Published by :
http://www.ijert.org

International Journal of Engineering Research & Technology (IJERT)
ISSN: 2278-0181
Vol. 5 Issue 01, January-2016

An Efficient Approach for Fractional Harry Dym Equation by Using Homotopy Analysis Method

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***Abstract* - In this article, a structured approach based on Homotopy analysis method is implemented to resolve nonlinear fractional Harry Dym equation. The fractional derivative is described in Caputo sense. The ultimate goal of this paper is to find error analysis which shows our final concluded result converge to the exact and approximate solution. The solution which was derived from convergent series form proves the effectiveness of the method in resolving different type of fractional differential equation. Henceforth this technique is efficient and less time consuming when compare to other methods of finding approximate and exact solutions for nonlinear partial differential equations.**

Keywords: Harry Dym Equation, Caputo derivative, Homotopy analysis method, approximate solution

1. INTRODUCTION

Fractional calculus is a separate study of investigating the characteristics of integral and derivatives of non integers order. This fractional calculus is specifically popular for solving differential equation involving fractional derivatives of unknown function which is also called fractional differential equation. The number of problems in science and engineering involving in fractional derivatives is growing continuously and perhaps the fractional calculus of the twenty –five century[1-6].The idea appeared in a letter of Leibniz to L'Hospital in 1965.The analytical results related to fractional differential equation have been studied by many authors[7,8].During the last decades, multiple number of numerical and analytical methods have been suggested to resolve fractional differential equations. The most accepted methods are fractional difference method[9,10],Adomin decomposition method[11],variational iteration method[12,13].Recently the Homotopy perturbation method and Lagrange multiplier method were used extensively to solve multiorder fractional differential equation[14] .

In this paper, we refer the below nonlinear time fractional Harry Dym equation of the form

$$D_t^{\propto} U(x,t) = U^3(x,t) D_x^3 U(x,t),\ 0 < \propto \leq 1 \quad \text{-----1.1}$$

With the initial condition

$$U(x,0) = \left(a - \frac{3\sqrt{b}}{2}x\right)^{2/3} \quad \text{-------1.2}$$

Where $\propto$ denotes the order of the fractional derivative and

$U(x,t)$ refers to a function of x and t.The fractional derivative is present in the Caputo sense.

In case of $\propto$ =1, the equation(fractional Harry Dym equation) simplify into classical nonlinear Harry Dym equation. The exact solution of the Harry Dym equation is given by[15]

$$U(x,t) = \left(a - \frac{3\sqrt{b}}{2}(x+bt)\right)^{2/3} \quad \text{------1.3}$$

Where a and b are suitable constant.

This Harry Dym Dynamical equation is used to identify applications in several physical systems. It is also appeared(Harry Dym equation) in Kruskal and Moser[16] and is attributed in an unpublished paper by Harry Dym in 1973-1974.Harry Dym is a completely integrable nonlinear evolution equation. Meanwhile it obeys an infinite number of conversion laws; but it doesn't possess the Painleve property. The primary development of fractional calculus was achieved with the help of book by Oldhamnand Spanier[17].The essential results related with the solution of fractional differential equation may be found in book[18-21].The Homotopy analysis method was developed by liao[22-31].It made an analytical study on fractional Kdv, K(2,2),Burgers, BBM-Burgers, Cubic Boussineq, Boussinesq-like B(m,n) equation and coubled Kdv.

In the present paper, we implemented the Homotopy analysis method to determine the approximate solution of nonlinear fractional differential equation. Our aim of this paper is to extend the application of Homotopy analysis method to obtain analytic and approximate solution to the nonlinear time fractional Harry Dym equation. HAM provides the solutions in terms of convergent series with easily computable components in a direct way which is not achievable using linearization, perturbation. The error analysis will be made by comparing the approximate and the exact solution. The results are represented in the form of graph.

2. BASIC DEFINITIONS OF FRACTIONAL CALCULUS

In this part, we give some definitions and properties of the fractional calculus[32]

Definition 2.1: A real function f(t), t > 0, is said to be in the space $C_\mu, \mu \in R$,if there exists a real number $p > \mu$, such that $f(t) = t^p f_1(t)$ where $f_1(t) \in C(0,\infty)$ and it is said to be in the space C_μ^n, if and only if $f^{(n)} \in C_\mu, n \in N$ [32].

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Definition 2.2: The Riemann-Liouville fractional integral operator $(J^{\alpha})$of order $\alpha \geq 0$, of a function $f \in C_{\lambda}$, $\lambda \geq -1$ if defined as [32]

$$J^{\alpha}f(t) = D^{-\alpha}f(t) =$$

$$\frac{1}{\Gamma(\alpha)}\int_0^1 (t-\tau)^{\alpha-1}f(\tau)d\tau, \quad (\alpha > 0) \quad \text{----2.1}$$

$$J^0 f(t) = f(t) \quad \text{----------2.2}$$

Where $\Gamma(z)$ is the well known Gamma function.Some of the properties of the operator (J^{α}), according to our requirement are given below.

For $f \in C_{\lambda}$, $\lambda \geq -1$, $\alpha, \beta \geq 0$ *and* $\gamma \geq -1$:

(1) $J^{\alpha}J^{\beta}f(t) = J^{\alpha+\beta}f(t)$
(2) $J^{\alpha}J^{\beta}f(t) = J^{\beta}J^{\alpha}f(t)$
(3) $J^{\alpha}J^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}t^{\alpha+\gamma}$

Definition 2.3: The fractional derivative D_t^{α} of $f(t)$ in Caputo sense defined as

$$D_t^{\alpha}f(t) = \frac{1}{\Gamma(m-\alpha)}\int_0^t \frac{f^{(m)}(\tau)}{(t-\alpha)^{\alpha+1-m}}d\tau,$$

where $m-1 < \alpha \leq m$, $m \in N$, $t > 0, f \in C_{-1}^{m}$

The two basic properties of the Caputo's fractional derivative are given below

Lemma 2.1: If $m-1 < \alpha \leq m$, $m \in N$, *and* $f \in C_{\mu}^{m}$, $\mu \geq -1$ then

$$(D_t^{\alpha}J_t^{\alpha})f(t) = f(t),$$

$$(J_t^{\alpha}D_t^{\alpha})f(t) = f(t) - \sum_{i=0}^{m-1} f^{i}(0^{+})\frac{t^{i}}{i!}$$

3. BASIC HOMOTOPY ANALYSIS METHOD

The Homotopy analysis method (HAM) has been proposed by Liao in [22]. The application of HAM is use to calculate the exact solution and the approximate solutions of linear and nonlinear Differential equations. Consider the non-linear differential equation

$$N[X(t)]=0; \quad (3.1)$$

Where N is a nonlinear auxiliary operator and $X(t)$ is an unknown function. The boundary and initial conditions are ignored, which can be treated in the similar way. The zero-order deformation equation is constructed as

$$(1-q)L[X_n(t)-x_0(t)]=q\ \hbar N[X(t),A], \quad (3.2)$$

Where $q \in [0,1] \rightarrow$ the embedding parameter,
$\hbar \neq 0 \rightarrow$ a nonzero auxiliary parameter,
$L \rightarrow$ an auxiliary linear operator,
$X(t) \rightarrow$ an unknown function.
Also we can choose whatever auxiliary unknown in HAM because of its properties. By Taylor's theorem, $X(t)$ can be expanded with respect to the embedding parameter q as

$$X(t) = X_0(t) + \sum_{n=1}^{\infty} X_n(t)q^n \quad (3.3)$$

$$X_n(t) = \frac{1}{n!}\frac{\partial^n X(t)}{\partial q^n}|_{q=0} \quad (3.4)$$

Differentiating the Zeroth-Order deformation equation n-times with respect to q at $q = 0$ and then dividing it by n,we have the following nth-order deformation equation

$$L[Xn(t)- \chi_n X_{n-1}(t)] = \hbar R_n(\overrightarrow{X_{n-1}}(t)); \quad (3.5)$$

$$R_n(X_{n-1}(t)) = \frac{1}{(n-1)!}\frac{\partial^{n-1}N(X(t))}{\partial q^{n-1}}|_{q=0} \quad (3.6)$$

$$\text{and } \chi_n = \begin{cases} 0, & n \leq 1 \\ 1, & x > 1 \end{cases} \quad (3.7)$$

Here $\overrightarrow{X_{n-1}} = [X_0(t), X_1(t), X_2(t), X_3(t) \ldots\ldots X_{m-1}(t)]$

If the series (3) converges at $q = 1$ we have

$$X(t) = X_0(t) + \sum_{n=1}^{\infty} X_n(t) \quad (3.8)$$

4.HAM SOLUTION OF HARRY DYM EQUATION:

Consider the Harry Dym equation:

$$D_t^{\alpha}U(x,t) = U^3(x,t)D_x^3U(x,t), \ 0 < \alpha \leq 1 \quad \text{------ 4.1}$$

With the initial condition

$$U(x,0) = (a - \frac{3\sqrt{b}}{2}x)^{2/3} \quad \text{-------4.2}$$

Where $\propto$ denotes the order of the fractional derivative and

$U(x,t)$ refers to a function of x and t.

Solution:

Step 1: Let us consider the initial approximation from the initial condition (2)

$$U_0(x,t) = (a - \frac{3\sqrt{b}}{2}x)^{2/3} \quad \text{--------4.3}$$

Step 2: Let us consider the nonlinear operator

$$N(\varphi(x,t;q)) = D_t^{\alpha}\varphi(x,t;q) - \varphi^3(x,t;q)D_x^3\varphi(x,t;q)\text{-} \quad \text{-------4.4}$$

Step 3: Let us consider the Linear operator

$$L(\varphi(x,t;q)) = D_t^{\alpha}\varphi(x,t;q) \quad \text{--------4.5}$$

Step 4: Using the above definition, we construct the zeroth order deformation equation

$$(1-q)L[\varphi(x,t:q) - U_0(x,t)] = q\hbar N[\varphi(x,t;q)] \text{ --4.6}$$

Obviously when q=0 and q=1, we can write from equation (4.6)

$$\varphi(x;0) = U_0(x) = U(0) \text{ and}$$

International Journal of Engineering Research & Technology (IJERT)
ISSN: 2278-0181
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$$\varphi(x;1) = U(x)$$

Step 5: According to equation 3.4 and 3.7 we obtain m[th] order deformation equation

$$L[U_m(x,t) - \chi_m U_{m-1}(x)] = hR_m[\overrightarrow{U_{m-1}(x)}]$$

--------4.7

Where $R_m\left[\overrightarrow{U_{m-1}(x)}\right] = \frac{1}{(m-1)!}\left[\frac{\partial^{m-1}N[\varphi(x;q)]}{\partial^{m-1}}\right]_{q=0}$

$$= \frac{1}{(m-1)!}\left[\frac{\partial^{m-1}}{\partial q^{m-1}}\{D_t^{\alpha}\varphi(x,t;q) - \varphi^3(x,t;q)D_x^3\varphi(x,t;q)\}\right]_{q=0}$$

$$=[(D_t^{\alpha}U_{m-1} - \sum_{k=1}^{m}\sum_{n=1}^{k}U_{k-n}\sum_{s=1}^{n}U_{n-s}U_s\ D^3U_{m-k}]$$

-----------4.8

Step 6: From equation 4.7 and 4.8 we can write

$$L[U_m(x,t) - \chi_m U_{m-1}(x)] = h[(D_t^{\alpha}U_{m-1} - \sum_{k=1}^{m}\sum_{n=1}^{k}U_{k-n}\sum_{s=1}^{n}U_{n-s}U_s\ D^3U_{m-k}]$$

$$\Rightarrow U_m = \chi_m U_{m-1}(x) + L^{-1}[hD_t^{\alpha}U_{m-1}] - L^{-1}[h\sum_{k=1}^{m}\sum_{n=1}^{k}U_{k-n}\sum_{s=1}^{n}U_{n-s}U_s\ D^3U_{m-k}]$$

$$\Rightarrow U_m = \chi_m U_{m-1}(x) + J_t^{\alpha}[hD_t^{\alpha}U_{m-1}] - J_t^{\alpha}[h\sum_{k=1}^{m}\sum_{n=1}^{k}U_{k-n}\sum_{s=1}^{n}U_{n-s}U_s\ D^3U_{m-k}]$$

-------------------4.8a

Step 7: For m=1, equation 4.8a becomes

$$U_1 = -h\,J_t^{\alpha}[U_0^3D_x^3U_0]$$

For m=2, equation 4.8a becomes

$$U_2 = U_1 + [hU_1] - h\,J_t^{\alpha}[U_0^3D_x^3U_1 + 3U_0^2U_1D_x^3U_0]$$

For m=3, equation 4.8a becomes

$$U_3 = U_2 + [hU_2] - h\,J_t^{\alpha}[U_0^3D_x^3U_2 + 3U_0^2U_1D_x^3U_1 + (3U_0U_1^2 + 3U_0^2U_2)D_x^3U_0]$$

For $m \geq 3$,equation 4.8 becomes

$$U_m = U_m + [hU_m] - h\,J_t^{\alpha}[\sum_{k=1}^{m}\sum_{n=1}^{k}U_{k-n}\sum_{s=1}^{n}U_{n-s}U_s\ D^3U_{m-k}]$$ -------4.9

Step 8: Taking $h = -1$, We get from equation 4.3,4.7,4.8 and 4.9 after some rigorous calculation,

$$U_0(x,t) = (a - \tfrac{3\sqrt{b}}{2}x)^{2/3}$$

$$U_1(x,t) = -\frac{t^{\alpha}}{\Gamma(\alpha+1)}b^{3/2}(a - \tfrac{3\sqrt{b}}{2}x)^{-1/3}$$

$$U_2(x,t) = -\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\frac{b^3}{2}(a - \tfrac{3\sqrt{b}}{2}x)^{-4/3}$$

$$U_3(x,t) = \frac{t^{3\alpha}}{\Gamma(3\alpha+1)}\,b^{9/2}(a - \tfrac{3\sqrt{b}}{2}x)^{-7/3}\left[\frac{15\,\Gamma(2\alpha+1)}{2\,2(\Gamma(\alpha+1)^2)} - 16\right]$$

and so on.

Step 9: By using equation 3.3, we obtain

$$U(x) = U_0(x) + \sum_{m=1}^{\infty}U_m(x)\,, \quad \text{for } q=1$$

(i) The second order approximation of $U(x)$ is given

$$U_{HAM}(x,t) = U_0(x) + \sum_{m=1}^{1}U_m(x)$$

$$=U_0(x) + U_1(x)$$

$$=(a - \tfrac{3\sqrt{b}}{2}x)^{2/3} - b^{3/2}(a - \tfrac{3\sqrt{b}}{2}x)^{-1/3}\frac{t^{\alpha}}{\Gamma(\alpha+1)}$$

(ii) The third order approximation of $U(x)$ is given

$$U_{HAM}(x,t) = U_0(x) + \sum_{m=1}^{2}U_m(x)$$

$$=U_0(x) + U_1(x) + U_2(x)$$

$$=(a - \tfrac{3\sqrt{b}}{2}x)^{2/3} - b^{3/2}(a - \tfrac{3\sqrt{b}}{2}x)^{-1/3}\frac{t^{\alpha}}{\Gamma(\alpha+1)} - \frac{b^3}{2}(a - \tfrac{3\sqrt{b}}{2}x)^{-4/3}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$$

(iii) The third order approximation of $U(x)$ is given

$$U_{HAM}(x,t) = U_0(x) + \sum_{m=1}^{3}U_m(x)$$

$$=U_0(x) + U_1(x) + U_2(x) + U_3(x)$$

$$=(a - \tfrac{3\sqrt{b}}{2}x)^{2/3} - b^{3/2}(a - \tfrac{3\sqrt{b}}{2}x)^{-1/3}\frac{t^{\alpha}}{\Gamma(\alpha+1)} - \frac{b^3}{2}(a - \tfrac{3\sqrt{b}}{2}x)^{-4/3}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + b^{9/2}(a - \tfrac{3\sqrt{b}}{2}x)^{-7/3}\left[\frac{15\,\Gamma(2\alpha+1)}{2\,2(\Gamma(\alpha+1)^2)} - 16\right]\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}$$

TABLE 1: Comparison study between HPSTM,ADM,HAM and Exact solution, when α=1 and for constant values of a=4 and b=1

x	T	HPSTM[33]	ADM[33]	HAM	Exact solution
0	1	1.843946953	1.843946953	1.843946953	1.843946953
0.2	1	1.694117377	1.694117377	1.6918972844	1.691538112
0.4	1	1.537581542	1.537581542	1.5365890571	1.534036644
0.6	1	1.373028020	1.373028020	1.3737638298	1.367980757
0.8	1	1.198654865	1.198654865	1.1949462024	1.191138425
1	1	1.011880649	1.011880649	1.0113697992	1.000000000

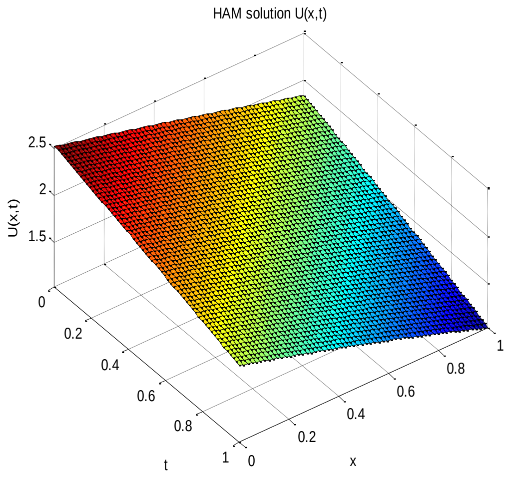

Figure 1: HAM solution of U(x,t) when alpha=1

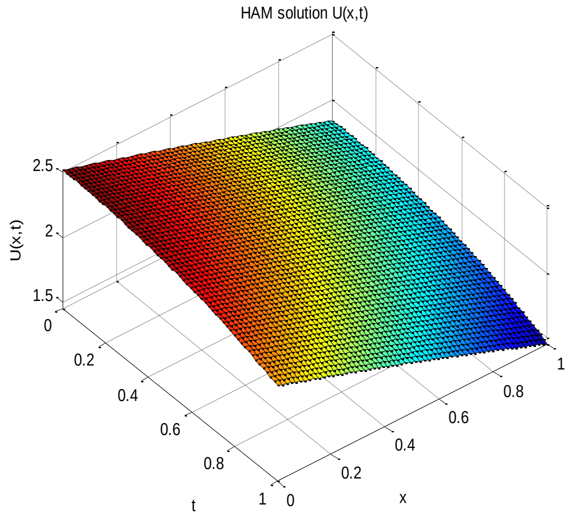

Figure 2: HAM solution of U(x,t) when alpha=2

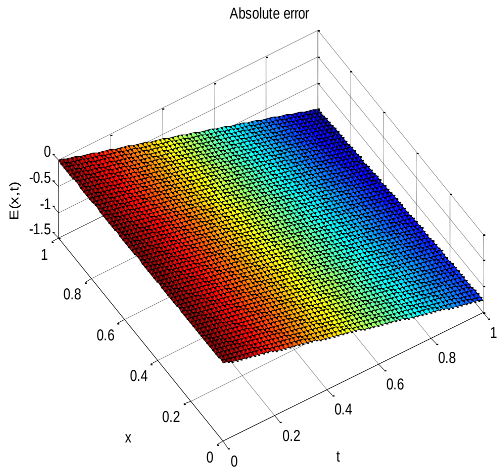

Figure 3: Absolute error E(x,t)

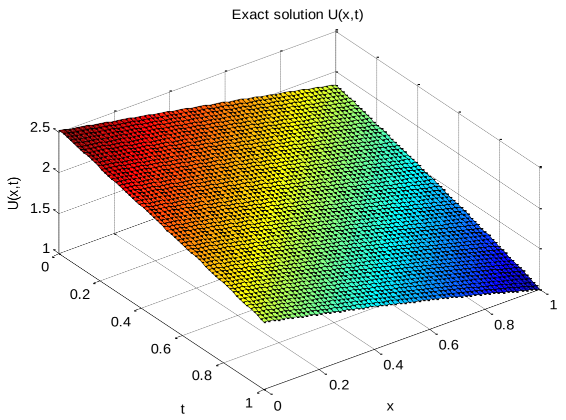

Figure 4: Exact solution of U(x,t)

International Journal of Engineering Research & Technology (IJERT)
ISSN: 2278-0181
Vol. 5 Issue 01, January-2016

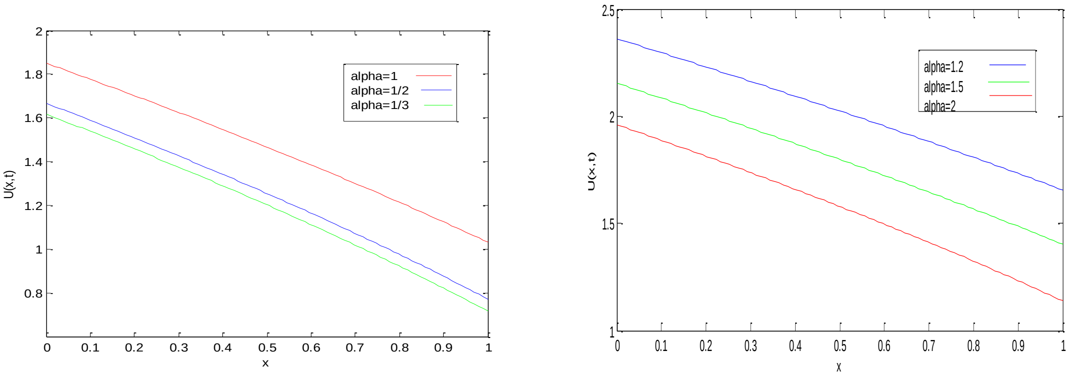

Figure 5&6: Plot of U(x,t)versus x at t=1 for different value of alpha

5. RESULTS & DISCUSSION

In this article, detail study has been made on error analysis between exact and approximate solution which is specified by through figure 3 with high accuracy. The easiest and accuracy of the suggested method is depicted by calculating the absolute error $E_5(x,t) = |U(x,t) - U_5(x,t)|$ at the constant value (a=4,b=1) where $U(x,t)$ and $U_5(x,t)$ are the exact solution and approximate solution of equation (4.1) respectively. Figure 3 shows the error analysis between the exact and approximate solution which indicates significantly small convergence of the series solution very quickly. Hence during all numerical evolution we are going to take only third order of the series solution By introducing more terms of approximate solution, the accuracy of error analytical result will be improved. The characteristics of the approximate solution is specified in the figure 5 for $U_5(x,t)$ for different value of ∝= 1, ½ ,1/3 for standard Harry Dym equation. As per the Table 1 and graphs we inferred that HAM solution made a good agreement with the HPSTM and ADM solution. The figure shows that the accurate solution can only be improve introducing more terms of HAM solution. This is due to quick convergence of the HAM.

6. CONCLUSION

In this paper HAM is efficiently applied to get approximate solution of time fractional Harry Dym equation. In HAM a Homotopy with embedding small parameter $q \in [0,1]$ which is helpful to implement full advantages of traditional perturbation method and Homotopy technique.HAM is different from other analytical method by providing the way to as to control the convergence region of solution series by with auxiliary parameter $\hbar$ .This is the main useful features of HAM.HAM does not require any other methods or transformation techniques. The results of the compare solution (HAM third order solution, HPSTM, ADM) are specified in the table 1.From the table 1 it is observed that there is a good understanding between the HSPTM, ADM and exact solution. From the derived results it shows the reliability of the algorithm and it is greatly suitable for Non-linear fractional partial differential equation.

ACKNOWLDGEMENTS

The authors are grateful to the reviewers for interpretation this paper and for their valuable comments and suggestion for the improvement of the article.

REFERENCES

[1] J. S. Leszczynski, An Itorduction to Fractional Mechanics, Czestochowa University of Technology, Częstochowa, Poland, 2011.

[2] K. S. Miller and B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations, John Wiley & Sons Inc., New York, NY, USA, 1993.

[3] A. Oustaloup, La Derivation Non Entiere, Hermes, Paris, France, 1995.

[4] J. A. T. Machado, "Analysis and design of fractional-order digital control systems," Systems Analysis Modelling Simulation, vol. 27, no. 2-3, pp. 107–122, 1997. View at Google Scholar

[5] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, Theory and Applications of Fractional Differential Equations, vol. 204, Elsevier Science B.V., Amsterdam, The Netherlands, 2006.

[6] J. T. Machado, V. Kiryakova, and F. Mainardi, "Recent history of fractional calculus," Communications in Nonlinear Science and Numerical Simulation, vol. 16, no. 3, pp. 1140–1153, 2011. View at Publisher ·View at Google Scholar

[7] K. Deithelm, N. J. Ford, et al., "Analysis of fractional differential equation," Numerical Analysis Report 377, The University of Manchester, 2003. View at Google Scholar

[8] R. W. Ibrahim and S. Momani, "On the existence and uniqueness of solutions of a class of fractional differential equations," Journal of Mathematical Analysis and Applications, vol. 334, no. 1, pp. 1–10, 2007. View at Publisher · View at Google Scholar

[9] I. Podlubny, Fractional Differential Equations, vol. 198, Academic Press Inc., San Diego, Calif, USA, 1999.

[10] K. B. Oldham and J. Spanier, The Fractional Calculus, Academic Press, New York. NY, USA, 1974.

International Journal of Engineering Research & Technology (IJERT)
ISSN: 2278-0181
Vol. 5 Issue 01, January-2016

[11] S. Momani, "Analytical approximate solution for fractional heat-like and wave-like equations with variable coefficients using the decomposition method," Applied Mathematics and Computation, vol. 165, no. 2, pp. 459–472, 2005. View at Publisher · View at Google Scholar

[12] S. Momani, Z. Odibat, and A. Alawneh, "Variational iteration method for solving the space- and time-fractional KdV equation," Numerical Methods for Partial Differential Equations, vol. 24, no. 1, pp. 262–271, 2008. View at Publisher · View at Google Scholar

[13] Z. Odibat and S. Momani, "The variational iteration method: an efficient scheme for handling fractional partial differential equations in fluid mechanics," Computers & Mathematics with Applications. An International Journal, vol. 58, no. 11-12, pp. 2199–2208, 2009. View at Publisher · View at Google Scholar

[14] N. H. Sweilam, M. M. Khader, and R. F. Al-Bar, "Numerical studies for a multi-order fractional differential equation," Physics Letters A, vol. 371, no. 1-2, pp. 26–33, 2007. View at Publisher · View at Google Scholar

[15] R. Mokhtari, "Exact solutions of the Harry-Dym equation," *Communications in Theoretical Physics*, vol. 55, no. 2, pp. 204–208, 2011.

[16] M. D. Kruskal and J. Moser, *Dynamical Systems, Theory and Applications*, Lecturer Notes Physics, Springer, Berlin, Germany,1975

[17] Oldham KB, Spanier J. The fractional calculus. New York:Academic Press; 1974.

[18] .Miller KS, Ross B. An introduction to the fractional calculus andfractional differential equations. New York: John Willey and Sons, Inc.; 2003.

[19] .Podlubny I. Fractional differential equations. New York: Academic Press; 1999.

[20] .Kilbas AA, Srivastava HM, Trujillo JJ. Theory and application of fractional differential equations. Amsterdam: Elsevier; 2006.

[21] .H. Xu and C. Jie, Analysis of a time fractional wave-like equation with the homotopy analysis method,Phys Lett A 372 (2008), 1250–1255.

[22] S. J. Liao, The proposed homotopy analysis technique for the solution of nonlinear problems, PhD thesis, Shanghai Jiao Tong University, 1992.

[23] S. J. Liao, Series solutions of unsteady boundary-layer flows over a stretching flat plate, Studies Appl Math 117 (2006), 2529–2539.

[24] S. J. Liao, An approximate solution technique which does not depend upon small parameters: a special example, Int J Nonlinear Mech 30 (1995), 371–380.

[25] S. J. Liao, An approximate solution technique which does not depend upon small parameters (II): an application in fluid mechanics, Int J Nonlinear Mech 32 (1997), 815–822.

[26] S. J. Liao, An explicit, totally analytic approximation of Blasius viscous flow problems, Int J Nonlinear Mech 34 (1999), 759–778.

[27] S. J. Liao, Beyond perturbation: introduction to the homotopy analysis method, Chapman and Hall, CRC Press, Boca Rator, 2003.

[28] S. J. Liao, On the homotopy analysis method for nonlinear problems, Appl Math Comput 147 (2004), 499–513.

[29] S. J. Liao and A. Campo, Analytic solutions of the temperature distribution in Blasius viscous flow problems, J Fluid Mech 453 (2002), 411–425.

[30] S. J. Liao, On the analytic solution of magnetohydrodynamic flows of non-Newtonian fluids over a stretching sheet, J Fluid Mech 488 (2003), 189–212.

[31] S. J. Liao, An analytic approximate approach for free oscillations of self-excited systems, Int J Nonlinear Mech 39 (2004), 271–280

[32] I. Podlubny, Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications, Academic Press, New York, 1999

[33] An Efficient Approach for Fractional Harry Dym Equation by Using Sumudu Transform by Devendra Kumar,Jagdev Singh.