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Analysis and Design of Cantilever Springs

Hemendra Singh Shekhawat, Hong Zhou
Department of Mechanical Engineering
Texas A&M University-Kingsville
Kingsville, Texas, USA

***Abstract*—Cantilever springs are clamped at one end and loaded at the other end. They are widely used in suspension systems such as railway carriages and automobiles. A cantilever spring usually takes the form of long slender beam with rectangular cross section, and is mainly subjected to bending. Although a cantilever spring with constant rectangular cross section is simple and easy to manufacture, its material is not efficiently utilized especially in the region that is close to the loading end. That is because the bending stress is proportional to the distance from the loading end. The bending stress is zero at the loading end and reaches its maximum at the clamping end. To improve the material utilization efficiency of a cantilever spring, its width or thickness or both can be made variable. It is not trivial to design a cantilever spring with variable width or thickness for a practical need. In this paper, variable width and thickness of cantilever springs are described by linear or spline interpolation. A systematic approach is introduced to analyze and design cantilever springs. Examples on analyzing and designing cantilever springs with different variable widths and thicknesses are presented in the paper to verify the effectiveness and demonstrate the procedure of the introduced method.**

Keywords—Cantilever Spring; Spring Rate; Analysis; Design; Optimization.

I. INTRODUCTION

Cantilever springs are clamped at one end and loaded at the other end. They are widely used in suspension systems such as railway carriages and automobiles. A typical example of cantilever springs is the diving board in a swimming pool. The diving board is fixed at one end and free at the other end, and is typically of uniform rectangular cross section. The board is initially straight and horizontal. A diver stands at the free end of the board, initiates an up and down swing of the board, and then utilizes the spring action of the board for jumping.

Although a cantilever spring with constant rectangular cross section is simple and easy to manufacture, its material is not efficiently utilized especially in the region that is close to the loading end [1]. This is because the primary stress of a cantilever beam is from its bending and the bending stress is proportional to the distance from the loading end. The bending stress is zero at the loading end and reaches its maximum at the clamping end. Figure 1 shows a cantilever beam with uniform rectangular cross section. The bending stress along the beam axis can be calculated by the following formula [2].

$$\sigma(x) = \frac{6Px}{bt^2} \quad (1)$$

b and t here are the width and thickness of the beam, respectively. P is the vertical force applied at the free end of the beam. The maximum bending stress ($\sigma_{\max} = 6PL/bt^2$) occurs at the fixed end of the beam. L is the total length of the beam. Bending stress is tensile on the top of the beam and compressive on the bottom of the beam.

If beam width b is allowed to vary along the beam axis, bending stress can be made constant. As shown in equation (1), the ratio $b(x)/x$ has to be constant if $\sigma(x)$ is constant and beam thickness t does not change. The cantilever beam with a constant bending stress has a shape of triangular width, which is shown in Figure 2. If b_0 is the maximum width at the fixed end of the triangular beam, $b(x)$ can be represented as follows.

$$b(x) = \frac{b_0 x}{L} \quad (2)$$

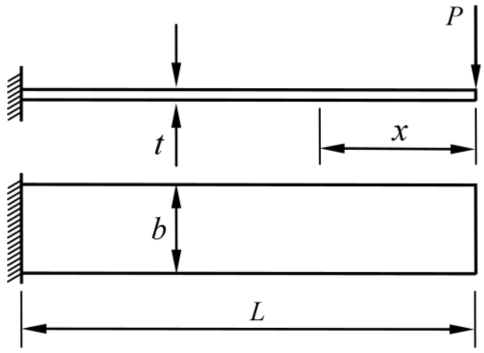

Fig. 1 A cantilever beam with uniform cross section.

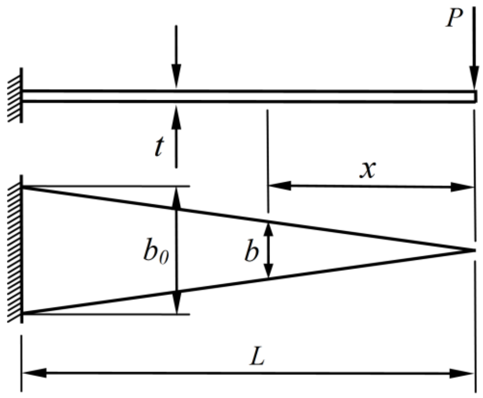

Fig. 2 A cantilever beam with triangular width.

If beam width b does not change, beam thickness t must vary parabolically with x to make bending stress constant along the beam axis [3]. As shown in equation (1), the ratio $t^2(x)/x$ has to be constant if $\sigma(x)$ is constant and beam width b is uniform. Figure 3 shows a cantilever beam with parabolic thickness in which t_0 is the maximum thickness at the fixed end of the parabolic beam. $t(x)$ can be represented by the following equation.

$$t(x) = t_0\sqrt{\frac{x}{L}} \tag{3}$$

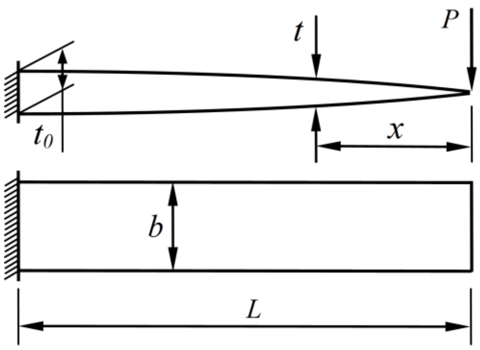

Fig. 3 A cantilever beam with parabolic thickness.

In addition to normal stress from bending, there is also shear stress on each cross section of a cantilever beam. For rectangular cross section, the maximum shear stress occurs at the neutral axis of the beam and can be calculated by the following formula.

$$\tau = \frac{3P}{2bt} \tag{4}$$

At the loading end, the triangular beam shown in Figure 2 has zero width while the parabolic beam in Figure 3 has zero thickness. When width b or thickness t in equation (4) is zero, shear stress reaches infinity. Because of the zero width or thickness, the loading end becomes a line and has zero area in Figures 2 and 3. To make the shear stress reasonable at the loading end, its width in Figure 2 or thickness in Figure 3 has to be nonzero. Then comes this question: what is the best width for a cantilever beam spring with uniform thickness to meet practical needs and efficiently utilize its material, or what is the best thickness for a cantilever beam spring with uniform width to meet practical needs and efficiently utilize its material? If both width and thickness are allowed to vary, the question becomes: what is the best shape for a cantilever beam spring to meet practical needs and efficiently utilize its material? The motivation of this paper is to provide answers to these questions. The research objective of this paper is to introduce a systematic approach to analyze and design cantilever beam springs.

The remainder of the paper is organized as follows. The analysis on cantilever beam springs is presented in section II. The design optimization procedure is provided in section III. Section IV is on the design of cantilever beam springs. Conclusions are drawn in section V.

II. ANALYSIS OF CANTILEVER BEAM SPRINGS

The cantilever beam in Figure 1 has uniform rectangular cross section. Its deflection (δ) at the loading end can be calculated by the following formula.

$$\delta = \frac{4PL^3}{Ebt^3} \tag{5}$$

E in equation (4) is the Young's modulus of the cantilever beam material.

When the cantilever beam is used as a spring, its spring stiffness or rate (k) is the ratio of P and δ.

$$k = P/\delta = \frac{Ebt^3}{4L^3} \tag{6}$$

Cantilever springs are made of ductile materials for which distortion energy theory has been introduced to explain their yielding failure and von Mises stress has been widely used as a criterion in determining the onset of yielding failure. The von Mises stress (σ_v) that combines bending normal stress (σ) and shear stress (τ) acting at the same point of a cross-section of a cantilever beam spring can be calculated as follows [4].

$$\sigma_v = \sqrt{\sigma^2 + 3\tau^2} \tag{7}$$

If the width at the loading end of the cantilever beam shown in Figure 2 is nonzero, its triangular width becomes trapezoidal width, which is shown in Figure 4.

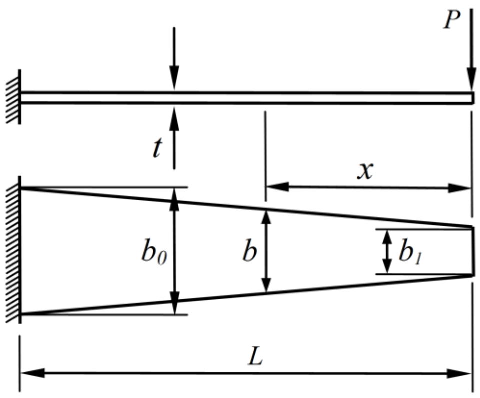

Fig. 4 A cantilever beam with trapezoidal width.

The width function, $b(x)$, now becomes:

$$b(x) = b_1 + \frac{(b_0 - b_1)\,x}{L} \tag{8}$$

The moment of inertia of the cross-section about the neutral axis of the cantilever beam is then:

$$I(x) = \frac{b(x)\,t^3}{12} \tag{9}$$

If the vertical deflection along the beam axis is represented as $v(x)$, and the slope $dv(x)/dx$ of the deflected beam is very small, $v(x)$ can be derived from the following equation.

$$\frac{d^2v(x)}{d^2x} = \frac{M(x)}{EI(x)} \tag{10}$$

$M(x)$ is the bending moment and equals to Px. The two needed boundary conditions for solving equation (10) are: $v(x)=0$ and $dv(x)/dx=0$ when $x=L$. For a specific width ratio (b_1/b_0), equation (10) can be solved, and the deflection (δ) at the loading end and the beam spring stiffness (k) can be obtained.

Correction factor (β) has been introduced and used to calculate the deflection at the loading end for trapezoidal beam springs [5].

$$\delta = \frac{4\beta PL^3}{Ebt^3} \tag{11}$$

β depends on the beam width ratio of b_1/b_0 and can be found from related chart [5].

The thickness at the loading end of the cantilever beam shown in Figure 3 can be made nonzero to make its shear stress reasonable. If the two symmetric parabolic thickness curves are simplified as symmetric straight lines, the beam spring with variable thickness is shown in Figure 5.

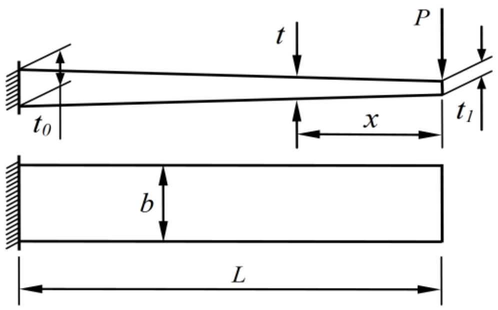

Fig. 5 A cantilever beam with variable thickness.

The variable thickness along the cantilever beam axis can be represented as

$$t(x) = t_1 + \frac{(t_0 - t_1)x}{L} \tag{12}$$

With the variable thickness, the moment of inertia of the cross-section about the neutral axis of the cantilever beam is then:

$$I(x) = \frac{bt(x)^3}{12} \tag{13}$$

The deflection $v(x)$ of the cantilever beam with variable thickness can still be derived from equation (10), but the moment of inertia of the cross-section is now calculated by equation (13).

If both width and thickness are made linearly variable along the beam axis, the moment of inertia of the cross-section about the neutral axis of the cantilever beam becomes:

$$I(x) = \frac{b(x)t(x)^3}{12} \tag{14}$$

Equations (8) and (12) are applicable for calculating $b(x)$ and $t(x)$, respectively. The derivation of beam deflection can still rely on equation (10).

The width or thickness change along the beam axis does not have to be linear or quadratic. It can be different based on the practical requirements for the designed beam spring. Variable beam width is used here as an example to explain the procedure to establish a general change. Figure 6 shows a cantilever beam with variable width that is described by two symmetric spline curves.

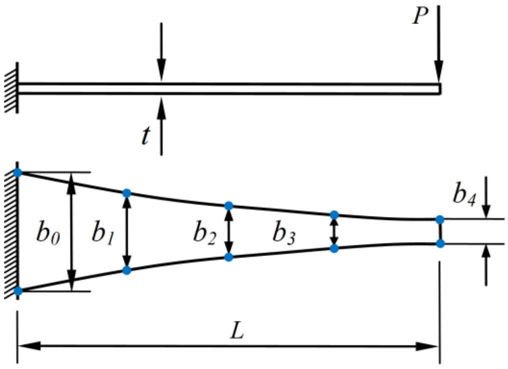

Fig. 6 The variable width described by spline interpolation.

Each spline curve in Figure 6 interpolates five points along the horizontal beam axis. Because of the symmetry of the two spline curves, the line segment formed by two corresponding interpolation points on the two curves is in the vertical direction and has the value of beam width at that position. The variable beam width shown in Figure 6 can thus be considered as the spline interpolation of five beam widths along the beam axis (b_0, b_1, b_2, b_3 and b_4). The five interpolation beam widths in Figure 6 are evenly distributed along the beam axis. Generally, the number of interpolation points and the distribution of the interpolation points are based on specific design situations. The linear change of width in Figure 4 or thickness in Figure 5 is just a special case of general spline interpolation and has only two width or thickness values to interpolate, which is a linear interpolation.

Spline interpolation uses piecewise polynomials to interpolate all required points. Different from Lagrange interpolation, the number of interpolation points and the degree of polynomials in spline interpolation are independence each other, which leads to smooth and tight interpolation curve that passes through all interpolation points in the desired order [6-7]. Cubic spline interpolation has been widely adopted for different applications and is composed of a set of third degree polynomials. Any two neighboring polynomials are smoothly connected and have continuous slope and curvature at their shared internal interpolation point. The two end interpolation points of a cubic spline curve can have different conditions that include natural end conditions (two end curvatures are set as zero), not-a-knot end conditions (the third derivative is continuous at both the first and last internal points) or clamped end conditions (two end slopes are specified).

Both width and thickness changes of cantilever beam springs can be described by linear or cubic spline interpolations.

III. PARAMETER OPTIMIZATION OF CANTILEVER BEAM SPRINGS

The deflection, stress and material volume of cantilever beam springs are analyzed in the paper by using popular finite element analysis software ANSYS [8-9]. To analyze a beam spring, its solid model has to be created first. The Design Modeler of ANSYS is used in this work to establish solid models of analyzed beam springs. ANSYS Design Modeler [10] is an ANSYS Workbench application that provides modeling functions for analyses and simulations that include detailed geometry creation, CAD geometry modification, simplification and concept model creation tools. A beam spring can also be first generated by a solid modeling software such as SolidWorks [11] and then imported to ANSYS for analysis.

The configuration of a designed beam spring is modeled based on its geometric control parameters, which are the independent design parameters to describe its variable width or thickness or both. These geometric control parameters fully define and control a beam spring. The design of a beam spring can be systematized as optimizing its geometric control parameters for its specific application and requirements.

The geometric control parameters of a designed beam are optimized in this paper based on the Design Exploration of ANSYS [12]. One of ANSYS Workbench applications is ANSYS Design Exploration, which supports design optimization through ANSYS simulation results. Simulations are based on ANSYS Mechanical that is also a Workbench application to perform engineering simulations including stress, thermal, vibration simulations. The Direct Optimization toolbox of ANSYS Design Exploration is one of the goal driven optimization systems and is chosen for parameter optimization of cantilever beam springs in the paper. The Adaptive Multiple-Objective (AMO) method is used for parameter optimization. AMO supports multiple objectives and aims at finding the global optimal solution [13].

IV. DESIGN OF CANTILEVER BEAM SPRINGS

The horizontal cantilever beam spring shown in Figure 7 has uniform cross section. Its length, width and thickness are 100 mm, 10 mm and 2 mm, respectively. The material of the spring is structural steel that has Young's modulus of 200000 MPa, Poisson' ratio of 0.3, yield strength of 250 MPa. The left end of the spring is fixed. The downward force that is applied on its right end is 10 N. Figure 8 shows the finite element mesh of the beam spring.

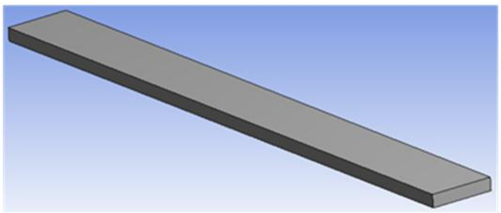

Fig. 7 The solid model of a cantilever beam spring with uniform cross section.

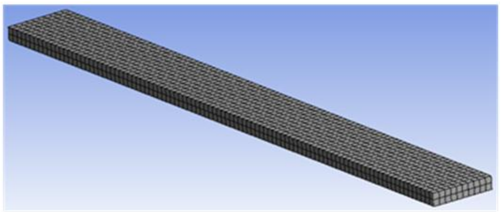

Fig. 8 The finite element mesh of the cantilever beam spring with uniform cross section.

The maximum vertical deflection at the free end is 2.48 mm while the maximum von Mises stress within the beam spring is 155.22 MPa, which are shown in Figures 9 and 10, respectively. With the force of 10 N at the free end and its corresponding deflection of 2.48 mm at the loading end, the spring rate is 4.03 N/mm. The spring rate from Equation (6) is 4.0 N/mm, which is very close to that from finite element analysis result. ANSYS Workbench 15 is used here to analyze the beam spring.

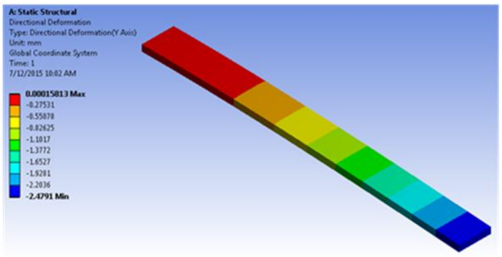

Fig. 9 The vertical deflection of the cantilever beam spring with uniform cross section.

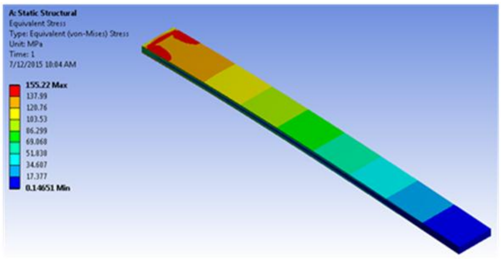

Fig. 10 The von Mises Stress of the cantilever beam spring with uniform cross section.

The material volume of the uniform beam spring is 2000 mm^3. To improve the material utilization efficiency of the beam spring, we first make its width variable and keep its thickness fixed at 2 mm. The beam spring with variable width has the same length (100 mm), free end load (10 N) and spring rate (4.03 N/mm) as the beam spring with uniform width, so the beam spring with variable width can fulfil the same function as that with the uniform width. In addition, the maximum von Mises within the beam spring with variable width is constrained to be no higher than that within the beam spring with uniform width, which is 155.22 MPa.

For trapezoidal width, there are two independent parameters to optimize, b_0 and b_1. If their range is set from 5 mm to 13 mm, their optimal values from ANSYS Design Exploration are found to be 11.80 mm and 5.28 mm. The solid model and finite element mesh of the optimal trapezoidal beam spring are shown in Figures 11 and 12, respectively.

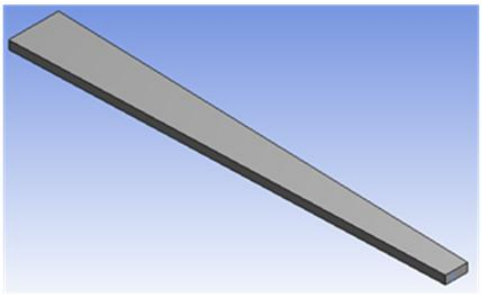

Fig. 11 The cantilever beam spring with trapezoidal width.

The deflection at the free end is 2.48 mm and the maximum von Mises stress is 134.67 MPa, which are shown in Figures 13 and 14. The material volume of the trapezoidal beam spring is 1706.6 mm^3, which is decreased by 14.7% compared with that of the uniform beam spring.

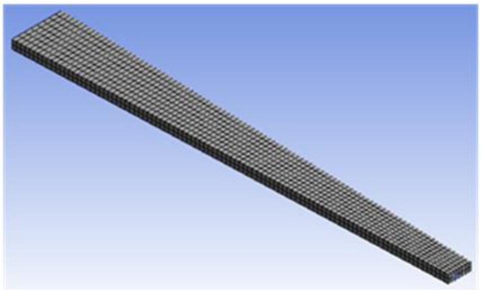

Fig. 12 The finite element mesh of the cantilever beam spring with trapezoidal width.

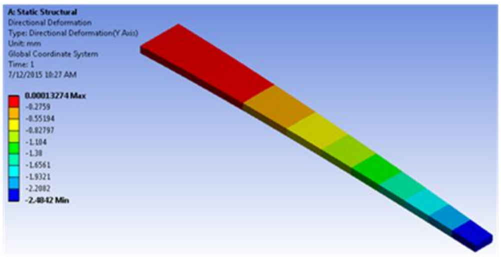

Fig. 13 The deflection of the cantilever beam spring with trapezoidal width.

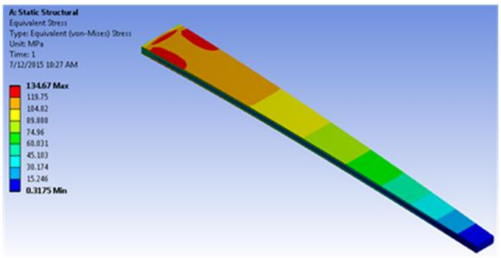

Fig. 14 The von Mises Stress of the cantilever beam spring with trapezoidal width.

To further improve the material utilization efficiency, the straight lines of trapezoidal width can be replaced by spline interpolation curves as shown in Figure 6. There are now five independent width parameters (b_0, b_1, b_2, b_3 and b_4) to be optimized. Their optimal values from ANSYS Design Exploration are (11.98, 10.63, 7.84, 6.42 and 5.17). The solid model and finite element mesh of the beam spring are shown in Figures 15 and 16, respectively.

The deflection at the free end is 2.48 mm and the maximum von Mises stress is 133.34 MPa, which are shown in Figures 17 and 18. The material volume of the beam spring with spline width is 1693.1 mm^3, which is a little below that of the trapezoidal beam spring. The material volume difference between spline and trapezoidal beam springs is minor. Using spline interpolation curves to replace straight lines does not contribute a lot in this case, but trapezoidal beam spring is just a special case of general spline beam spring although spline width takes more parameters to describe.

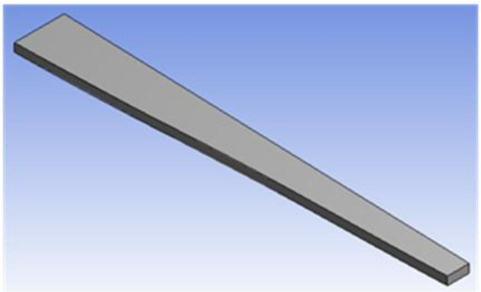

Fig. 15 The cantilever beam spring with spline width.

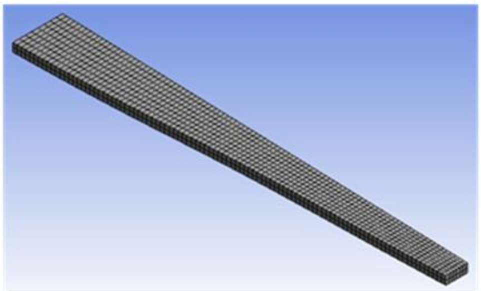

Fig. 16 The finite element mesh of the cantilever beam spring with spline width.

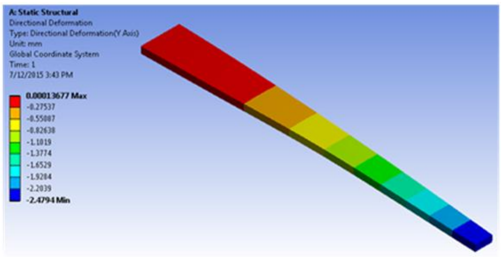

Fig. 17 The deflection of the cantilever beam spring with spline width.

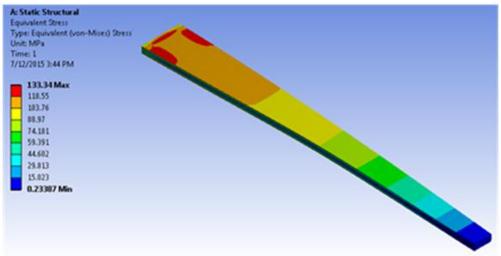

Fig. 18 The von Mises Stress of the cantilever beam spring with spline width.

If the width of a beam spring is kept constant, its material utilization can be improved by making its thickness variable. As shown in Figure 5, there are two independent parameters (t_0 and t_1) to be optimized for a tapered beam spring. If their range is set from 1.0 mm to 3.0 mm, their optimal values from ANSYS Design Exploration are 2.45 mm and 1.02 mm. The solid model and finite element mesh of the tapered beam spring are shown in Figures 19 and 20, respectively.

The deflection at the free end is 2.48 mm and the maximum von Mises stress is 104.06 MPa, which are shown in Figures 21 and 22. The material volume of the spline beam spring is 1738.6 mm^3, which is decreased by 13% compared with the original uniform beam spring. The maximum von Mises stress for tapered beam spring is much lower than that of the uniform beam spring.

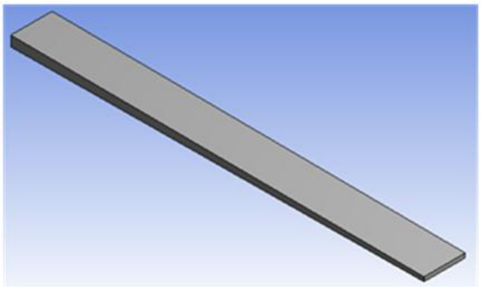

Fig. 19 The cantilever beam spring with tapered thickness.

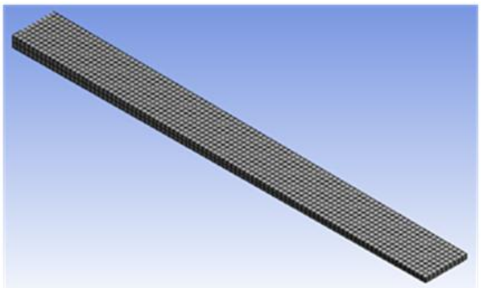

Fig. 20 The finite element mesh of the cantilever beam spring with tapered thickness.

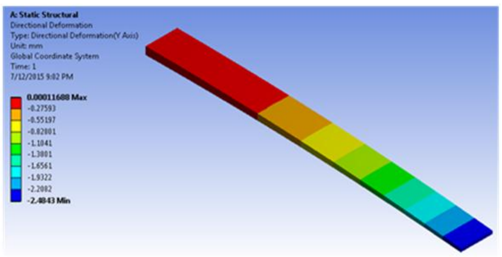

Fig. 21 The deflection of the cantilever beam spring with tapered thickness.

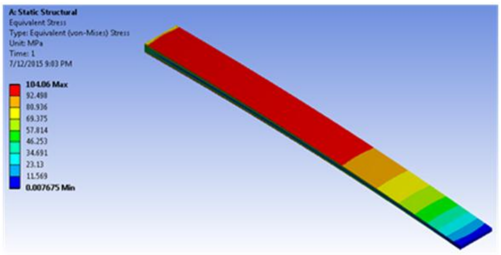

Fig. 22 The von Mises Stress of the cantilever beam spring with tapered thickness.

Both its width and thickness of a cantilever beam spring can be made variable. When trapezoidal width and tapered thickness are adopted for a beam spring, four parameters (b_0, b_1, t_0, t_1) are needed to define its geometry. Their optimal values from ANSYS Design Exploration are 10.63 mm, 5.00 mm, 2.43 mm, 1.33 mm. Their allowed change ranges are set as the same as above to get their optimal values. The solid model and finite element mesh of the beam spring with trapezoidal width and tapered thickness are shown in Figures 23 and 24, respectively.

The deflection at the free end is 2.48 mm and the maximum von Mises stress is 109.05 MPa, which are shown in Figures 25 and 26. The material volume of the beam spring is now 1520.2 mm^3, which is decreased by 24% compared with the original uniform beam spring and is the smallest among all designs.

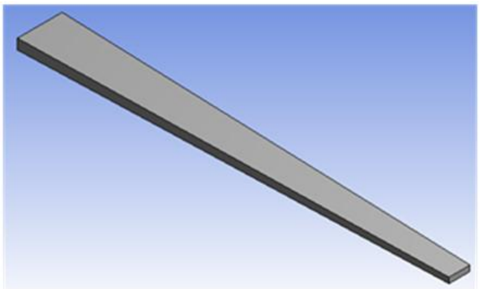

Fig. 23 The cantilever beam spring with trapezoidal width and tapered thickness.

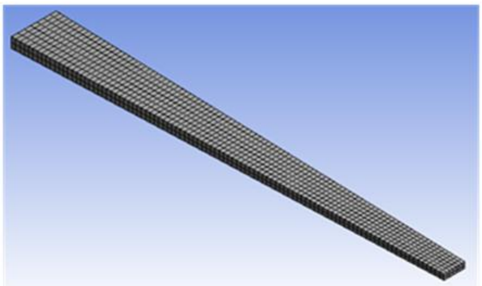

Fig. 24 The finite element mesh of the cantilever beam spring with trapezoidal width and tapered thickness.

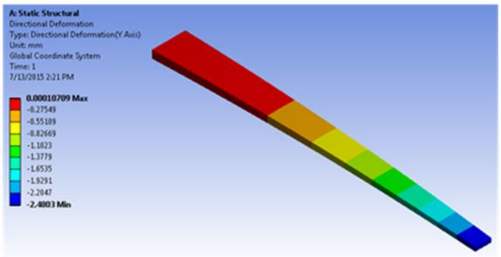

Fig. 25 The deflection of the cantilever beam spring with trapezoidal width and tapered thickness.

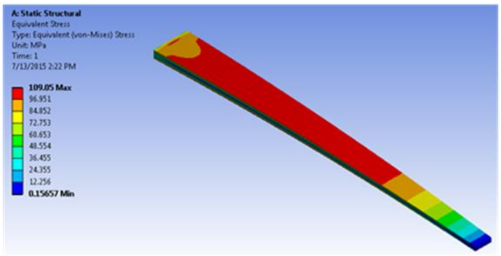

Fig. 26 The von Mises Stress of the cantilever beam spring with trapezoidal width and tapered thickness.

The design results from uniform cross section, trapezoidal width, tapered thickness, and combined trapezoidal width and tapered thickness are summarized in the following table. In each case, the beam spring has the same load (10 N), same deflection (-2.48 mm), and same spring rate (4.03 N/mm).

Table 1 Design results for the cantilever beam spring

Beam Spring Configuration	Material Volume	Maximum von Mises Stress
Uniform Cross Section	2000 mm^3	155.22 MPa
Trapezoidal Width	1706.6 mm^3	134.67 MPa
Tapered Thickness	1738.6 mm^3	104.06 MPa
Combined Trapezoidal Width and Tapered Thickness	1520.2 mm^3	109.05 MPa

As shown in Table 1, the beam spring with uniform cross section has the lowest material utilization efficiency and the highest maximum von Mises stress although it has the simplest shape and is the easiest to manufacture. The beam spring with combined trapezoidal width and tapered thickness has the highest material utilization efficiency, and its maximum von Mises stress is little bit higher than that of the beam spring with tapered thickness.

V. CONCLUSIONS

Although a cantilever beam spring with uniform cross section is simple and convenient for manufacturing, its material is not efficiently utilized. To improve the material utilization efficiency of a cantilever beam spring, its width or thickness or both can be made variable. A method for designing cantilever beam springs with nonuniform cross sections is presented in the paper. A designed cantilever beam spring is optimized for its minimum material volume under the constraints of desired spring rate, allowable stress, practical dimensions and external load.

To design a cantilever beam spring, its solid model is created based on its geometric control parameters in ANSYS Design Modeler. Finite element analysis of the modeled cantilever beam spring is then conducted in ANSYS Mechanical to simulate its performance. The geometric control parameters are optimized in ANSYS Design Exploration through simulation results. The AMO optimization method is adopted in the paper to obtain optimal solutions.

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