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An Analytical Approximate Solution for the Bratu Problem by using Nonlinearities Distribution Homotopy Perturbation Method

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***Abstract*— This work introduces the nonlinearities distribution homotopy perturbation method in order to get an accurate analytical approximate solution for the relevant Bratu problem which has many applications in science and engineering. We will show that the distribution of the nonlinearities in the different iterations of the proposed method is a convenient scheme which eases the obtaining of the approximation based on elementary integrals. We will conclude that the proposed method is very convenient for practical applications.**

Keywords:-Bratu problem, nonlineardifferential equation, approximate solution.

I. INTRODUCTION

The Bratu problem in one dimension is called Liouville-Bratu-Gelfand and has many applications in science and engineering where physical and chemical phenomena are modeled. One of the applications of this equation is in solid fuel ignition models of the thermal combustion theory [1-2]. Gelfand's differential equation with boundary value in one dimension in planar coordinates is given by [3-8].

$$u'' + \varepsilon e^{u(x)} = 0. \quad 0<x<1. \qquad (1)$$

with boundary conditions $u(0) = u(1) = 0$.

The analytical solution of (1)is given by [3, 4]

$$u(x) = -2\ln\left[\frac{\cosh\left(\left(x-\frac{1}{2}\right)\frac{\theta}{2}\right)}{\cosh\left(\frac{\theta}{4}\right)}\right], \qquad (2)$$

where θ is the solution of

$$\theta = \sqrt{2\varepsilon}\cosh\left(\frac{\theta}{4}\right). \qquad (3)$$

There are three cases for the solution; if $\varepsilon > \varepsilon_c$ then the problem has zero solution, if $\varepsilon = \varepsilon_c$ there is one solution, and $\varepsilon < \varepsilon_c$ there are two solutions, where $\varepsilon_c = 3.513830719$ is the critical value and satisfies the equation.

$$1 = \frac{1}{4}\sqrt{2\varepsilon_c}\sinh\left(\frac{\varepsilon_c}{4}\right). \qquad (4)$$

Different approximate methods have been applied to solve the Bratu problem. For example in [9] Chebyshev polynomials was employed in order to obtain an approximate solution for the proposed problem. In [10] the Adomian decomposition method was used to solve the equation. In [11] the B-spline method was proposed to solve the equation. In [12] the Leal-polynomials were proposed to find an accurate analytical approximate solution for (1). In [13] Perturbation Method was employed to get a handy approximate solution and so on.

This research paper is organized as follows In Section 2 we introduce the idea of Homotopy Perturbation Method (HPM). For Section 3 we briefly introduce Nonlinearities Distribution Homotopy Perturbation Method (NDHPM). Additionally Section 4 presents the application of the proposed method in the search for an analytical approximate solution for Bratu problem. Besides a discussion on the results is presented in Section 5. Finally, a brief conclusion is given in Section 6.

II. HOMOTOPY PERTURBATION METHOD

To show how homotopy perturbation method (HPM) works, first, consider a general nonlinear differential equation as follows

$$A(u) - f(r) = 0, \quad r \in \Omega \qquad (5)$$

With the boundary conditions

$$B(u, \partial u/\partial n) = 0, \quad r \in \Gamma, \qquad (6)$$

where A is a general differential operator, B is a boundary operator, $f(r)$ a known analytical function and Γ is the domain boundary for Ω. In general terms, A can be divided into two operators L and N, where L is linear and N is nonlinear; so that (5) can be rewritten as

$$L(u) + N(u) - f(r) = 0. \qquad (7)$$

A homotopy can be constructed as [14, 15]

$$H(U,p) = (1-p)[L(U) - L(u_0)] + p[L(U) + N(U) - f(r)] = 0, \quad r \in \Omega, \qquad (8)$$

where p is a homotopy parameter, which value varies continuously from 0 to 1, u_0 is the first approximation for the solution of (5) that satisfies the boundary conditions. The solution of (8) can be written as power series of p

$$U = v_0 + v_1 p + v_2 p^2 + \cdots. \quad (9)$$

Substituting (9) into (8) and equating identical powers of p terms, there can found values for $v_0, v_1, v_2, \cdots.$ when p tends to (5), we obtain the approximate solution for (1) in the form

$$U = v_0 + v_1 + v_2 + \cdots. \quad (10)$$

III. BASIC IDEA OF NONLINEARITIES DISTRIBUTION HOMOTOPY PERTURBATION METHOD

In [16] was introduced nonlinearities distribution homotopy perturbation method (NDHPM) with the purpose to reduce the complexity of solving differential equations in terms of power series.

To carry out the procedure a homotopy is written in the form (Vazquez-Leal et al. 2012b) [16]

$$H(U,p) = (1-p)[L(U) - L(u_0)] + p[L(U) + N(U,p) - f(r,p)] = 0, \qquad r \in \Omega. \quad (11)$$

Homotopy function (11) is essentially the same as (8), except because the non-linear operator N and the non homogeneous function f, contain embedded the homotopy parameter p. The rest of the procedure coincides with that explained in the last Section.
From [14,16] we have

$$U = v_0 + v_1 p + v_2 p^2 + \cdots. \quad (12)$$

When p tends to 1, it is expected to get an approximate solution for (7) in the form

$$U = v_0 + v_1 + v_2 + \cdots. \quad (13)$$

IV. APPLICATION OF NDHPM

Next, we will employ NDHPM in order to find an analytical approximate solution for the Bratu problem.

In accordance with the proposed problem we proposed the following homotopy:

$$(1-p)y'' + p\left(y'' + \varepsilon e^{py(x)}\right) = 0, \quad (14)$$

or

$$y'' + p\varepsilon e^{py(x)} = 0. \quad (15)$$

We note exponential function contain embedded the homotopy parameter p.

After using Taylor approximation for the exponential function we obtain the following approximation as follows:

$$e^{py(x)} = 1 + py + \frac{p^2y^2}{2} + \frac{p^3y^3}{6} + \dots \quad (16)$$

After substituting (16) into (15) we obtain

$$y'' + p\varepsilon\left(1 + py + \frac{p^2y^2}{2} + \frac{p^3y^3}{6} + \dots\right) = 0, \quad (18)$$

let it be

$$y = \sum_{n=0}^{\infty} p^n y_n(x), \quad (19)$$

so that after substituting (19) into (18) and equating identical powers of p terms we obtain

$$p^0)\ \ y_0'' = 0,$$

$$p^1)\ \ y_1'' + \varepsilon = 0,$$

$$p^2)\ \ y_2'' + \varepsilon y_0 = 0,$$

$$p^3)\ \ y_3'' + \frac{\varepsilon}{2} {y_0}^2 = 0, \quad (20)$$

and so on.

The solutions of equations (20) which satisfy the initial conditions of the problem in accordance with the proposed problem are:

$$p^0)\ y_0 = Ax,$$

$$p^1)\ y_1 = -\frac{\varepsilon x^2}{2} + cx,$$

$$p^2)\ y_2 = \frac{-\varepsilon Ax^3}{6} + Dx,$$

$$p^3)\ y_3 = \frac{-\varepsilon A^2x^4}{24} + Ex, \quad (21)$$

and so on.

Substituting (21) into (13)

$$y(x) = (A+K)x - \frac{\varepsilon x^2}{2} - \frac{\varepsilon Ax^3}{6} - \frac{\varepsilon A^2x^4}{24}, \quad (22)$$

where the constant K is given by

$$K = c + D + E. \quad (23)$$

With the end to get a better precision, we will determine the value of K, from the requiring that the approximate solution satisfy the real value for x=0.5. As a consequence (22) adopts the following form:

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$$y(x)=\left(\frac{A^2}{384}+0.0208A+0.256817\right)x-\frac{\varepsilon x^2}{2}-\frac{\varepsilon A x^3}{6}-\frac{\varepsilon A^2 x^4}{24}. \quad (24)$$

With the purpose to verify the accuracy of the proposed approximation we propose the value ε=0.5 in such a way that (24) assumes the form

$$y(x)=\left(\frac{A^2}{384}+0.0208A+0.256817\right)x-\frac{x^2}{4}-\frac{A x^3}{12}-\frac{A^2 x^4}{48}. \quad (25)$$

With the end to determinate A we apply the condition y(1)=0 to (25) to generate the following algebraic equation

$$\frac{7A^2}{384}+\frac{3A}{48}-0.006817=0. \quad (26)$$

The solution to this equation is substituted into (25) to obtain

$$y(x)=0.2590x-\frac{x^2}{4}-0.008817x^3-0.0002332x^4. \quad (27)$$

Figure 1 compares (27) with the numerical solution for (1) with ε = 0.5.

Although the precision of (27) is clear at sight, from Table 1 is clear the precision of the proposed solution.

V. DISCUSION

In this work NDHPM was employed in order to find an approximate solution, for the nonlinear ordinary differential equation that describes the relevant Bratu problem that models the solid fuel ignition which rises of the thermal combustion theory. One of the great advantages of the proposed method is that it systematically distribute the nonlinearity term in the different iterations which ease the obtaining of an analytical approximate solution. The procedure consisted in ensure that the approximate solution satisfy the real value for *x=0.5* and pose an algebraic equation for determining an unknown parameter A. The solution of this equation was obtained from the boundary condition $y(1)=0$ This solution provides the sought analytical approximation solution.

Figure 1 shows the comparison between numerical solution and approximate solution (27) for ε=0.5. It can be noticed that curves are in good agreement, whereby it is clear that the proposed method is potentially useful in the search for approximate solutions of nonlinear problems definite with boundary conditions.

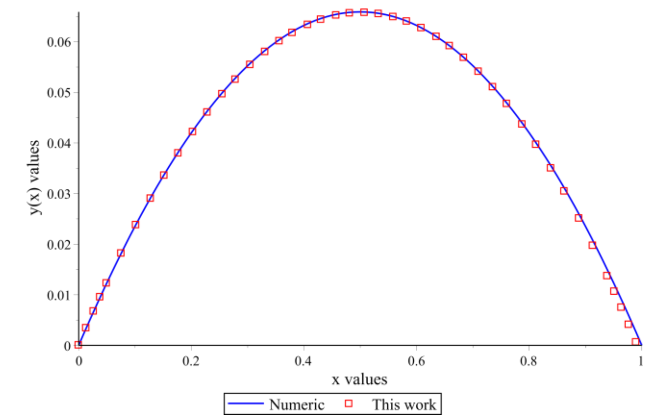

Figure 1. Comparison between numerical solution and approximate solution (27) for ε = 0.5

On the other hand, Table 1 shows a more detailed comparison between numerical solution for (1) and approximation (27) of this work for ε=0.5. The mentioned table shows that the relative error is less than one percent for most of the interval. It is clear that (27), is competitive taking into account that the proposed method required to solve essentially elementary integrals and an algebraic equation. From above it is very important to emphasize that, it is possible to improve the accuracy of our approximation performing more iterations from the homotopy (18).

V1. CONCLUSION

This work presented method NDHPM with the purpose to calculate an analytical approximate solution for the Bratu problem. The method basically works calculating elementary integrals after distributing the nonlinear term in the several iterations of the method in order to ease the procedure. As a matter of fact the method expressed the solution of a differential equation in terms of the solution of one or more algebraic equations. The application of the proposed method in this work showed the convenience of employ NDHPM as a practical tool with the purpose to obtain accuracy solutions for boundary value problems instead of using other more sophisticated and cumbersome procedures.

x	Exact	(19)	Error relative
0.1	0.0235388	0.0233909	0.62%
0.2	0.0420135	0.0417257	0.68%
0.3	0.0552701	0.05494305	0.59 %
0.4	0.0632473	0.06297601	0.42%
0.5	0.0659146	0.06575212	0.24%
0.6	0.0632473	0.06319330	0.085%
0.7	0.0552701	0.0552158	0.098%
0.8	0.0420135	0.04173050	0.67%
0.9	0.0235388	0.02264238	3.8%

Table 1: Comparison between (27), and exact solution using ε=0.5

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