An EOQ Inventory Model For Two Parameter Weibull Deterioration With Time Dependent Demand And Shortages

Anchal Agarwal¹ And S. R. Singh²

¹ Research Scholar, Banasthali Vidyapith Rajasthan India
² Associate Professor, Department Of Mathematics, D.N. College Meerut

Abstract
In this paper, we developed an inventory model with time dependent demand rate and Weibull deterioration. The demand rate is a linear function of time. Deterioration rate is taken as two parameter Weibull distribution deterioration. In this study, shortage is allowed and completely backlogged. The objective of this model is to maximize the total profit. The numerical example of the model is provided to illustrate the problem.

Key Words: Inventory system, Deterioration, Weibull Distribution of two parameters, Time Dependent Demand, Shortages.

Introduction
Deterioration has a significant role in many inventory systems. Deterioration is defined as damage, declension, worsening, corrosion and loss of the utility of the product. However certain types of commodities either deteriorate or become obsolete in the course of time and hence are unstable. For example, the commonly used goods like fruits, vegetables, meat, foodstuffs, gasoline, radioactive substances, electronic components etc. where deterioration is usually observed during the normal storage period. The loss due to deterioration can not be ignored.

In this model demand is taken as time dependent & deterioration is taken as weibull distribution with two parameter. Shortage are allowed & completely lost in this paper. A numerical example is also shown in this paper.

**Assumptions and Notations**
The proposed mathematical model of inventory replenishment policy is developed under the following notations and assumptions:

**Assumptions:**
- Lead time is zero
- The inventory system involves only one time.
- The demand rate is time dependent.
- Shortage are allowed & completely lost.
- The length of the cycle is T.
- \( \theta(t) = \alpha\beta t^{(\beta-1)} \) is the weibull two parameter deterioration rate, where \( 0 < \alpha < 1 \) and \( \beta > 0 \) are called scale and shape parameter respectively.

**Notations:**
- \( A \): Setup cost
- \( C_1 \): Shortage cost / Unit/ Unit time
- \( C_2 \): Deterioration cost / Unit/ Unit time
- \( h \): Holding cost/Unit/Unit time
- \( T \): Duration of a cycle
- \( t_i \): The time at which the inventory level becomes zero.
- \( S \): Selling Price
- \( \alpha \): Scale parameter
- \( \beta \): Shape parameter
- \( \theta \): Deterioration rate
- \( D(t) \): a+bt, where a>b

**Mathematical Formulation of the model:**
The length of the cycle is T. At the time \( t_i \) the inventory level becomes zero and shortages occurring in the period \( (t_i, T) \) which is completely backlogged. Let I (t) be the inventory level at time t \( (0 \leq t \leq t_i) \). The differential equations for the instatantaneous state over \( (0, T) \) are given by
\[
\frac{dI(t)}{dt} = \alpha \beta t^{(\beta-1)} I(t) = -(a + bt), 0 \leq t \leq t_1
\]

\[
\frac{dI(t)}{dt} = -(a + bt), t_1 \leq t \leq T
\]

With the boundary condition \( I(t_1) = 0 \) at \( t = t_1 \).

Solving equation (1) & (2) we get

\[
I(t) = (1 - \alpha \beta^\beta) \left[ a(t_1 - t) + \frac{b}{2} (t_1^2 - t^2) + \frac{\alpha \alpha}{\beta + 1} (t_1^{(\beta+1)} - t^{(\beta+1)}) + \frac{b \alpha}{\beta + 2} (t_1^{(\beta+2)} - t^{(\beta+2)}) \right], 0 \leq t \leq t_1
\]

\[
I(t) = [a(t_1 - t) + \frac{b}{2} (t_1^2 - t^2)], t_1 \leq t \leq T
\]

The total holding cost during the time period 0 to \( t_1 \)

\[
HC = h \int_0^{t_1} I(t) dt
\]

\[
HC = h \int_0^{t_1} (1 - \alpha \beta^\beta) \left[ a(t_1 - t) + \frac{b}{2} (t_1^2 - t^2) + \frac{\alpha \alpha}{\beta + 1} (t_1^{(\beta+1)} - t^{(\beta+1)}) + \frac{b \alpha}{\beta + 2} (t_1^{(\beta+2)} - t^{(\beta+2)}) \right] dt
\]

\[
HC = h \left[ \frac{at_1^2}{2} + \frac{bt_1^3}{3} + \frac{\alpha \alpha}{\beta + 1} (\beta + 2) t_1^{(\beta+2)} + \frac{b \alpha}{\beta + 2} (\beta + 3) t_1^{(\beta+3)} + \frac{\alpha \alpha}{2(\beta + 1)^2} t_1^{(2\beta+2)} \right]
\]

The total Shortage cost during the time period \( t_1 \) to \( T \) is given by

\[
SC = -c_i \int_{t_1}^{T} I(t) dt
\]

\[
SC = -c_i \int_{t_1}^{T} \left[ a(t_1 - t) + \frac{b}{2} (t_1^2 - t^2) \right] dt
\]

\[
SC = \left[ \frac{c_i a}{2} (t_1 - T)^2 + \frac{c_i b}{6} (2t_1^3 + T^3 - 3Tt_1^2) \right]
\]
The total Deterioration cost during the time period 0 to \( t_1 \) is given by
\[
DC = c_2 \int_0^{t_1} \theta(t)J(t)dt
\]
\[
DC = c_2 \int_0^{t_1} \alpha \beta \Gamma(\beta - 1) (1 - \alpha t^\beta) [a(t_1 - t) + \frac{b}{2} (t_1^2 - t^2) + \frac{a \alpha}{\beta + 1} (t_1^{(\beta + 1)} - t^{(\beta + 1)}) + \frac{b \alpha}{\beta + 2} (t_1^{(\beta + 2)} - t^{(\beta + 2)})] dt
\]
\[
DC = \frac{a t_1^{\beta + 1}}{\beta + 1} t_1^{(\beta + 1)} + \frac{3a \alpha}{2 \beta (2 \beta + 1)} t_1^{2 \beta + 1} + \frac{b}{\beta (\beta + 2)} t_1^{\beta + 2} - \frac{b \alpha}{\beta (2 \beta + 2)} t_1^{2 \beta + 2} - \frac{a \alpha^2}{2 \beta (3 \beta + 1)} t_1^{3 \beta + 1}
\]
\[
+ \frac{b \alpha^2}{2 \beta (3 \beta + 2)} t_1^{3 \beta + 1} \]
------------------- (7)

Sales Revenue = \( s \int_0^T (a + bt) dt = s(aT + b \frac{T^2}{2}) \) ---------------------- (8)

From equation (5), (6), (7) & (8) the total profit per unit time is

\[
P(T, t_1) = \frac{1}{T} [s(a + b \frac{T^2}{2}) - \frac{1}{T} (A + HC + SC + DC)]
\]
\[
P(T, t_1) = s(a + b \frac{T}{2}) - \frac{1}{T} [A + h \{ \frac{a t_1^2}{2} + \frac{b t_1^3}{3} + \frac{a \alpha^2}{(\beta + 1)(\beta + 2)} t_1^{(\beta + 2)} + \frac{b \alpha^2}{(\beta + 1)(\beta + 3)} t_1^{3 \beta + 3} + \frac{a \alpha t_1 t_1^{(\beta + 1)}}{2 \beta (2 \beta + 1)} + \frac{3 a \alpha}{2 \beta (2 \beta + 3)} t_1^{2 \beta + 1} + \frac{b}{\beta (\beta + 2)} t_1^{\beta + 2} - \frac{b \alpha}{\beta (2 \beta + 2)} t_1^{2 \beta + 2} - \frac{a \alpha^2}{2 \beta (3 \beta + 1)} t_1^{3 \beta + 1}
\]
\[
- \frac{b \alpha^2}{2 \beta (3 \beta + 2)} t_1^{3 \beta + 1} \}] \]
------------------- (9)

Our objective is to maximize the total profit. The necessary condition for maximize the profit are

\[
\frac{\partial p(T, t_1)}{\partial T} = 0 \quad \text{and} \quad \frac{\partial p(T, t_1)}{\partial t_1} = 0 \quad \text{then}
\]

\[
\frac{bs}{2} - \frac{- a c_1 (T + t_1) + \frac{1}{6} b c_1 (T^2 - 3 t_1^2)}{2 \beta (1 + 3 \beta)} + \frac{1}{T^2} (A + \frac{a t_1^2 + b t_1^3}{1 + \beta} + \frac{b t_1^{2 + \beta}}{\beta (2 + \beta)} + \frac{3 a c_1 t_1^{2 + \beta}}{2 \beta (1 + \beta)} - \frac{b a c_1 t_1^{2 + \beta}}{\beta (2 + \beta)})
\]
\[
- \frac{a \alpha^2 t_1^{1 + 3 \beta}}{2 \beta (1 + 3 \beta)} + \frac{b \alpha^2 t_1^{2 + 3 \beta}}{2 \beta (2 + 3 \beta)} + \frac{1}{2} a c_1 (T + t_1)^2 + \frac{1}{6} b c_1 (T^3 - 3 T t_1^2 + 2 t_1^3) + h(\frac{a t_1^2}{2} + \frac{b t_1^3}{3} + \frac{a \alpha c_1 t_1^{2 + \beta}}{(1 + \beta) (2 + \beta)})
\]

\[
\frac{- a \alpha^2 t_1^{1 + 3 \beta}}{2 \beta (1 + 3 \beta)} + \frac{b \alpha^2 t_1^{2 + 3 \beta}}{2 \beta (2 + 3 \beta)} + \frac{1}{2} a c_1 (T + t_1)^2 + \frac{1}{6} b c_1 (T^3 - 3 T t_1^2 + 2 t_1^3) + h(\frac{a t_1^2}{2} + \frac{b t_1^3}{3} + \frac{a \alpha c_1 t_1^{2 + \beta}}{(1 + \beta) (2 + \beta)})
\]
\[
\begin{align*}
&\frac{b\alpha b t_1^3 + \beta}{(1+\beta)(3+\beta)} + \frac{a\alpha t_1^2 + 2\beta}{2(1+\beta)^2} + \frac{b\alpha t_1^3 + 2\beta}{(1+\beta)(3+2\beta)} = 0 \\
&\quad\text{................................. (10)}
\end{align*}
\]

And
\[
\begin{align*}
&-\frac{1}{T}(\frac{3\alpha a t_1^2}{2\beta} - \frac{a^2 a t_1^3}{2\beta} + \frac{b t_1 + \beta}{\beta} + \frac{a(2 + \beta) t_1^3 + \beta}{(1+\beta)} - \frac{b\alpha t_1 + 2\beta}{2\beta} + \frac{b\alpha t_1 + 1 + 3\beta}{2\beta} + ac_i(-T + t_i) + \frac{1}{6}bc_i \\
&(-6T t_1 + 6t_1^2) + h(a t_1 + bt_1^2 + \frac{a\alpha b t_1^3 + \beta}{1+\beta} + \frac{a(2 + 2\beta) t_1^3 + 2\beta}{2(1+\beta)^2} + \frac{b\alpha t_1^3 + 2\beta}{1+\beta}) = 0 \\
&\quad\text{................................. (11)}
\end{align*}
\]

Provided
\[
\frac{\partial^2 P}{\partial T^2} < 0, \frac{\partial^2 P}{\partial t_1^2} < 0 \quad \text{And} \quad \frac{\partial^2 P}{\partial T^2},\frac{\partial^2 P}{\partial t_1^2} - \frac{\partial^2 P}{\partial T \partial t_1} > 0
\]

Using the software mathematica-5.2, we can find the optimal values of \( T^* \) and \( t_1^* \) by equation no. (10) & (11) simultaneously and also find the optimal value of \( P^*(T, t_1) \) by equation no. (9).

**Numerical Example & Sensitivity Analysis**

We consider \([A, a, b, s, c_1, c_2, h, \alpha, \beta] = [200, 100, 76, 2, 1.3, 1.1, 1.0, 0.1, 0.3]\) in proper units, where \( h, c_1 \& c_2 \) are holding cost, shortage cost and deterioration cost respectively.

The optimal value of \( T^* = 2.21261, t_1^* = 0.550549 \) & \( P^*(T, t_1) = 93.3771 \)
This is the concavity of the total cost w.r. to $T^*$ and $t_1^*$. 

The variation in the parameter is as follows

**Table- (1) Variation in parameter $a$**

<table>
<thead>
<tr>
<th>$a$</th>
<th>$t_1$</th>
<th>$T$</th>
<th>$P(T,t_1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>0.550549</td>
<td>2.21</td>
<td>93.3771</td>
</tr>
<tr>
<td>120</td>
<td>0.526357</td>
<td>2.08115</td>
<td>113.514</td>
</tr>
<tr>
<td>140</td>
<td>0.503424</td>
<td>1.96326</td>
<td>134.837</td>
</tr>
<tr>
<td>160</td>
<td>0.482039</td>
<td>1.85775</td>
<td>157.217</td>
</tr>
</tbody>
</table>

**Table- (2) Variation in parameter $b$**

<table>
<thead>
<tr>
<th>$b$</th>
<th>$t_1$</th>
<th>$T$</th>
<th>$P(T,t_1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>76</td>
<td>0.550549</td>
<td>2.21261</td>
<td>93.3771</td>
</tr>
<tr>
<td>78</td>
<td>0.549563</td>
<td>2.21476</td>
<td>95.6483</td>
</tr>
<tr>
<td>80</td>
<td>0.548588</td>
<td>2.21685</td>
<td>97.9206</td>
</tr>
<tr>
<td>82</td>
<td>0.547624</td>
<td>2.2189</td>
<td>100.194</td>
</tr>
</tbody>
</table>

**Table- (3) Variation in scale parameter $\alpha$**

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$t_1$</th>
<th>$T$</th>
<th>$P(T,t_1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.550549</td>
<td>2.21261</td>
<td>93.3771</td>
</tr>
<tr>
<td>0.2</td>
<td>0.504696</td>
<td>2.19272</td>
<td>88.9163</td>
</tr>
<tr>
<td>0.3</td>
<td>0.462959</td>
<td>2.17396</td>
<td>85.1296</td>
</tr>
<tr>
<td>0.4</td>
<td>0.425271</td>
<td>2.15656</td>
<td>81.916</td>
</tr>
</tbody>
</table>

**Table- (4) Variation in shape parameter $\beta$**

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$t_1$</th>
<th>$T$</th>
<th>$P(T,t_1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.3</td>
<td>0.550549</td>
<td>2.21261</td>
<td>93.3771</td>
</tr>
<tr>
<td>0.4</td>
<td>0.639018</td>
<td>2.25639</td>
<td>101.374</td>
</tr>
<tr>
<td>0.5</td>
<td>0.701465</td>
<td>2.28739</td>
<td>107.353</td>
</tr>
<tr>
<td>0.6</td>
<td>0.747829</td>
<td>2.3102</td>
<td>112.02</td>
</tr>
</tbody>
</table>
Table- (5) Variation in selling price $s$

<table>
<thead>
<tr>
<th>$s$</th>
<th>$t_1$</th>
<th>$T$</th>
<th>$P(T, t_1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.550549</td>
<td>2.2126</td>
<td>93.3771</td>
</tr>
<tr>
<td>2.5</td>
<td>0.634548</td>
<td>2.5282</td>
<td>188.313</td>
</tr>
<tr>
<td>3.0</td>
<td>0.738288</td>
<td>2.9094</td>
<td>289.867</td>
</tr>
<tr>
<td>3.5</td>
<td>0.862504</td>
<td>3.3541</td>
<td>399.276</td>
</tr>
</tbody>
</table>

**Observations:**
- From the table (1), we observed that total profit increases if we increase the parameter $a$.
- From the table (2), we observed that total profit increases if we increase the parameter $b$.
- From the table (3), we observed that total profit decrease if we increase the scale parameter $\alpha$.
- From the table (4), we observed that total profit increases if we increase the shape parameter $\beta$.
- From the table (5), we observed that total profit increases if we increase the selling price $s$.

**Concluding Remarks**
In this paper, we developed an inventory model for deteriorating items with time dependent demand and shortages. The rate of deterioration follows the Weibull distribution with two parameters. The demand rate is assumed of time dependent. The shortages are allowed and shortages are completely backlogged. The shortage cost, holding cost and, deterioration cost are considered in this model. A numerical example and sensitivity analysis are presented to illustrate the proposed model.

**References:**
[5] Peter Chu and Patrick S. Chen “On an inventory model for deteriorating items and
time-varying demand” Mathematical Methods of Operations Research c Springer-
Verlag, 53:297-307 (2001)
demand rate, shortages and decrease in demand. Journal of combinatorics 
7. Goyal, S.K., and Giri, B.C., Recent trends in modelling of deteriorating inventory, 
8. Roy, T., and Chaudhuri, K.S., A production-inventory model under stock-
Two-parameter Weibull Distribution Deterioration and Price-dependent Demand’, 
International Journal of Mathematical Education in Science and Technology, 36, 25– 
10. Vikas Sharma and Rekha Rani Chaudhary, “An inventory model for deteriorating 
items with weibull deterioration with time dependent demand and shortages” 
deterioration and backorder rates, European Journal of Operational Research, 208(3), 
[12] Deb M. and Chaudhuri K.S., An EOQ model for items with finite rate of 
production and variable rate of deterioration, Opsearch, 23, 175-181 (1986).
distribution deterioration shortages and trended demand, International Journal of 
[14] Chakraborti T. and Chaudhuri K.S., An EOQ model for deteriorating items with 
a linear trend in demand and shortages in all cycles, International Journal of 
shortages and trended demand an extension of Philip’s model, Computers and 