ISSN: 2278-0181
ICESMART-2015 Conference Proceedings

An Analytical Solution of Effect of Water Content on Solute Transport in Saturated and Unsaturated Porous Media

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Abstract:- **Most of the researchers use the coordinate transformation (x - ut) in order to evaluate the equation for advection-dispersion of a moving fluid in porous media. Further, the boundary conditions $C = 0$ at $x = \infty$ and $C = C_0$ at $x = -\infty$ for $t > 0$ are used, which results in a symmetrical concentration distribution. The objective of this paper is to find the solution of differential equation in longitudinal direction that avoids this transformation, thus giving rise to an asymmetrical concentration distribution. It is then shown that the solution approaches that given by symmetrical boundary conditions, provided the dispersion coefficient D is small and the region near the source will not be considered. The solution has been obtained for the dispersion model of longitudinal mixing with variable coefficients in a finite length initially solute free domain. In the beginning, homogeneous domain is studied for dependent advection-dispersion along with uniform flow. The solution has been obtained for the uniform velocity by considering spatially dependent due to the heterogeneity of the domain and the dispersion proportional to the square of the velocity. The velocity is linearly interpolated and small increment along the finite domain. The input condition has been considered continuous of uniform and of increasing nature both. The solutions are obtained for both the domains by using Duhamel's theorem and integral solution technique. The new independent space and time variables processes has been introduced. The effects of the dependency of dispersion with time and the heterogeneity of the domain on the solute transport are studied separately with the help of graphs.**

Key words: Advection, dispersion, adsorption, Integral transforms, Fick's law,Duhamel's theorem,

1. INTRODUCTION

In recent years, considerable interest and attention have been directed to dispersion phenomena in flow through porous media. Scheidegger (1954), deJong (1958), and Day (1956) have presented statistical means to establish the concentration distribution and the dispersion coefficient. Advection–dispersion equation explains the solute transport due to combined effect of convection and dispersion in a medium. It is a partial differential equation of parabolic type, derived on the principle of conservation of mass using diffusion equation. Due to the growing surface and subsurface hydro environment degradation, the advection–diffusion equation has drawn significant attention of hydrologists, civil engineers and mathematical modelers. Its analytical/numerical solutions along with an initial condition and two boundary conditions help to understand the contaminant or pollutant concentration distribution behavior through an open medium like air, rivers, lakes and porous medium like aquifer, on the basis of which remedial processes to reduce or eliminate the damages may be enforced. It has wide applications in other disciplines too, like soil physics, petroleum engineering, chemical engineering and biosciences.

In the initial works while obtaining the analytical solutions of dispersion problems in ideal conditions, the basic approach was to reduce the advection–dispersion equation into a diffusion equation by eliminating the convective term(s). It was done either by introducing moving co-ordinates (Ogata and Banks 1961; Harleman and Rumer 1963; Bear 1972; Guvanasen and Volker 1983; Aral and Liao 1996; Marshal *et al* 1996) or by introducing another dependent variable (Banks and Ali 1964 Ogata 1970; Lai and Jurinak 1971; Marino 1974 and Al-Niami and Rushton 1977). Then Laplace transformation technique has been used to get desired solutions.

Some of the one-dimensional solutions have been given (Tracy 1995, Sudheendra 2011) by transforming the non-linear advection–diffusion equation into a linear one for specific forms of the moisture content *vs.* pressure head and relative hydraulic conductivity *vs.* pressure head curves which allow both two-dimensional and three-dimensional solutions has been derived. A method has been given to solve the transport equations for a kinetically adsorbing solute in a porous medium with spatially varying velocity field and dispersion coefficients (Van Kooten 1996, Sudheendra 2012).

ISSN: 2278-0181

Later it has been shown that some large subsurface formations exhibit variable dispersivity properties, either as a function of time or as a function of distance (Matheron and deMarsily 1980; Sposito *et al* 1986; Gelhar *et al* 1992). Analytical solutions were developed for describing the transport of dissolved substances in heterogeneous semi infinite porous media with a distance dependent dispersion of exponential nature along the uniform flow (Yates 1990, 1992). The temporal moment solution for one dimensional advective-dispersive solute transport with linear equilibrium sorption and first order degradation for time pulse sources has been applied to analyze soil column experimental data (Pang *et al* 2003). An analytical approach was developed for non-equilibrium transport of reactive solutes in the unsaturated zone during an infiltration–redistribution cycle (Severino and Indelman 2004, Sudheendra 2014).

The solute is transported by advection and obeys linear kinetics. Analytical solutions were presented for solute transport in rivers including the effects of transient storage and first order decay (Smedt 2006, Sudheendra 2012). Pore flow velocity was assumed to be a non-divergence, free, unsteady and non-stationary random function of space and time for ground water contaminant transport in a heterogeneous media (Sirin 2006). A two-dimensional semi-analytical solution was presented to analyze stream–aquifer interactions in a coastal aquifer where groundwater level responds to tidal effects (Kim *et al* 2007).

A more direct method is presented here for solving the differential equation governing the process of dispersion. It is assumed that the porous medium is homogeneous and isotropic and that no mass transfer occurs between the solid and liquid phases. It is assumed also that the solute transport, across any fixed plane, due to microscopic velocity variations in the flow tubes, may be quantitatively expressed as the product of a dispersion coefficient and the concentration gradient. The flow in the medium is assumed to be unidirectional and the average velocity is taken to be constant throughout the length of the flow field. In this paper, the solutions are obtained for two solute dispersion problems in a longitudinal finite length, respectively. In the first problem time dependent solute dispersion of increasing or decreasing nature along a uniform flow through a homogeneous domain is studied. In the second problem the medium is considered heterogeneous, hence the velocity is considered dependent on position variable. The velocity is linearly interpolated in position variable which represents a small increment in the velocity from one end to the other end of the domain. This expression contains a parameter to represent a change in heterogeneous from one medium to other medium. Dispersion is assumed proportional to square of velocity. In each problem the domain is initially solute free. The input condition is of uniform and varying nature, respectively. Numerical solution has also been obtained for the case in which dispersion varies linearly with velocity and has been compared with the analytical solution obtained in the previous cases.

2. TEMPORALLY DEPENDENT DISPERSION ALONG UNIFORM FLOW

Because mass is conserved, the governing differential equation is determined to be

$$\frac{\partial C}{\partial t} = \frac{\partial}{\partial x}\left(D(x,t)\frac{\partial C}{\partial x} - u(x,t)\,C \right) \tag{1}$$

where C is solute concentration at position x along the longitudinal direction at time t, D is dispersion coefficient and u is the average velocity of fluid or superficial velocity. To study the temporally dependent solute dispersion of a uniform input concentration of continuous nature in an initially solute free finite domain, we consider

$$D(x,t) = D_0 f(mt) \text{ and } u(x,t) = u_0 \tag{2}$$

When m is a coefficient whose dimension is inverse of the time variable. Thus $f(mt)$ is an expression in non-dimensional variable (mt). The expression of $f(mt) = 1$ for $m = 0$ or $t = 0$. The former case represents the uniform solute dispersion and the latter case represents the initial dispersion. The coefficients D_0 and u_0 in equation (2) may be defined as initial dispersion coefficient and uniform flow velocity, respectively. Thus the partial differential equation (1) along with initial condition and boundary conditions may be written as:

$$\frac{\partial C}{\partial t} = D_0 f(mt)\frac{\partial^2 C}{\partial x^2} - u_0\frac{\partial C}{\partial x} \tag{3}$$

Initially, saturated flow of fluid of concentration, $C = 0$, takes place in the medium. At $t = 0$, the concentration of the plane source is instantaneously changed to $C = C_0$. Thus, the appropriate boundary conditions are

$$\left.\begin{aligned} C(x,0) &= 0 & x \ge 0 \\ C(0,t) &= C_0\left(1-e^{-\gamma t}\right) & t \ge 0 \\ C(\infty,t) &= 0 & t \ge 0 \end{aligned}\right\}. \tag{4}$$

The problem then is to characterize the concentration as a function of x and t. where the input condition is assumed at the origin and a second type or flux type homogeneous condition is assumed. C_0 is initial concentration. To reduce equation (3) to a more familiar form, we take

$$C(x,t) = \Gamma(x,t)\exp\left(\frac{u_0 x}{2D_0 f(mt)} - \left\{ \frac{u_0^2 t}{4D_0 f(mt)} + \frac{k_d(1-n)t}{n} \right\} \right) \tag{5}$$

Substituting equation (5) into equation (3) gives

$$\frac{\partial \Gamma}{\partial t} = D_0 f(mt)\frac{\partial^2 \Gamma}{\partial x^2} \tag{6}$$

The initial and boundary conditions (3) transform to

$$\left.\begin{array}{ll}\Gamma(0,t)=C_0\left(1-e^{-\gamma t}\right)\exp\left[\dfrac{u_0^2 t}{4D_0 f(mt)}+\dfrac{k_d(1-n)t}{n}\right] & t\geq 0\\ \Gamma(x,0)=0 & x\geq 0\\ \Gamma(\infty,t)=0 & t\geq 0\end{array}\right\} \quad (7)$$

It is thus required that equation (6) may be solved for a time dependent influx of the fluid at x = 0. The solution of equation (5) may be obtained readily by use of Duhamel's theorem (Carslaw and Jaeger, 1947).

If $C=F(x,\ y,\ z,\ t)$ is the solution of the diffusion equation for semi-infinite media in which the initial concentration is zero and its surface is maintained at concentration unity, then the solution of the problem in which the surface is maintained at temperature $\phi(t)$ is

$$C=\int_0^t \phi(\lambda)\frac{\partial}{\partial t}F(x,\ y,\ z,\ t-\lambda)\,d\lambda$$

This theorem is used principally for heat conduction problems, but the above has been specialized to fit this specific case of interest. Consider now the problem in which initial concentration is zero and the boundary is maintained at concentration unity. The boundary conditions are

$$\left.\begin{array}{ll}\Gamma(x,0)=0 & x\geq 0\\ \Gamma(0,t)=1 & t\geq 0\\ \Gamma(\infty,t)=0 & t\geq 0\end{array}\right\} \quad (8)$$

The problem is readily solved by application of the Laplace transform which is defined as

$$L[\Gamma(x,t)]=\overline{\Gamma}(x,\ p)=\int_0^\infty e^{-pt}\,\Gamma(x,t)\,dt \quad (9)$$

Hence, if equation (6) is multiplied by e^{-pt} and integrated term by term it is reduced to an ordinary differential equation

$$\frac{d^2\overline{\Gamma}}{dx^2}=\frac{p}{D_0 f(mt)}\overline{\Gamma}\,.$$

The solution of the above equation is

$\Gamma=C_1\,e^{-qx}+C_2\,e^{qx}$ where, $q=\sqrt{\dfrac{p}{d}}$.

The boundary condition as $x\rightarrow\infty$ requires that $C_2=0$ and boundary condition at $x=0$ requires that $C_1=\dfrac{1}{p}$ thus the particular solution of the Laplace transformed equation is

$$\Gamma=\frac{1}{p}e^{-qx}$$

The inversion of the above function is given in any table of Laplace transforms. The result is

$$\Gamma=1-erf\left(\frac{x}{2\sqrt{D_0 f(mt)t}}\right)=\frac{2}{\sqrt{\pi}}\int_{\frac{x}{2\sqrt{Dt}}}^{\infty}e^{-\eta^2}d\eta$$

Utilizing Duhamel's theorem, the solution of the problem with initial concentration zero and the time dependent surface condition at *x =0* is

$$\Gamma=\int_0^t\phi(\tau)\frac{\partial}{\partial t}\left[\frac{2}{\sqrt{\pi}}\int_{\frac{x}{2\sqrt{D_0 f(mt)(t-\tau)}}}^{\infty}e^{-\eta^2}d\eta\right]d\tau \quad (10)$$

Since $e^{-\eta^2}$ is a continuous function, it is possible to differentiate under the integral, which gives

$$\frac{2}{\sqrt{\pi}}\frac{\partial}{\partial t}\int_{\frac{x}{2\sqrt{D_0 f(mt)(t-\tau)}}}^{\infty}e^{-\eta^2}d\eta=\frac{x}{2\sqrt{\pi D_0 f(mt)}(t-\tau)^{3/2}}\exp\left[\frac{-x^2}{4D_0 f(mt)(t-\tau)}\right] \quad (11)$$

The solution to the problem is

$$\Gamma=\frac{x}{2\sqrt{\pi D_0 f(mt)}}\int_0^t\phi(\tau)\exp\left[\frac{-x^2}{4D_0 f(mt)(t-\tau)}\right]\frac{d\tau}{(t-\tau)^{3/2}} \quad (12)$$

Putting $\lambda=\dfrac{x}{2\sqrt{D_0 f(mt)(t-\tau)}}$ then the equation (12) can be written as

$$\Gamma=\frac{2}{\sqrt{\pi}}\int_{\frac{x}{2\sqrt{D_0 f(mt)t}}}^{\infty}\phi\left(t-\frac{x^2}{4D_0 f(mt)\lambda^2}\right)e^{-\lambda^2}d\lambda \quad (13)$$

Since

$\phi(t)=C_0\left(1-e^{-\gamma t}\right)\exp\left(\dfrac{u_0^2 t}{4D_0 f(mt)}-\dfrac{k_d(1-n)t}{n}\right)$ the particular solution of the problem may be written as

$$\Gamma(x,t)=\frac{2C_0\left(1-e^{-\gamma t}\right)}{\sqrt{\pi}}\exp\left(\frac{u_0^2 t}{4D_0 f(mt)}-\frac{k_d(1-n)t}{n}\right)\left\{\int_0^\infty\exp\left(-\lambda^2-\frac{\varepsilon^2}{\lambda^2}\right)d\lambda-\int_0^\alpha\exp\left(-\lambda^2-\frac{\varepsilon^2}{\lambda^2}\right)d\lambda\right\} \quad (14)$$

where, $\alpha=\dfrac{x}{2\sqrt{D_0 f(mt)\,t}}$ and $\varepsilon=\dfrac{u_0 x}{4D_0 f(mt)}$.

ISSN: 2278-0181

ICESMART-2015 Conference Proceedings

Evaluation of the integral solution

The integration of the first term of equation (14) gives

$$\int_0^{\infty}\exp\left(-\lambda^2-\frac{\varepsilon^2}{\lambda^2}\right)d\lambda=\frac{\sqrt{\pi}}{2}e^{-2\varepsilon} \qquad (15)$$

For convenience the second integral may be expressed on terms of error function (Horenstein, 1945), because this function is well tabulated.
Noting that

$$-\lambda^2-\frac{\varepsilon^2}{\lambda^2}=-\left(\lambda+\frac{\varepsilon}{\lambda}\right)^2+2\varepsilon=-\left(\lambda-\frac{\varepsilon}{\lambda}\right)^2-2\varepsilon$$

The second integral of equation (14) may be written as

$$I=\int_0^{\alpha}\exp\left(-\lambda^2-\frac{\varepsilon^2}{\lambda^2}\right)d\lambda=$$

$$\frac{1}{2}\left\{e^{2\epsilon}\int_0^{\alpha}\exp\left[-\left(\lambda+\frac{\varepsilon}{\lambda}\right)^2\right]d\lambda+e^{-2\epsilon}\int_0^{\alpha}\exp\left[-\left(\lambda-\frac{\varepsilon}{\lambda}\right)^2\right]d\lambda\right\} \qquad (16)$$

Since the method of reducing integral to a tabulated function is the same for both integrals in the right side of equation (16), only the first term is considered. Let $z=\varepsilon/\lambda$ and adding and subtracting

$$e^{2\varepsilon}\int_{\varepsilon/\alpha}^{\infty}\exp\left[-\left(\frac{\varepsilon}{z}+z\right)^2\right]dz$$

The integral may be expressed as

$$I_1=e^{2\varepsilon}\int_0^{\alpha}\exp\left[-\left(\lambda+\frac{\varepsilon}{\lambda}\right)^2\right]d\lambda$$

$$=-e^{2\varepsilon}\int_{\varepsilon/\alpha}^{\infty}\left(1-\frac{\varepsilon}{z^2}\right)\exp\left[-\left(\frac{\varepsilon}{z}+z\right)^2\right]dz+e^{2\varepsilon}\int_{\varepsilon/\alpha}^{\infty}\exp\left[-\left(\frac{\varepsilon}{z}+z\right)^2\right]dz$$

(17)

Further, let, $\beta=\left(\frac{\varepsilon}{z}+z\right)$ in the first term of the above equation, then

$$I_1=-e^{2\varepsilon}\int_{\alpha+\frac{\varepsilon}{\alpha}}^{\infty}e^{-\beta^2}d\beta+e^{2\varepsilon}\int_{\frac{\varepsilon}{\alpha}}^{\infty}\exp\left[-\left(\frac{\varepsilon}{z}+z\right)^2\right]dz. \qquad (18)$$

Similar evaluation of the second integral of equation (16) gives

$$I_2=e^{-2\varepsilon}\int_{\varepsilon/\alpha}^{\infty}\exp\left[-\left(\frac{\varepsilon}{z}-z\right)^2\right]dz-e^{-2\varepsilon}\int_{\varepsilon/\alpha}^{\infty}\exp\left[-\left(\frac{\varepsilon}{z}-z\right)^2\right]dz\cdot$$

Again substituting $-\beta=\frac{\varepsilon}{z}-z$ into the first term, the result is

$$I_2=e^{-2\varepsilon}\int_{\frac{\varepsilon}{\alpha}-\alpha}^{\infty}e^{-\beta^2}d\beta-e^{2\varepsilon}\int_{\varepsilon/\alpha}^{\infty}\exp\left[-\left(\frac{\varepsilon}{z}-z\right)^2\right]dz$$

Noting that

[illegible]

Substitution into equation (10) gives

$$I=e^{-2\varepsilon}\int_{\frac{\varepsilon}{\alpha}-\alpha}^{\infty}e^{-\beta^2}d\beta-e^{2\varepsilon}\int_{\alpha+\frac{\varepsilon}{\alpha}}^{\infty}e^{-\beta^2}d\beta. \qquad (19)$$

Thus, equation (14) may be expressed as

$$\Gamma(x,t)=\frac{2C_0\left(1-e^{-\gamma t}\right)}{\sqrt{\pi}}\exp\left(\frac{u^2t}{4D_0\,f(mt)}-\frac{k_d(1-n)t}{n}\right) \qquad (20)$$

$$\left\{\frac{\sqrt{\pi}}{2}e^{-2\varepsilon}-\frac{1}{2}\left[\int_{\frac{\varepsilon}{\alpha}-\alpha}^{\infty}e^{-\beta^2}d\beta-e^{2\varepsilon}\int_{\alpha+\frac{\varepsilon}{\alpha}}^{\infty}e^{-\beta^2}d\beta\right]\right\}$$

However, by definition,

$$e^{2\varepsilon}\int_{\alpha+\frac{\varepsilon}{\alpha}}^{\infty}e^{-\beta^2}d\beta=\frac{\sqrt{\pi}}{2}e^{2\varepsilon}\,erfc\left(\alpha+\frac{\varepsilon}{\alpha}\right)$$

Also,

$$e^{-2\epsilon}\int_{\frac{\varepsilon}{\alpha}-\alpha}^{\infty}e^{-\beta^2}d\beta=\frac{\sqrt{\pi}}{2}e^{-2\varepsilon}erfc\left(\alpha-\frac{\varepsilon}{\alpha}\right).$$

Writing equation (20) in terms of error functions, we get

$$\Gamma(x,t)=\frac{C_0\left(1-e^{-\gamma t}\right)}{2}\exp\left(\frac{u^2t}{4D_0\,f(mt)}-\frac{k_d(1-n)t}{n}\right) \qquad (21)$$

$$\left[e^{2\varepsilon}erfc\left(\alpha+\frac{\varepsilon}{\alpha}\right)+e^{-2\varepsilon}erfc\left(\alpha-\frac{\varepsilon}{\alpha}\right)\right]$$

Thus, Substitution into equation (5) the solution is

$$\frac{C(x,t)}{C_0\left(1-e^{-\gamma t}\right)}=\frac{1}{2}\left[erfc\left(\alpha-\frac{\varepsilon}{\alpha}\right)+e^{4\varepsilon}erfc\left(\alpha+\frac{\varepsilon}{\alpha}\right)\right]$$

Re-substituting for ε and α gives

$$\frac{C(x,t)}{C_0\left(1-e^{-\gamma t}\right)}=\frac{1}{2}\left\{\begin{matrix}erfc\left(\frac{x-ut}{2\sqrt{D_0\,f(mt)\,t}}\right)+\\ \exp\left(\frac{ux}{D_0\,f(mt)}\right)erfc\left(\frac{x+ut}{2\sqrt{D_0\,f(mt)\,t}}\right)\end{matrix}\right\}$$

Re-substitute the value of the u in terms of u_0, we get

$$\frac{C(x,t)}{C_0\left(1-e^{-\gamma t}\right)}=\frac{1}{2}\left\{\begin{matrix}erfc\left(\frac{x-u_0t}{2\sqrt{D_0\;f(mt)\,t}}\right)+\\ \exp\left(\frac{u_0x}{D_0\;f(mt)}\right)erfc\left(\frac{x+u_0t}{2\sqrt{D_0\;f(mt)\,t}}\right)\end{matrix}\right\} \qquad (22)$$

where boundaries are symmetrical the solution of the problem is given by the first term the equation (22). The second term is equation (22) is thus due to the asymmetric boundary imposed in the more general problem. However, it should be noted also that if a point a great distance away from the source is considered, then it is possible to approximate the boundary condition by $C(-\infty, t) = C_0$, which leads to a symmetrical solution.

3. SPATIALLY DEPENDENT DISPERSION ALONG NON-UNIFORM FLOW

The heterogeneity of porous domain was defined by scale dependent dispersion and flow through the medium has been considered uniform Yates (1992) but the flow velocity may also depend upon position variable in case the domain is heterogeneous. Zoppou and Knight (1997) have considered the velocity as $u = \beta x$, and the solute dispersion proportional to square of velocity, i.e., as $D = \alpha x^2$; in a semi-infinite domain $x_0 \le x < \infty$. But these expressions do not reflect real variations due to heterogeneity of the medium because as $x \to \infty$, dispersion and velocity also become too large. In fact the variation in velocity due to heterogeneity should be small so that the velocity at each position satisfies the Darcy's law in case the medium is porous or satisfies the laminar condition of the flow in a non-porous medium, an essential conditions for the velocity parameter, *u* in the advection-diffusion equation. This factor is taken care of in the present work and velocity is linearly interpolated in position variable such that it increases from a value u_0 at $x = 0$ to a value $(1+b)u_0$ at $x = L$, where b may be a real constant. Thus

$$u(x,t) = u_0(1+ax), \qquad (23a)$$

Where $a = b/L$, is the parameter accounting for the heterogeneity of the medium. It should be small so that the increase in velocity is of small order. Solute dispersion is assumed proportional to square of the velocity so we consider

$$D(x,t) = D_0(1+ax)^2 \qquad (23b)$$

As ax is a non-dimensional term hence D_0 and u_0 are dispersion coefficient and velocity, respectively at the origin ($x = 0$) of the medium. The domain is assumed initially solute free. An input concentration is assumed at the origin and a flux type homogeneous condition is assume at the other end of the domain. Then advection-diffusion equation assumes the form

$$\frac{\partial C}{\partial t} = D_0(1+ax)^2 \frac{\partial^2 C}{\partial x^2} - u_0(1+ax)\frac{\partial C}{\partial x} \qquad (24)$$

It is further reduced into a partial differential equation with constant coefficients by using a transformation. Ultimately we use the same initial and boundary conditions to solve the above dispersion problem for dependent dispersion non-uniform. The procedure is same as solved in the earlier case. Then the desired solution may be written as

$$\frac{C(x,t)}{C_0} = \frac{1}{2}\left\{ \begin{array}{l} erfc\left(\dfrac{x - u_0(1+ax)t}{2(1+ax)\sqrt{D_0 t}}\right) + \\ \exp\left(\dfrac{u_0(1+ax)x}{D_0(1+ax)^2}\right) erfc\left(\dfrac{x + u_0(1+ax)t}{2(1+ax)\sqrt{D_0 t}}\right) \end{array} \right\} \qquad (25)$$

A plot of logarithmic probability graph of the above solution is given for various values of the dimensionless group $\eta = D_0/u_0 x$. The figure shows that as η becomes small the concentration distribution becomes nearly symmetrical about the value ξ = 1 (i.e., $\xi = u_0 t/x$). However, for large values of η asymmetrical concentration distributions become noticeable. This indicates that for large value of D or small values of distance x the contribution of the second term in equation (25) becomes significant as ξ approaches unity.

4. RESULTS AND DISCUSSIONS

Concentration values are evaluated from the four analytical solutions discussed in a finite domain at times *t* (years) = 1.0, 2.0, 3.0 and 4.0, for input values C_0 = 1.0, u_0 = 0.11 (km/year), D_0 = 50 (km^2/year). Figures 1 represents temporal dependent concentration dispersion pattern of uniform input and input of increasing nature, respectively along a uniform flow through a homogeneous medium, described by the analytical solutions, equation (22), respectively. In figure 1, the uniform input concentration value is 1.0 at all times and the concentration value at $x = 0$ increases with time. Thus the respective input boundary conditions are satisfied. In the figure the dotted curves represents the solutions for an expression $f(mt) = \exp(-mt)$ which is of decreasing nature. In the figures the solid curve represents the respective solutions at t = 1.0 (year), for another expression $f(mt) = \exp(mt)$, which is of increasing nature. It may be observed that in case of uniform input the concentration value at a particular position is higher for the latter expression of $f(mt)$ than that for the former expression of $f(mt)$. The difference increases with the distance along the domain. But in case of an input concentration of increasing nature its value is less for increasing nature of $f(mt)$ than that for decreasing nature of $f(mt)$. This trend is of diminishing nature up to x = 2.0, beyond which the trend reverses. For all the curves drawn in figure 1, a value m(year)−1 = 1.0 is chosen. Both the analytical solutions of section 2 may be solved using other expressions of $f(mt)$ which satisfy the conditions stated at the outset of the section 2.

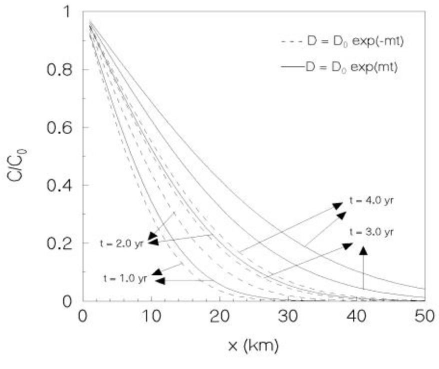

Figure 1: Temporal dependent solute dispersion along uniform flow of uniform input described by solution (equation 22).

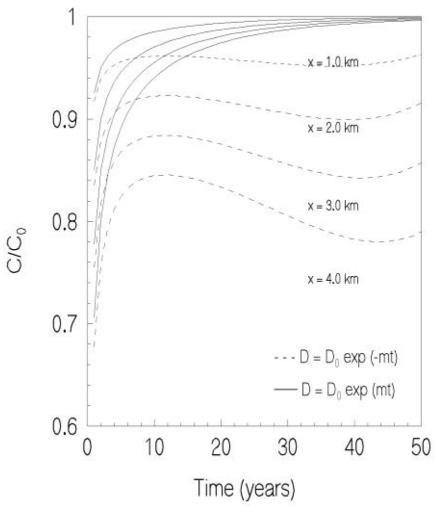

Figure 2: Break through curve for dispersion along with uniform flow.

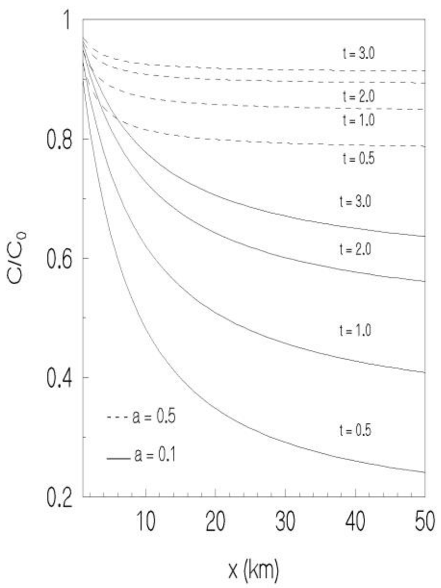

Figure 3: Spatially dependent solute dispersion along with non-uniform flow input described by solution (equation 25).

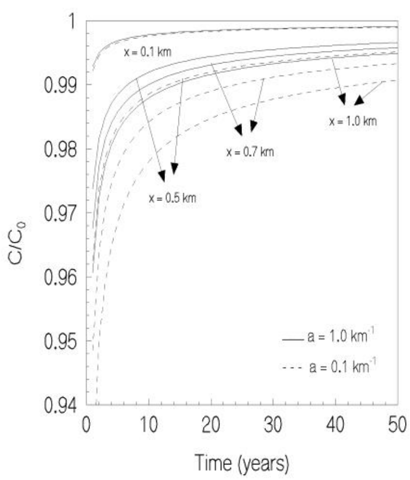

Figure 4: Break through curve for dispersion along with non-uniform flow.

The distribution is symmetrical for values of x chosen some distance from the source. An example of break through curves obtained for dispersion in a cylindrical vertical column is shown as Figure 2. The theoretical curve was obtained by neglecting the second term of equation (22).

Figure 3 gives the concentration values evaluated from analytical solutions (equations 25) for spatially dependent dispersion of uniform input and input of increasing nature, respectively, along non-uniform flow, through an heterogeneous domain. The solid curves in figure 3 represent the solution (equation 30) in which a value a = 1.0 (km^{-1}) is taken. Using expressions it may be evaluated that due to the heterogeneity of the medium, the velocity u varies from a value of 0.11 (km/year) to a value of 0.22 (km/year) and dispersion D varies from a value of 0.21 (km/year) to a value of 0.42 (km/year), along the domain $0 \leq x$(km) ≤ 1. This figure also shows the effect of heterogeneity on the dispersion pattern. A dotted curve is drawn for the value a = 0.1 (km^{-1}). It may be observed that the concentration values evaluated from the solution (equation 25) along a medium of lesser heterogeneity (which introduces lesser variation in velocity and dispersion along the column), are slightly higher than those at the respective positions of a medium of higher heterogeneity, near the origin but decrease at faster rate as the other end of the medium is approached. This comparison is done at t = 2.0 (year). This value is chosen to ensure that the factor $(u_0 - aD_0)$ in condition remains positive for the values chosen for u_0 and D_0. The distribution is symmetrical for values of x chosen some distance from the source. A break through curve is obtained for dispersion in for different depth as shown in Figure 4. The theoretical curve was obtained by neglecting the second term of equation (25).

5. CONCLUSIONS

Consideration of the governing differential equation for dispersion in flow through porous media give rise to a solution that is not symmetrical about $x = u_0 t$ for large values of η. Experimental evidence, however, reveals that D_0 is small. This indicates that, unless the region close to the source is considered, the concentration distribution is approximately symmetrical. Theoretically, $\frac{C}{C_0} \to \frac{1}{2}$ only as $\eta \to 0$; however, only errors of the order of magnitude of experimental errors are introduced in the ordinary experiments if a symmetrical solution is assumed

The solution is obtained for one dimensional advection – diffusion equation with variable coefficients along with two set of boundary conditions in an initially solute free finite domain have been obtained in two cases:

1. temporal dependent dispersion along with uniform flow through homogeneous medium and
2. spatially dependent dispersion along non-uniform flow through heterogeneous medium which solute dispersion is assumed proportional to the square of velocity.

The application of a new transformation which introduces another space variable, on the advection-diffusion equation makes it possible to use Laplace transformation technique in getting the solution. Numerical solution has been obtained using a two-level explicit finite difference scheme. The respective analytical and numerical solutions have also been compared and very good agreement between the two has been found. The analytical solution of the second problem in case of uniform input has been compared with the numerical solution of same problem but assuming dispersion varying with velocity. Such analytical solutions may serve as tools in validating numerical solutions in more realistic dispersion problems facilitating to assess the transport of pollutants solute concentration away from its source along a flow through soil medium, through aquifers and through oil reservoirs.

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