}\left(1-\frac{r_1}{R}\right)^{1/n}-\left(1-\left(\frac{r_1}{R}\right)^2\right)\left(1-\frac{r_1}{R}\right)^{\frac{1}{n}-1}\right)}{\left(1-\left(\frac{r_1}{R}\right)^2+2\left(\frac{r_1}{R}\right)^2\ln\left(\frac{r_1}{R}\right)\right)} \tag{48}$$

$$B = \frac{\left(\frac{u_{max}}{nR^2}\right)\left(-n\left(1-\frac{r_1}{R}\right)^{1/n}-\left(\frac{r_1}{R}\right)\ln\left(\frac{r_1}{R}\right)\left(1-\frac{r_1}{R}\right)^{\frac{1}{n}-1}\right)}{\left(1-\left(\frac{r_1}{R}\right)^2+2\left(\frac{r_1}{R}\right)^2\ln\left(\frac{r_1}{R}\right)\right)} \tag{49}$$

$$C = \frac{\left(\frac{u_{max}}{n}\right)\left(n\left(1-2\left(\frac{r_1}{R}\right)^2\ln(R)\right)\left(1-\frac{r_1}{R}\right)^{1/n}+\left(\left(\frac{r_1}{R}\right)^2\ln\left(\frac{r_1}{R}\right)-\left(\frac{r_1}{R}\right)^3\ln(R)\right)\left(1-\frac{r_1}{R}\right)^{\frac{1}{n}-1}\right)}{\left(1-\left(\frac{r_1}{R}\right)^2+2\left(\frac{r_1}{R}\right)^2\ln\left(\frac{r_1}{R}\right)\right)} \tag{50}$$

$$E = \frac{\left(\frac{u_{max}}{nR^2}\right)\left(-n\left(1-\frac{r_1}{R}\right)^{1/n}-\left(\frac{r_1}{R}\right)\ln\left(\frac{r_1}{R}\right)\left(1-\frac{r_1}{R}\right)^{\frac{1}{n}-1}-\frac{1}{2}\left(\frac{r_1}{R}\right)\left(2n\frac{r_1}{R}\left(1-\frac{r_1}{R}\right)^{1/n}-\left(1-\left(\frac{r_1}{R}\right)^2\right)\left(1-\frac{r_1}{R}\right)^{\frac{1}{n}-1}\right)\right)}{\left(1-\left(\frac{r_1}{R}\right)^2+2\left(\frac{r_1}{R}\right)^2\ln\left(\frac{r_1}{R}\right)\right)} \quad (51)$$

$$F = \frac{\frac{2}{R}\left(\frac{u_{max}}{n}\right)\left(\frac{r_1}{R}\right)\left(2n\frac{r_1}{R}\left(1-\frac{r_1}{R}\right)^{1/n}-\left(1-\left(\frac{r_1}{R}\right)^2\right)\left(1-\frac{r_1}{R}\right)^{\frac{1}{n}-1}\right)}{\left(1-\left(\frac{r_1}{R}\right)^2+2\left(\frac{r_1}{R}\right)^2\ln\left(\frac{r_1}{R}\right)\right)} \quad (52)$$

$$G = -\frac{\left(\frac{u_{max}}{n}\right)\left(-n\left(1-\frac{r_1}{R}\right)^{1/n}-\left(\frac{r_1}{R}\right)\ln\left(\frac{r_1}{R}\right)\left(1-\frac{r_1}{R}\right)^{\frac{1}{n}-1}-\frac{1}{2}\left(\frac{r_1}{R}\right)\left(2n\frac{r_1}{R}\left(1-\frac{r_1}{R}\right)^{1/n}-\left(1-\left(\frac{r_1}{R}\right)^2\right)\left(1-\frac{r_1}{R}\right)^{\frac{1}{n}-1}\right)\right)}{\left(1-\left(\frac{r_1}{R}\right)^2+2\left(\frac{r_1}{R}\right)^2\ln\left(\frac{r_1}{R}\right)\right)} - \frac{2\left(\frac{u_{max}}{n}\right)\left(\frac{r_1}{R}\right)\left(2n\frac{r_1}{R}\left(1-\frac{r_1}{R}\right)^{1/n}-\left(1-\left(\frac{r_1}{R}\right)^2\right)\left(1-\frac{r_1}{R}\right)^{\frac{1}{n}-1}\right)}{\left(1-\left(\frac{r_1}{R}\right)^2+2\left(\frac{r_1}{R}\right)^2\ln\left(\frac{r_1}{R}\right)\right)} \quad (53)$$

Just as was done for **case 1**, u_{max} can be found from the average velocity value as shown below

$$V = \frac{\int_0^{r_0} u_0(r)2\pi r dr + \int_{r_0}^{R} u_1(r)2\pi r dr}{\pi R^2} = \frac{\int_0^{r_0} u_{max}\left(1-\left(\frac{r}{R}\right)^2\right)2\pi r dr + \int_{r_0}^{R} u_{max}\left(1-\frac{r}{R}\right)^{1/n}2\pi r dr}{\pi R^2}$$

Again, it is assumed that the error in using u_1 to overlap u_2 and u_3 is negligible since integration is involved and $(R-r_1)$ and ε are much smaller than r_1. Therefore, integrating and solving for u_{max} gives

$$u_{max} = 1.4611Vn^{0.8885} \quad (25)$$

Finding r_1 for case 2 is more challenging than case 1, since we have assumed that the fluid structure has been discontinued due to the protruding rough spikes. Therefore, the friction coefficient has to be treated separately for the rough region and for the clear region. Consider the electrical analogy in Fig 1.

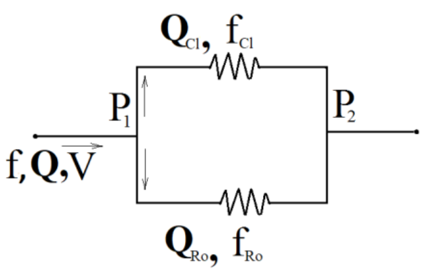

Fig 1: Electrical analogy of separate friction effect in rough region and clear region.

The pressure change is given by Darcy Weisbach equation as [1]

$$\Delta P = \frac{\rho L f}{2}\frac{V^2}{D} = \frac{8\rho L f Q^2}{\pi^2 D^5} \quad (54)$$

Summing the two flow rates

$$Q = Q_{Cl} + Q_{Ro} \quad (55)$$

Equation (54) into equation (55) gives

$$\left(\frac{\pi^2 D^5 \Delta P}{8\rho L f}\right)^{\frac{1}{2}} = \left(\frac{\pi^2 D_{Cl}^5 \Delta P}{8\rho L f_{Cl}}\right)^{\frac{1}{2}} + \left(\frac{\pi^2 D\left(D^2-D_{Cl}^2\right)^2 \Delta P}{8\rho L f_{Ro}}\right)^{\frac{1}{2}} \quad (56)$$

Using $D = 2R$ and $D_{Cl} = 2(R-\varepsilon) = D - 2\varepsilon$, and dividing through by common terms reduces equation (56) to

$$\left(\frac{1}{f}\right)^{\frac{1}{2}} = \left(\frac{\left(1-2\frac{\varepsilon}{D}\right)^5}{f_{Cl}}\right)^{\frac{1}{2}} + \left(\frac{16\left(\left(\frac{\varepsilon}{D}\right)^2-2\left(\frac{\varepsilon}{D}\right)^3+\left(\frac{\varepsilon}{D}\right)^4\right)}{f_{Ro}}\right)^{\frac{1}{2}} \quad (57)$$

In order to estimate the average velocity in the clear region

$$V_{Cl} = \frac{\int_0^{r_0} u_0(r)2\pi r dr + \int_{r_0}^{R-\varepsilon} u_1(r)2\pi r dr}{\pi(R-\varepsilon)^2} = \frac{\int_0^{r_0} u_{max}\left(1-\left(\frac{r}{R}\right)^2\right)2\pi r dr + \int_{r_0}^{R-\varepsilon} u_{max}\left(1-\frac{r}{R}\right)^{1/n}2\pi r dr}{\pi(R-\varepsilon)^2} \quad (58)$$

Similarly, u_2 overlaps over u_3 for approximation and u_{max} is used from equation (25) so integrating equation (58) gives

$$V_{Cl} = V\frac{\left[0.9642n^{0.0122}-\frac{2.9222n^{1.8885}}{(1+2n)(1+n)}\left(\frac{\varepsilon}{R}\right)^{1+\frac{1}{n}}\left(n\left(2-\frac{\varepsilon}{R}\right)+1-\frac{\varepsilon}{R}\right)\right]}{\left(1-\frac{\varepsilon}{R}\right)^2} \quad (59)$$

$Q = VA$ into equation (55) gives

$$VA = V_{Cl}A_{Cl} + V_{Ro}A_{Ro} \quad (60)$$

Which expands to

$$V\pi R^2 = V_{Cl}\pi(R-\varepsilon)^2 + V_{Ro}\left(\pi R^2 - \pi(R-\varepsilon)^2\right) \quad (61)$$

Therefore

$$V_{Ro} = \frac{V - V_{Cl}\left(1-\frac{\varepsilon}{R}\right)^2}{\left(2\frac{\varepsilon}{R}-\left(\frac{\varepsilon}{R}\right)^2\right)} = V\frac{\left(1-0.9642n^{0.0122}-\frac{2.9222n^{1.8885}}{(1+2n)(1+n)}\left(\frac{\varepsilon}{R}\right)^{1+\frac{1}{n}}\left(n\left(2-\frac{\varepsilon}{R}\right)+1-\frac{\varepsilon}{R}\right)\right)}{\left(2\frac{\varepsilon}{R}-\left(\frac{\varepsilon}{R}\right)^2\right)} \quad (62)$$

f_{Cl} is estimated from

$$\tau_{Cl} = \frac{1}{8}\rho f_{Cl}V_{Cl}^2 = -\mu\left.\frac{du_2}{dr}\right|_{r=R-\varepsilon} = -\mu\left(\frac{A}{R-\varepsilon}+2B(R-\varepsilon)\right) = -\mu\left(\frac{A}{R\left(1-\frac{\varepsilon}{R}\right)}+2BR\left(1-\frac{\varepsilon}{R}\right)\right) \quad (63)$$

As

$$f_{Cl} = \frac{-\mu}{\frac{1}{8}\rho V_{Cl}^2}\left(\frac{A}{R\left(1-\frac{\varepsilon}{R}\right)}+2BR\left(1-\frac{\varepsilon}{R}\right)\right) \quad (64)$$

$$f_{Cl} = \frac{-16}{Re}\left(\frac{\left(\frac{1.4611n^{0.8885}}{n}\right)\left(\frac{r_1}{R}\right)\left(2n\frac{r_1}{R}\left(1-\frac{r_1}{R}\right)^{1/n}-\left(1-\left(\frac{r_1}{R}\right)^2\right)\left(1-\frac{r_1}{R}\right)^{\frac{1}{n}-1}\right)\left(1-\frac{\varepsilon}{R}\right)^3}{\left(1-\left(\frac{r_1}{R}\right)^2+2\left(\frac{r_1}{R}\right)^2\ln\left(\frac{r_1}{R}\right)\right)\left[0.9642n^{0.0122}-\frac{2.9222n^{1.8885}}{(1+2n)(1+n)}\left(\frac{\varepsilon}{R}\right)^{1+\frac{1}{n}}\left(n\left(2-\frac{\varepsilon}{R}\right)+1-\frac{\varepsilon}{R}\right)\right]^2} + 2\left(\frac{\left(\frac{1.4611n^{0.8885}}{n}\right)\left(-n\left(1-\frac{r_1}{R}\right)^{1/n}-\left(\frac{r_1}{R}\right)\ln\left(\frac{r_1}{R}\right)\left(1-\frac{r_1}{R}\right)^{\frac{1}{n}-1}\right)}{\left(1-\left(\frac{r_1}{R}\right)^2+2\left(\frac{r_1}{R}\right)^2\ln\left(\frac{r_1}{R}\right)\right)\left[0.9642n^{0.0122}-\frac{2.9222n^{1.8885}}{(1+2n)(1+n)}\left(\frac{\varepsilon}{R}\right)^{1+\frac{1}{n}}\left(n\left(2-\frac{\varepsilon}{R}\right)+1-\frac{\varepsilon}{R}\right)\right]^2}\right)\left(1-\frac{\varepsilon}{R}\right)^4\right) \quad (65)$$

Assuming that r_1 falls in the wall region such that r_1/R ranges from 0.805 to 1, equation (65) can be simplified to

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$$f_{Cl} = \frac{16}{Re}\left(\frac{1.1284n\left(1-\frac{r_1}{R}\right)^{-1.8}\left(1-\frac{\varepsilon}{R}\right)^4 - 0.6225n\left(1-\frac{r_1}{R}\right)^{-2}\left(1-\frac{\varepsilon}{R}\right)^3}{\left[0.9642n^{0.0122} - \frac{2.9222n^{1.8885}}{(1+2n)(1+n)}\left(\frac{\varepsilon}{R}\right)^{1+\frac{1}{n}}\left(n\left(2-\frac{\varepsilon}{R}\right)+1-\frac{\varepsilon}{R}\right)\right]^2}\right) \quad (66)$$

f_{Ro} is estimated from

$$\tau_{Ro} = \frac{1}{8}\rho f_{Ro} V_{Ro}^2 = -\mu\left(\left.\frac{du_3}{dr}\right|_{r=R-\varepsilon} - \left.\frac{du_3}{dr}\right|_{r=R}\right) = 2\mu\varepsilon E \quad (67)$$

Therefore,

$$f_{Ro} = \frac{2\mu\varepsilon E}{\frac{1}{8}\rho V_{Ro}^2} \quad (68)$$

$$f_{Ro} = \frac{32}{Re}\frac{\left(\frac{1.4611n^{0.8885}}{n}\right)\left(-n\left(1-\frac{r_1}{R}\right)^{1/n} - \left(\frac{r_1}{R}\right)\ln\left(\frac{r_1}{R}\right)\left(1-\frac{r_1}{R}\right)^{\frac{1}{n}-1} - \frac{1}{2}\left(\frac{r_1}{R}\right)\left(2n\frac{r_1}{R}\left(1-\frac{r_1}{R}\right)^{1/n} - \left(1-\left(\frac{r_1}{R}\right)^2\right)\left(1-\frac{r_1}{R}\right)^{\frac{1}{n}-1}\right)\right)\left(\frac{\varepsilon}{R}\right)\left(2\frac{\varepsilon}{R}-\left(\frac{\varepsilon}{R}\right)^2\right)^2}{\left(1-\left(\frac{r_1}{R}\right)^2+2\left(\frac{r_1}{R}\right)^2\ln\left(\frac{r_1}{R}\right)\right)\left(1-0.9642n^{0.0122} - \frac{2.9222n^{1.8885}}{(1+2n)(1+n)}\left(\frac{\varepsilon}{R}\right)^{1+\frac{1}{n}}\left(n\left(2-\frac{\varepsilon}{R}\right)+1-\frac{\varepsilon}{R}\right)\right)^2} \quad (69)$$

Assuming that r_1 falls in the wall region such that r_1/R ranges from 0.805 to 1, equation (69) can be simplified to

$$f_{Ro} = \frac{32}{Re}\left(\frac{\left(0.8619n\left(1-\frac{r_1}{R}\right)^{-1.9}\right)\left(\frac{\varepsilon}{R}\right)\left(2\frac{\varepsilon}{R}-\left(\frac{\varepsilon}{R}\right)^2\right)^2}{\left(1-0.9642n^{0.0122} - \frac{2.9222n^{1.8885}}{(1+2n)(1+n)}\left(\frac{\varepsilon}{R}\right)^{1+\frac{1}{n}}\left(n\left(2-\frac{\varepsilon}{R}\right)+1-\frac{\varepsilon}{R}\right)\right)^2}\right) \quad (70)$$

Equations (66) and (70) into equation (57) gives

$$n = \left(\frac{Re}{16}\left(\frac{\left(1-2\frac{\varepsilon}{2R}\right)^5\left[0.9642n^{0.0122} - \frac{2.9222n^{1.8885}}{(1+2n)(1+n)}\left(\frac{\varepsilon}{R}\right)^{1+\frac{1}{n}}\left(n\left(2-\frac{\varepsilon}{R}\right)+1-\frac{\varepsilon}{R}\right)\right]^2}{1.1284n\left(1-\frac{r_1}{R}\right)^{-1.8}\left(1-\frac{\varepsilon}{R}\right)^4 - 0.6225n\left(1-\frac{r_1}{R}\right)^{-2}\left(1-\frac{\varepsilon}{R}\right)^3}\right)\right)^{\frac{1}{2}} +$$
$$\left(\frac{Re}{2}\left(\frac{\left(\left(\frac{\varepsilon}{2R}\right)^2 - 2\left(\frac{\varepsilon}{2R}\right)^3 + \left(\frac{\varepsilon}{2R}\right)^4\right)\left(1-0.9642n^{0.0122} - \frac{2.9222n^{1.8885}}{(1+2n)(1+n)}\left(\frac{\varepsilon}{R}\right)^{1+\frac{1}{n}}\left(n\left(2-\frac{\varepsilon}{R}\right)+1-\frac{\varepsilon}{R}\right)\right)^2}{\left(0.8619n\left(1-\frac{r_1}{R}\right)^{-1.9}\right)\left(\frac{\varepsilon}{R}\right)\left(2\frac{\varepsilon}{R}-\left(\frac{\varepsilon}{R}\right)^2\right)^2}\right)\right)^{\frac{1}{2}} \quad (71)$$

For the purpose of estimation, equation (71) can be reduced to a solvable equation

$$\frac{1}{Re^{0.5}\left(1-\frac{r_1}{R}\right)^{0.95}} =$$
$$\left(\frac{1}{16}\left(\frac{\left(1-\frac{\varepsilon}{R}\right)^5\left[0.9642n^{0.0122} - \frac{2.9222n^{1.8885}}{(1+2n)(1+n)}\left(\frac{\varepsilon}{R}\right)^{1+\frac{1}{n}}\left(n\left(2-\frac{\varepsilon}{R}\right)+1-\frac{\varepsilon}{R}\right)\right]^2}{1.1284n^3\left(1-\frac{\varepsilon}{R}\right)^4 - 0.6225n^3\left(1-\frac{\varepsilon}{R}\right)^3}\right)\right)^{\frac{1}{2}} +$$
$$\left(\frac{1}{2}\left(\frac{\left(\left(\frac{\varepsilon}{2R}\right)^2 - 2\left(\frac{\varepsilon}{2R}\right)^3 + \left(\frac{\varepsilon}{2R}\right)^4\right)\left(1-0.9642n^{0.0122} - \frac{2.9222n^{1.8885}}{(1+2n)(1+n)}\left(\frac{\varepsilon}{R}\right)^{1+\frac{1}{n}}\left(n\left(2-\frac{\varepsilon}{R}\right)+1-\frac{\varepsilon}{R}\right)\right)^2}{\left(0.8619n^3\right)\left(\frac{\varepsilon}{R}\right)\left(2\frac{\varepsilon}{R}-\left(\frac{\varepsilon}{R}\right)^2\right)^2}\right)\right)^{\frac{1}{2}} \quad (72)$$

Assuming that ε/D falls with the range of 0.0000001 to 0.1, equation (72) can be simplified to

$$\frac{1}{Re^{0.5}\left(1-\frac{r_1}{R}\right)^{0.95}} = \left(-0.2163n^2 + 6.4502n - 23.4472\right)\frac{\varepsilon}{D} - 0.0035n^3 + 0.0982n^2 - 0.8934n + 2.7457 \quad (73)$$

Therefore,

$$\frac{r_1}{R} = 1 - \frac{1}{Re^{0.5263}\left(\left(-0.2163n^2 + 6.4502n - 23.4472\right)\frac{\varepsilon}{D} - 0.0035n^3 + 0.0982n^2 - 0.8934n + 2.7457\right)^{1.0526}} \quad (74)$$

Equation (74) can also be used to estimate r_1/R by adopting the total friction given by [20]

$$n = \frac{1}{\sqrt{f}} = -1.8\,\mathrm{Log}\left[\frac{6.9}{Re} + \left(\frac{\varepsilon/D}{3.7}\right)^{1.11}\right] = -0.7817\ln\left[\frac{6.9}{Re} + \left(\frac{\varepsilon/D}{3.7}\right)^{1.11}\right] \quad (75)$$

The value of y^+ corresponding to r_1 and ε can be found from equation (32).

III. RESULTS

Numerical values for Reynolds number were selected and used to check the results obtained for the case of turbulent flow in a smooth pipe [See Table 1.] The calculation results showed that y+ at r1 ranges from 21 to 295 for the Re range covered by equation (31). Similarly, numerical values for Re and ε were selected for the case of turbulent flow in rough pipe, and the resulting values r1/D and y+ are presented in Table 2. In order to obtain extreme values for turbulent flow in a rough pipe, Re = 4000 with low and high values of ε/D and Re = 10^8 with low and high values of ε/D. The values of ε/D were selected to keep n within 5 and 10 where most of the developed equations are valid.

Table 1: r_1/R and corresponding y^+ for a smooth pipe at selected Re values

Re	**ε/D**	***f* (31)**	**r_1/R (30)**	**y^+ @ r_1 (32)**
4.00E+03	-	0.040205	0.849896	21.28207
4.00E+04	-	0.022001	0.953207	49.07798
1.00E+05	-	0.018065	0.969406	72.69179
1.00E+06	-	0.011829	0.992243	149.1419
1.00E+07	-	0.008341	0.998509	240.6645
3.60E+07	-	0.007032	0.999446	295.6495

Table 2: r_1/R and ε along with corresponding y^+ values for a rough pipe at selected Re and ε values

Re	ε/D	*f* (75)	r_1/R (74)	Y @ r_1/D	y^+ @ r_1 (32)	y^+ @ ε (32)
1.00 E+04	0.01	0.043	0.976668	0.011666	8.558	7.336
4.00 E+04	0.006	0.034	0.984487	0.007757	20.23	15.651
6.50 E+05	0.0004	0.017	0.989747	0.005127	152.3	11.88
6.50 E+05	0.004	0.029	0.995294	0.002353	91.5	155.6
7.00 E+05	0.01	0.038	0.997164	0.001418	68.47	482.8
8.00 E+06	0.0000001	0.008	0.998186	0.000907	235.1	0.026
2.50 E+07	0.00001	0.009	0.999006	0.000497	405.0	8.145

Table 3: r_1/R and ε along with corresponding y^+ values for a rough pipe with Re and ε values for extreme y^+

Re	ε/D	*f* (75)	n	r_1/R	(y@r_1)/D	y^+ @ r1	y^+ @ ε
4.00 E+03	0.0000001	0.0404	4.97	0.9550	0.0225	6.4000	2.8435 E-05
4.00 E+03	0.001	0.0412	4.93	0.9568	0.0216	6.2088	0.2871
1.00 E+08	0.00003	0.0097	10.1	0.9995	0.0003	881.99	104.52
1.00 E+08	0.013	0.0416	4.90	0.9998	9.24 E-05	666.22	93735.0

IV. DISCUSSION

The results in Table 1 and Table 2 show that r_1 is a separation between two flow regions. r_1 clearly separates the outer region from the wall region. Table 1 shows that for turbulent flows in smooth pipes, the normalized distance from the wall at which the boundary between the two regions occur is always greater than 30 and increases logarithmically with increase in Reynolds number. Turbulent flow in a smooth pipe is therefore expected to have both a viscous sublayer and a buffer zone. Experimental investigators have reported that the wall/outer region boundary location is dependent on Reynolds number [1] [2], but not as detailed as the result presented here. For turbulent flows in rough pipes, the results of the analysis shows in Table 2 that the normalized distance from the wall at which wall/outer region boundary exists is a function of the relative roughness of the pipe. In more details, the normalized distance from the wall at which the wall/outer region exists may increase or decrease depending on the combination of variation of the Reynolds number and relative roughness. The distance y represented by non-dimensional values y/D and y/R decreases with either increase in relative roughness or increase in Reynolds number. The observed trends can be explained by the relation between the variables as presented in equation (31) which shows that y^+ is directly proportional to the product of Re and the square root of f and f reduces with increase in Re before the flow is completely turbulent but always increases with increase in ε/D. Extreme values of y^+ obtained (see Table 3) shows that y^+ can go as low as 6.2, so for a turbulent flow in a rough pipe, there may not always be a buffer zone ($5 < y^+ < 30$).

Though $y^+(\varepsilon)$ may be greater than $y^+(r_1)$ of $y^+ = 5$, it should not be used as a criteria for determining whether the smooth pipe friction coefficient equation should be used, rather the assumption should be made only when ε/D is extremely low such that

$$(\varepsilon/D)(Re^{0.725}) \leq 0.5 \qquad (76)$$

in order to ensure that the error due to the assumption is less than 2.5%. It is assumed that empirical friction coefficient equation such as equation (75) is developed from experimental data that absorbed the effect of whether $y^+(\varepsilon)$ is greater or less than 5.

At the pipe wall, the turbulent flow is characterized by bursting from the rough wall [14] and sweeping from upstream [21] which all contribute to the turbulence production process. Enough information on the wall boundary condition is not yet available for a direct solution of the flow equations to find r_1 and f directly. **Case 1** analysis for the smooth pipe is more accurate than **case 2** analysis for the rough pipe, since the effect of ε required more work and approximation as presented. The equations obtained for **case 2** could not be used to check that it drops to the equations of **case 1** by writing ε/D = 0 since the validity of the empirical equations used did not cover ε/D = 0 in its range of validity without significant error.

Additionally, selected equations which were presented in small fonts to fit the half page columns in section II have been expanded in font size and present in a full page column in the appendix.

CONCLUSION

Theoretical analysis of turbulent flow profile in pipes has been made. Equations, which include the power-law and other forms of solution to the Navier-Stokes equation, were fitted to sections of the flow field to eliminate the limitations of the power-law at the center and at the wall. The outer region and outer region boundary or interface location was found for both cases of turbulent pipe flow in a smooth pipe and in a rough pipe. The boundary layer location trend and variation with Reynolds number and relative roughness was presented. The normalized distance from the wall to the outer/wall region boundary was found to occur at 21- 295 for smooth pipes and 6 – 882 for rough pipes with the combination of Re and ε/D determining its exact location.

This work calls for further experimentation and measurements in the near-wall region and at the wall that will provide information for further analysis. The result of this work can also be used as a guide during experimentation to know where to focus on for measurements.

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APPENDIX

A. *Expanded form selected equations which were compressed to fit the required two column pages*

$$\frac{1}{8}\rho f V^2 = -\mu\left(\frac{1}{R}\left(\frac{\left(\frac{1.4611Vn^{0.8885}}{n}\right)\left(\frac{r_1}{R}\right)\left(2n\frac{r_1}{R}\left(1-\frac{r_1}{R}\right)^{1/n}-\left(1-\left(\frac{r_1}{R}\right)^2\right)\left(1-\frac{r_1}{R}\right)^{\frac{1}{n}-1}\right)}{\left(1-\left(\frac{r_1}{R}\right)^2+2\left(\frac{r_1}{R}\right)^2\ln\left(\frac{r_1}{R}\right)\right)}\right)+2\left(\frac{\left(\frac{1.4611Vn^{0.8885}}{nR^2}\right)\left(-n\left(1-\frac{r_1}{R}\right)^{1/n}-\left(\frac{r_1}{R}\right)\ln\left(\frac{r_1}{R}\right)\left(1-\frac{r_1}{R}\right)^{\frac{1}{n}-1}\right)}{\left(1-\left(\frac{r_1}{R}\right)^2+2\left(\frac{r_1}{R}\right)^2\ln\left(\frac{r_1}{R}\right)\right)}\right)R\right) \quad (27)$$

$$\frac{Re}{23.3776} = -n^2\left(\left(\frac{\left(\frac{r_1}{R}\right)\left(2n\frac{r_1}{R}\left(1-\frac{r_1}{R}\right)^{1/n}-\left(1-\left(\frac{r_1}{R}\right)^2\right)\left(1-\frac{r_1}{R}\right)^{\frac{1}{n}-1}\right)}{n^{0.1115}\left(1-\left(\frac{r_1}{R}\right)^2+2\left(\frac{r_1}{R}\right)^2\ln\left(\frac{r_1}{R}\right)\right)}\right)+2\left(\frac{\left(-n\left(1-\frac{r_1}{R}\right)^{1/n}-\left(\frac{r_1}{R}\right)\ln\left(\frac{r_1}{R}\right)\left(1-\frac{r_1}{R}\right)^{\frac{1}{n}-1}\right)}{n^{0.1115}\left(1-\left(\frac{r_1}{R}\right)^2+2\left(\frac{r_1}{R}\right)^2\ln\left(\frac{r_1}{R}\right)\right)}\right)\right) \quad (28)$$

$$G = -\frac{\left(\frac{u_{max}}{n}\right)\left(-n\left(1-\frac{r_1}{R}\right)^{1/n}-\left(\frac{r_1}{R}\right)\ln\left(\frac{r_1}{R}\right)\left(1-\frac{r_1}{R}\right)^{\frac{1}{n}-1}-\frac{1}{2}\left(\frac{r_1}{R}\right)\left(2n\frac{r_1}{R}\left(1-\frac{r_1}{R}\right)^{1/n}-\left(1-\left(\frac{r_1}{R}\right)^2\right)\left(1-\frac{r_1}{R}\right)^{\frac{1}{n}-1}\right)\right)}{\left(1-\left(\frac{r_1}{R}\right)^2+2\left(\frac{r_1}{R}\right)^2\ln\left(\frac{r_1}{R}\right)\right)}-\frac{2\left(\frac{u_{max}}{n}\right)\left(\frac{r_1}{R}\right)\left(2n\frac{r_1}{R}\left(1-\frac{r_1}{R}\right)^{1/n}-\left(1-\left(\frac{r_1}{R}\right)^2\right)\left(1-\frac{r_1}{R}\right)^{\frac{1}{n}-1}\right)}{\left(1-\left(\frac{r_1}{R}\right)^2+2\left(\frac{r_1}{R}\right)^2\ln\left(\frac{r_1}{R}\right)\right)} \quad (53)$$

$$f_{Cl} = \frac{-16}{Re}\left(\frac{\left(\frac{1.4611n^{0.8885}}{n}\right)\left(\frac{r_1}{R}\right)\left(2n\frac{r_1}{R}\left(1-\frac{r_1}{R}\right)^{1/n}-\left(1-\left(\frac{r_1}{R}\right)^2\right)\left(1-\frac{r_1}{R}\right)^{\frac{1}{n}-1}\right)\left(1-\frac{\varepsilon}{R}\right)^3}{\left(1-\left(\frac{r_1}{R}\right)^2+2\left(\frac{r_1}{R}\right)^2\ln\left(\frac{r_1}{R}\right)\right)\left[0.9642n^{0.0122}-\frac{2.9222n^{1.8885}}{(1+2n)(1+n)}\left(\frac{\varepsilon}{R}\right)^{1+\frac{1}{n}}\left(n\left(2-\frac{\varepsilon}{R}\right)+1-\frac{\varepsilon}{R}\right)\right]^2}+2\left(\frac{\left(\frac{1.4611n^{0.8885}}{n}\right)\left(-n\left(1-\frac{r_1}{R}\right)^{1/n}-\left(\frac{r_1}{R}\right)\ln\left(\frac{r_1}{R}\right)\left(1-\frac{r_1}{R}\right)^{\frac{1}{n}-1}\right)}{\left(1-\left(\frac{r_1}{R}\right)^2+2\left(\frac{r_1}{R}\right)^2\ln\left(\frac{r_1}{R}\right)\right)\left[0.9642n^{0.0122}-\frac{2.9222n^{1.8885}}{(1+2n)(1+n)}\left(\frac{\varepsilon}{R}\right)^{1+\frac{1}{n}}\left(n\left(2-\frac{\varepsilon}{R}\right)+1-\frac{\varepsilon}{R}\right)\right]^2}\right)\left(1-\frac{\varepsilon}{R}\right)^4\right) \quad (65)$$

$$f_{Ro} = \frac{32}{Re}\frac{\left(\frac{1.4611n^{0.8885}}{n}\right)\left(-n\left(1-\frac{r_1}{R}\right)^{1/n}-\left(\frac{r_1}{R}\right)\ln\left(\frac{r_1}{R}\right)\left(1-\frac{r_1}{R}\right)^{\frac{1}{n}-1}-\frac{1}{2}\left(\frac{r_1}{R}\right)\left(2n\frac{r_1}{R}\left(1-\frac{r_1}{R}\right)^{1/n}-\left(1-\left(\frac{r_1}{R}\right)^2\right)\left(1-\frac{r_1}{R}\right)^{\frac{1}{n}-1}\right)\right)\left(\frac{\varepsilon}{R}\right)\left(2\frac{\varepsilon}{R}-\left(\frac{\varepsilon}{R}\right)^2\right)^2}{\left(1-\left(\frac{r_1}{R}\right)^2+2\left(\frac{r_1}{R}\right)^2\ln\left(\frac{r_1}{R}\right)\right)\left(1-0.9642n^{0.0122}-\frac{2.9222n^{1.8885}}{(1+2n)(1+n)}\left(\frac{\varepsilon}{R}\right)^{1+\frac{1}{n}}\left(n\left(2-\frac{\varepsilon}{R}\right)+1-\frac{\varepsilon}{R}\right)\right)^2} \quad (69)$$

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$$n=\left(\frac{Re}{16}\left(\frac{\left(1-2\frac{\varepsilon}{2R}\right)^{5}\left[0.9642n^{0.0122}-\frac{2.9222n^{1.8885}}{(1+2n)(1+n)}\left(\frac{\varepsilon}{R}\right)^{1+\frac{1}{n}}\left(n\left(2-\frac{\varepsilon}{R}\right)+1-\frac{\varepsilon}{R}\right)\right]^{2}}{1.1284n\left(1-\frac{r_1}{R}\right)^{-1.8}\left(1-\frac{\varepsilon}{R}\right)^{4}-0.6225n\left(1-\frac{r_1}{R}\right)^{-2}\left(1-\frac{\varepsilon}{R}\right)^{3}}\right)\right)^{\frac{1}{2}}+\left(\frac{Re}{2}\left(\frac{\left(\left(\frac{\varepsilon}{2R}\right)^{2}-2\left(\frac{\varepsilon}{2R}\right)^{3}+\left(\frac{\varepsilon}{2R}\right)^{4}\right)\left(1-0.9642n^{0.0122}-\frac{2.9222n^{1.8885}}{(1+2n)(1+n)}\left(\frac{\varepsilon}{R}\right)^{1+\frac{1}{n}}\left(n\left(2-\frac{\varepsilon}{R}\right)+1-\frac{\varepsilon}{R}\right)\right)^{2}}{\left(0.8619n\left(1-\frac{r_1}{R}\right)^{-1.9}\right)\left(\frac{\varepsilon}{R}\right)\left(2\frac{\varepsilon}{R}-\left(\frac{\varepsilon}{R}\right)^{2}\right)^{2}}\right)\right)^{\frac{1}{2}} \tag{71}$$

$$\frac{1}{Re^{0.5}\left(1-\frac{r_1}{R}\right)^{0.95}}=\left(\frac{1}{16}\left(\frac{\left(1-\frac{\varepsilon}{R}\right)^{5}\left[0.9642n^{0.0122}-\frac{2.9222n^{1.8885}}{(1+2n)(1+n)}\left(\frac{\varepsilon}{R}\right)^{1+\frac{1}{n}}\left(n\left(2-\frac{\varepsilon}{R}\right)+1-\frac{\varepsilon}{R}\right)\right]^{2}}{1.1284n^{3}\left(1-\frac{\varepsilon}{R}\right)^{4}-0.6225n^{3}\left(1-\frac{\varepsilon}{R}\right)^{3}}\right)\right)^{\frac{1}{2}}+\left(\frac{1}{2}\left(\frac{\left(\left(\frac{\varepsilon}{2R}\right)^{2}-2\left(\frac{\varepsilon}{2R}\right)^{3}+\left(\frac{\varepsilon}{2R}\right)^{4}\right)\left(1-0.9642n^{0.0122}-\frac{2.9222n^{1.8885}}{(1+2n)(1+n)}\left(\frac{\varepsilon}{R}\right)^{1+\frac{1}{n}}\left(n\left(2-\frac{\varepsilon}{R}\right)+1-\frac{\varepsilon}{R}\right)\right)^{2}}{(0.8619n^{3})\left(\frac{\varepsilon}{R}\right)\left(2\frac{\varepsilon}{R}-\left(\frac{\varepsilon}{R}\right)^{2}\right)^{2}}\right)\right)^{\frac{1}{2}} \tag{72}$$