An approach of Laplace transform for solving various fractional differential equations and its applications

Lalita Mistry
Asst. Professor
Department of Mathematics
JIET, Jodhpur
lalita.nagal@gmail.com

Dr. Arif Khan
Professor
Department of Mathematics
JIET, Jodhpur
arif.khan@jietjodhpur.com

Abstract— One of the major advantages of **fractional calculus** is that it can be considered as a super set of integer-order calculus. Thus, fractional calculus has the potential to accomplish what integer-order calculus cannot. It has been believe that many of the great future developments will come from the applications of fractional calculus to different fields.

In many recent works, many researchers have demonstrated the usefulness of fractional calculus in the derivation of particular solutions of a significantly large number of linear ordinary and partial differential equations of the second and higher orders. **Laplace** decomposition method is applied to obtain series solutions of nonlinear fractional differential equation. One of the main objective that has been found in the literature survey is to show how this simple fractional calculus method to the solutions of some families of fractional differential equations would lead naturally to several interesting consequences, which include (for example) a generalization of the classical Frobenius method. The methodology used is based chiefly upon some general theorems on (explicit) particular solutions of some families of fractional differential equations with the Laplace transform and the expansion coefficients of binomial series.

The Laplace transform is a powerful tool in applied mathematics and engineering. It will allow us to transform fractional differential equations into algebraic equations and then by solving these algebraic equations. The unknown function by using the Inverse Laplace Transform can be obtained.

.

Keywords: Fractional-order differential equations; Laplace transform of the fractional derivative;

I. INTRODUCTION

A numerical method for solving LNFODE (Linear Non-homogenous Fractional Ordinary Differential Equation) is based on Bernstein polynomials approximation is also reported in the literature. The operational matrices of integration, differentiation and products are introduced and utilized to reduce the LNFODE problem in order to solve algebraic equations. The method is general, easy to implement and yields very accurate results

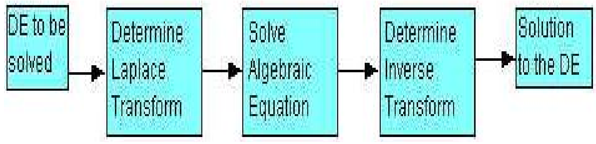
Laplace transform is yet another operational tool for solving constant coefficients linear differential equations. The process of solution consists of three main steps:

_ The given hard problem is transformed into a simple equation.

_ This simple equation is solved by purely algebraic manipulations.

_ The solution of the simple equation is transformed back to obtain the solution of the given problem.

In this way the Laplace transformation reduces the problem of solving a differential equation to an algebraic problem. The third step is made easier by tables, whose role is similar to that of integral tables in integration. The above procedure can be summarized by Figure



The Laplace transform can be used to solve differential equations. Besides being a different and efficient alternative to variation of parameters and undetermined coefficients, the Laplace method is particularly advantageous for input terms that are piecewise-defined, periodic or impulsive.

Fractional differential equations is a generalization of ordinary differential equations and integration to arbitrary non integer orders. The origin of fractional calculus goes back to Newton and Leibniz in the seventieth century. It is widely and efficiently used to describe many phenomena arising in engineering, physics, economy, and science. Recent investigations have shown that many physical systems can be represented more accurately through fractional derivative formulation.

Fractional differential equations, therefore find numerous applications in the field of visco-elasticity, feed back amplifiers, electrical circuits, electro analytical chemistry, fractional multipoles, neuron modelling encompassing different branches of physics, chemistry and biological sciences. There have been many excellent books and monographs available on this field. Most recent and up-to-date developments on fractional differential and fractional integro-differential equations with applications involving many different potentially useful operators of fractional calculus was given by many. In a recent work by Jaimini et.al. the authors have given the corresponding Leibnitz rule for fractional calculus. For the history of fractional calculus, interested reader may see the recent review paper by Machado et. al.

Many physical processes appear to exhibit fractional order behavior that may vary with time or space. The fractional calculus has allowed the operations of integration and differentiation to any fractional order. The order may take on any real or imaginary value. Recently theory of fractional differential equations attracted many scientists and mathematicians to work on. The results have been obtained by using fixed point theorems like Picard's, Schauder fixed-point theorem and Banach contraction mapping principle. About the development of existence theorems for fractional functional differential equations, many contribution exists. Many applications of fractional calculus amount to replacing the time derivative in a given evolution equation by a derivative of fractional order. The results of several studies clearly stated that the fractional derivatives seem to arise generally and universally from important mathematical reasons. Recently, interesting attempts have been made to give the physical meaning to the initial conditions for fractional differential equations with Riemann-Liouville fractional derivatives were proposed. Ahmed et. al. considered the fractional order predator-prey model and the fractional order rabies model. They have shown the existence and uniqueness of solutions of the model system and also studied the stability of equilibrium points. Lakshmikantham and Vatsala and Lakshmikantham defined and proved existence of the solution of fractional initial value problems.

Fractional calculus is a field of mathematics study that qrows out of the traditional definitions of calculus integral and derivative operators in much the same way fractional exponents is an outgrowth of exponents with integer value. The concept of fractional calculus(fractional derivatives and fractional integral) is not new. In 1695 *L'Hospital* asked the question as to the meaning of d^ny/dx^n if n = 1/2; that is " what if n is fractional?". *Leibniz* replied that "$d^{1/2}x$ will be equal to $x\sqrt{dx} : x$".

It is generally known that integer-order derivatives and integrals have clear physical and geometric interpretations. However, in case of fractional-order integration and differentiation, which represent a rapidly qrowing field both in theory and in applications to real world problems, it is not so.

II. Basic Definition of Laplace Transform

If f(t) is defined for t≥0 the (unilateral) Laplace transform (Pierre-Simon Laplace) and its inverse $\mathcal{L}^{-1}$ are defined by:

$$\mathcal{L}: \quad f(t) \mapsto F(s) \quad = \mathcal{L}\{f(t)\}(s) = \int_0^\infty e^{-st} f(t)dt,$$
$$\mathcal{L}^{-1}: \quad F(s) \mapsto f(t) \quad = \mathcal{L}^{-1}\{F(s)\}(t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} F(s)e^{st}ds.$$

Note that if f(t) $e^{-\sigma_0 t} \to 0$ as $t \to \infty$ then the first integral converges for all complex numbers s with real part greater than s0, and in the second integral we then demand that $a > \sigma_0$.

whenever the limit exists (as a finite number). When it does, the integral is said to *converge*. If the limit does not exist, the integral is said to *diverge* and there is no Laplace transform defined for *f*. The notation $\mathcal{L}(f)$ will also be used to denote the Laplace transform of *f*, and the integral is the ordinary Riemann (improper) integral.

The parameter *s* belongs to some domain on the real line or in the complex plane. The *s* will be chosen appropriately so as to ensure the convergence of the Laplace integral. In a mathematical and technical sense, the domain of *s* is quite important. However, in a practical sense, when differential equations are solved, the domain of *s* is routinely ignored. When *s* is complex, we will always use the notation $s = x + iy$.

The symbol *L* is the *Laplace transformation*, which acts on functions $f = f(t)$ and generates a new function, $F(s) = \mathcal{L}(f(t))$.

Convergence

Although the Laplace operator can be applied to a great many functions, there are some for which the integral does not converge.

Example For the function

$$\lim_{\tau\to\infty}\int_0^\tau e^{-st}e^{t^2}dt = \lim_{\tau\to\infty}\int_0^\tau e^{t^2-st}dt = \infty$$

for any choice of the variable *s*, since the integrand grows without bound as $\tau \to \infty$.

In order to go beyond the superficial aspects of the Laplace transform, it needs to distinguish two special modes of convergence of the Laplace integral. The integral is said to be *absolutely convergent* if

$$\lim_{\tau\to\infty}\int_0^\tau |e^{-st}f(t)|\,dt$$

exists. If $\mathcal{L}(f(t))$ does converge absolutely, then

$$\left|\int_\tau^{\tau'} e^{-st}f(t)\,dt\right| \le \int_\tau^{\tau'} |e^{-st}f(t)|dt \to 0$$

as $\tau \to \infty$, for all $\tau' > \tau$. This then implies that $\mathcal{L}(f(t))$ also converges in the ordinary sense of (3.1).

There is another form of convergence that is of the utmost importance from a mathematical perspective. The integral (3.1) is said to *converge uniformly* for s in some domain Ω in the complex plane if for any $\varepsilon > 0$, there exists some number τ_0 such that if $\tau \geq \tau_0$, then

$$\left| \int_{\tau}^{\infty} e^{-st} f(t)\, dt \right| < \varepsilon$$

for all s in Ω. The point here is that τ0 can be chosen sufficiently large in order to make the "tail" of the integral arbitrarily small, *independent of s*.

Continuity Requirements

Since we can compute the Laplace transform for some functions and not others, such as $e(t2)$, we would like to know that there is a large class of functions that do have a Laplace tranform. There is such a class once we make a few restrictions on the functions we wish to consider.

Definition : A function f has a **jump discontinuity at a point** t_0 if both the limits

$$\lim_{t \to t_0^-} f(t) = f(t_0^-) \qquad \text{and} \qquad \lim_{t \to t_0^+} f(t) = f(t_0^+)$$

exist (as finite numbers) and $f(t_0^-) = f(t_0^+)$. Here, $t \to t^-$ and $t \to t^+$ mean that $t \to t_0$ from the left and right, respectively (Figure 3.2).

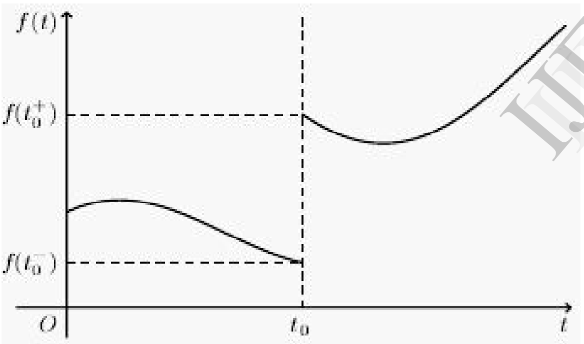


Figure 3.2

Exponential Order

The second consideration of our class of functions possessing a well defined Laplace transform has to do with the growth rate of the functions. In the definition

$$\mathcal{L}(f(t)) = \int_0^{\infty} e^{-st} f(t)\, dt,$$

whenwe take $s>0$ (or $Re(s) > 0$), the integral will converge as long as f does not grow too rapidly.

III. Some Important Properties of Laplace Transforms

The Laplace transforms of different functions can be found in most of the mathematics and engineering books and hence, is not included in this paper. Some of the very important properties of Laplace transforms which will be used in its applications to be discussed later on are described as follows:

a) Linearity

The Laplace transform of the linear sum of two Laplace transformable functions f(t) + g(t) is given by

$$L(f(t) + g(t)) = F(s) + G(s)$$

b) Differentiation

If the function f(t) is piecewise continuous so that it has a continuous derivative fn-1(t) of order n-1 and a sectionally continuous derivative fn(t) in every finite interval $0 \leq t \leq T$, then let, f(t) and all its derivatives through fn-1(t) be of exponential order ect as $t \rightarrow \infty$.

Then, the transform of fn(t) exists when Re(s) > c and has the following form:

$$Lf^n(t) = s^n F(s) - s^{n-1} f(0+) - s^{n-2} f^{(1)}(0+) - \ldots\ldots - s^{n-1} f^{(n-1)}(0+)$$

c) Time delay

The substitution of t -λ ¸ for the variable t in the transform Lf(t) corresponds to the multiplication of the function F(s) by e-λs, that is

$$L(f(t - \lambda)) = e^{-s\lambda} F(s)$$

d) The *t*-Integral Rule: Let $g(t)$ be of exponential order and continuous for $t \geq 0$. Then

$$\mathcal{L}\left(\int_0^t g(x)\, dx \right) = \frac{1}{s} \mathcal{L}(g(t)).$$

e) First Shifting Rule: Let $f(t)$ be of exponential order and $\infty < a < \infty$. Then

$$\mathcal{L}(e^{at} f(t)) = \mathcal{L}(f(t))|_{s \to (s-a)}$$

f) Second Shifting Rule: Let $f(t)$ and $g(t)$ be of exponential order and assume $a \geq 0$. Then

$$\text{(a)} \quad \mathcal{L}(f(t-a)H(t-a)) = e^{-as}\mathcal{L}(f(t)),$$
$$\text{(b)} \quad \mathcal{L}(g(t)H(t-a)) = e^{-as}\mathcal{L}(g(t+a)).$$

g) Periodic Function Rule: Let $f(t)$ be of exponential order and satisfy $f(t + P) = f(t)$. Then

$$\mathcal{L}(f(t)) = \frac{\int_0^P f(t) e^{-st} dt}{1 - e^{-Ps}}.$$

h) Convolution Rule: Let *f*(*t*) and *g*(*t*) be of exponential order. Then

$$\mathcal{L}(f(t))\mathcal{L}(g(t)) = \mathcal{L}\left(\int_0^t f(x)g(t-x)dx\right)$$

IV. Application of the Laplace transform

It is common in engineering education to find the perspective that the Laplace transform is just a theoretical and mathematical concept (outside of the real world) without any application in others areas. The transforms are considered as a tool to make mathematical calculations easier. However, it is important to notice that "frequency domain" is possible appreciate also in the real world and applied in other areas like, for example, economics.

"The most popular application of the Laplace transform is in electronic engineering, but it has also been applied to the economic and managerial problems, and most recently, to Materials Requirement Planning (MRP)" Yu and Grubbstr¨om (2001)

The article of Grubbstr¨om (1967) shows the application the Laplace transform to:

- Deterministic Economic Process
- Stochastic Economic Processes

It is pointed out that the metod of the Laplace transform has found an increasing number of applications in the fields of physics and technology. For example, the possibility of solving problems in the area of discounting with the aid of this method. Without any loss of general validity, it is shown that the discount factor can always be written in an exponential manner which implies that the present value of a "cash-flow" will obtain a very simple form in the Laplace transform terminology. This simplicity holds good for stochastic as well as for deterministic economic processes. Also it could be applied to all mathematical simplifications of reality. Grubbstr¨om (1996) consider a stochastic inventory process in which demand is generated by individuals separated by independent stochastic time intervals whereas production takes place in batches of possibily varying sizes at different points in time. The resulting processes are analyzed using the Laplace transform methodology. Then Grubbstrom and Molinder (1994) designed a generalized input-matrix to incorporate requirements as well as production lead times by means of z-transform methodology in a discrete time model.

The theory is extended to continuous time using the Laplace transform, which enables it to incorporate the possibility of batch production at finite production rates. Also Grubbstrom and Molinder (1996) developed a basic method of how such safety master production plans can be determinated in simple cases using the Laplace transform.

Other application: A telephone or simple inter-communicator does not need to be modulated, it is only necessary to have a couple of machines that transform the pressure waves into electric energy, the electric energy is sending by a couple of cables of copper (Cu) and in the receptor side is used a similar machine that convert the variations of electricity in variations of pressure. A microphone and speaker are built in same way.

We talk to the microphone by the diaphragm and we make the current in cables variabe, by other hand, we put variable current in the cables and the diaphragm produces sound. We can use the space instead of the cable to send signals. There is a lot of difference between send a signal trough the cable and to send it by the empthy. It is like to travel trough the flat road or to go through rough road. The air is not as conductive as the copper, otherwise we would be electrocuted. The alternating current (amplitude variable) has the capacity to travel through the space. To transmitt a signal from point A to point B trough space we need power and a very high frequency (almost radiating).

For lower frequencies we need more power and in the extreme case (continous current), it does not matter how much power is supplied, it radiates nothing. In certain special frequencies, the higher level of the atmosphere (ionosphere) that act as reflectors and rebound in them is possible to get a signal in large distances. This is "the short wave". If the frequency is too high, it passes across of the large and is lost in the exterior space; if it is too low, does not arrive and it is absorbed by the earth. The problem is that the frequncy of the sounds that we are able to hear is between 20Hz and 18 Khz (in people good hearing). If we directly convert the waves of the sound to electricity, they will be of a very low frequency that only can be transmitted by cable. The solution to transmit sounds by the air trough large distances, consist in to send a signal with high enough frequency so that it can radiate, but modifiying it in a proportional way to the variations of the sound that we want to send. The name of this frequency is "carrier" and the low is "modulation". There are different things that we can modify in a carrier and for it exist different methods of modulation: amplitude (AM), frequency (FM), phase (PM), etc.

The sounds are variations of pressure in the air and exist pure sounds and composed.

The pure sounds consist in only frequency (for example the "beep" of a computer) and are not common in the nature. The majority of the "real" sounds consist in thousand of different fequencies emitted at the same time. This let us to distinguish between a natural sound (rich in resonance) from an artificial (with not many components). The natural sounds like our voice, the music, the noice, etc. they do not consist in a frequency but in a band of frequencies with many fundamentals and other harmonies that produce rebound and resonance in the first.

V. Case

Eltayeb. A.M., et.al. "*Laplace Transform Method Solution of Fractional Ordinary Differential Equations*"

Summary: The Laplace transform method has been applied for solving the fractional ordinary differential equations with constant and variable coefficients. The solutions are expressed in terms of Mittage-Leffller functions, and then written in a compact simplified form. As special case, when the order of the derivative is two the result is simplified to that of second order equation.

In this paper they intend to apply Laplace transform method to solve fractional ordinary differential equations with constant coefficients. To achieve this task, special formulas of Mittage-Leffler function are derived and expressed in terms of elementary functions (power, exponential and error functions), instead of an infinite series. Also special formulations of inverse Laplace transformation are obtained, in terms of Mittage-Leffler functions, which already derived. In obtaining the inverse Laplace transform, some simplified elementary Algebra, relevant to the derived results is used. This work is based on some basic elements of fractional calculus, with special emphasis on the Riemann-Liouville type. For simplicity, they mainly used equations of order (2,2) with constant coefficients to illustrate this approach. However equation with variable coefficients of order α, where $1<\alpha\leq 2$, is considered. The obtained solution agrees with the solution of the classical ordinary differential equation, when $\alpha = 2$

Conclsuion: The Laplace transformation method has been successfully applied to find an exact solution of fractional ordinary differential equations, with constant and variable coefficients. Some theorems are introduced; also special formulas of Mittage-Leffler function are derived with their proofs. The method is applied in a direct way without using any assumptions. The results show that the Laplace transformation method needs small size of computations compared to the Adomain decomposition method (ADM), variational iteration method (VIM) and homotopy perturbation method (HPM). The numerical example for equation of variable coefficients shows that the solution in agreement with classical second order equation for

It is concluded that the Laplace transformation method is a powerful, efficient and reliable tool for the solution of fractional linear ordinary differential equations.

VI. Conclusions

To teach the Laplace transform as a separate mathematical topic seems to make it an obstacle for learning. From the survey it has been concluded that it is important to teach simple concepts that ought to have been understood earlier, especially at extreme values, e.g. abstracting an open circuit to an infinite resistance or a short circuit to a zero resistance.

Presented survey show the links between theoretical issues and the real circuits have to be made explicitly, something that is also shown in other studies and in the symposium Interaction in Labwork - linking the object/event world to the theory/model world. It is common that students learn to make mathematical operations without understanding what they are doing. They just repeat procedures that they have learned to solve the problem.

The Laplace transform is one of the many fields that have a teaching contents where it is very easy to disassociate the form and the meaning; the application and understanding of mechanic rules. The idea of some students concerning the Laplace transform is that it is a knowledge of strict and unquestionable rules that are applied to problems with just one solution, problems very far of the reality.

The disconnection between the application and understanding of procedures in specific situations can be dreadful in engineering education because some students think: mathematics is not necessary understand but it is necessary to know the adequate procedure to solve the problem. For this reason some students use superficial techniques to solve specific circumstances and there is not estrange to notice not motivation and absurd to make just mathematic calculus to pass the subject. It is necessary understand the process: "to go" and "come back" between the formal character, the strict mathematic language and it intuitive and contextual meaning.

A person who only can understand a transformation as an action can only make that action, reacting to external indications that give him exact detail about the steps that he has to do. For example, a student that is not able to interpret a situation like a function, with the exception that he has a formula to obtain values, he is restricted to a concept of action of a function. In this case the student cannot make many things with this function, except to evaluate it in specific points and manipulate the formula.

It is necessary to mark that mathematics is not disconnected of calculations but it is important do not do the routine calculations without understand the reality. The mathematic is not just a description group of elements.

References

[1] R.B. Chukleva, A.B. Dishliev, K.G. Dishlieva, Stability of the differential equations with variable structue and non fixed impulsive moments using sequences of Lyapunov's functions, International Journal of Differential Equations and Applications, 11, No. 1 (2012), 57-80.

[2] K.G. Dishlieva, A.B. Dishliev, S.A. Petkova, Death of the solutions of systems differential equations with variable structure and impulses, International Journal of Differential Equations and Applications, 11, No. 3 (2012), 169-181.

[3] M.B. Dimitrova, V.I. Donev, Oscillation criteria for the solutions of a first order neutral nonconstant delay impulsive differential equations with variable coefficients, International Journal of Pure and Applied Mathematics, 73, No. 1 (2011), 13-28.

[4] M.B. Dimitrova, V.I. Donev, On the non oscillation and oscillation of the solutions of a first order neutral non constant delay impulsive differential equations with variable or oscillating coefficients, International Journal of Pure and Applied Mathematics, 73, No. 1 (2011), 111-128.

[5] Tarig M. Elzaki, Salih M. Ezaki, On the Elzaki transform and ordinary differential equation with variable coefficients, The Advances in Theoretical and Applied Mathematics, 6, No. 1 (2011), 41-46.

[6] S. Huff, G. Olumolode, N. Pennington, A. Peterson, Oscillation of an Euler-Cauchy daynamic equation, In: Proceedings on the Fourth International Conference on Dynamical Systems and Differential Equations, May 24-27 (2002), 423-431.

[7] E. Kreyszig, Advanced Engineering Mathematics, John Willy & Sons, Inc., New York (2012).

[8] R. Kent Nagle, Edward B. Saff, A.D. Snider, Fundamental of Differential Equation, Pearson, Inc., Boston (1989).

[9] A. Kumar, D.K. Jaiswal, N. Kumar, Analytical solutions to one-dimensional advection-diffusion equation with variable coefficients in semi- infinite media, Journal of Hydrology, 380, No-s: 3-4 (2009), 330-337.

[10] S.R.S. Varadhan, On the behavior of the fundamental solution of the heat equation with variable coefficients, Comunication on Pure and Applied Mathematics (1967), 431-55.

[11] Y. Zhang, New grammian solutions for a variable-coefficient MKPI equation, International Journal of Pure and Applied Mathematics, 79, No. 3 (2012), 375-379.